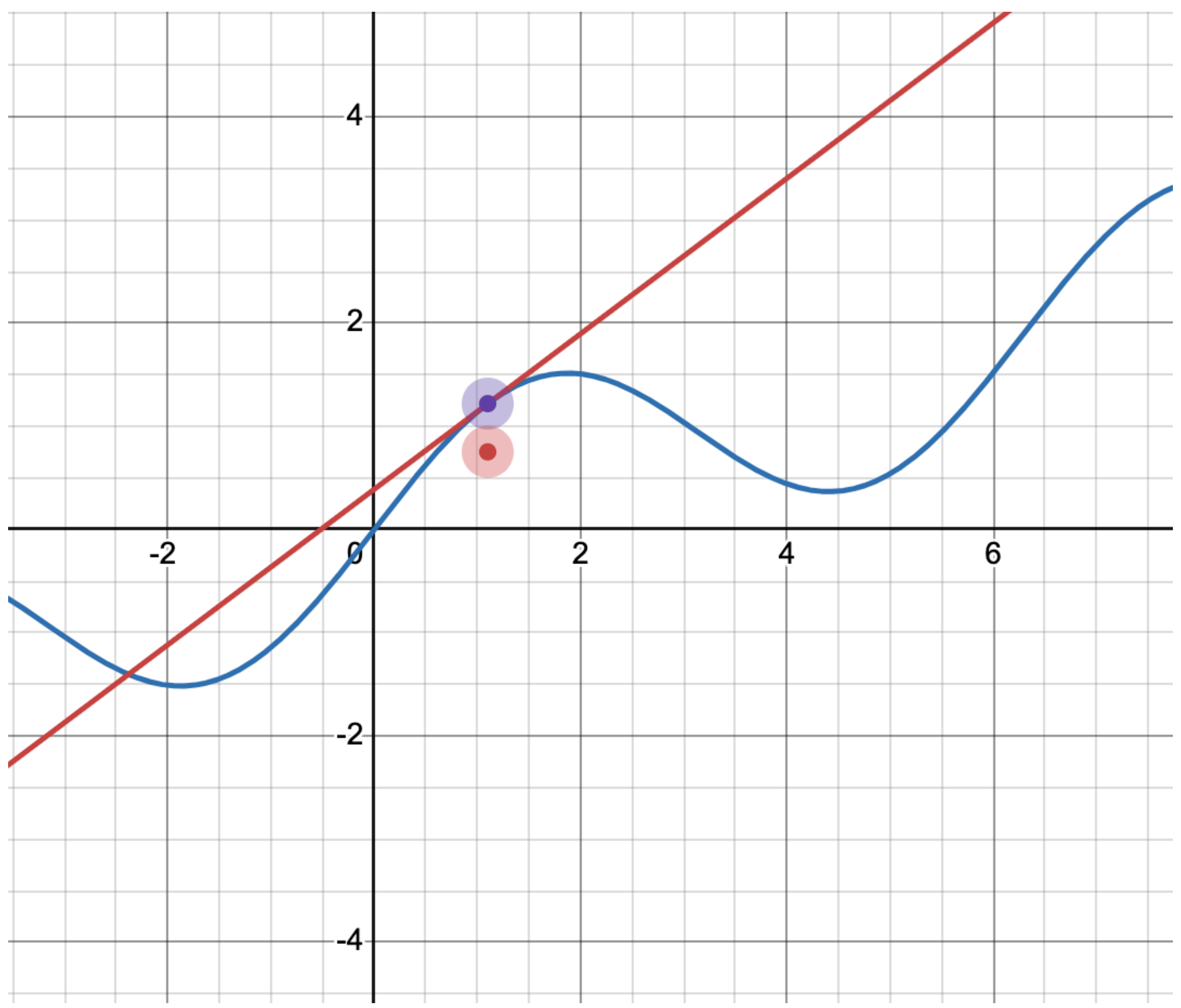
Differential Calculus

Chris Palmer, July 9th, 2024

Definition with limits

- When you first learn derivatives you had to use this (and find a way to factor out h from the numerator) to compute derivatives.
- $\frac{df}{dx}(x = a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$
- This relates directly to the graphical definition of the slope of the tangent line at $x = a$.
- One should always be able to start from an infinite table of numbers and draw an infinite number of derivatives.
    - No model needed.



https://www.desmos.com/calculator/4pf1dxxzq2

Derivative to remember

- Polynomials
  - $f(x) = ax^n$ $\qquad \dfrac{df}{dx} = ?$
- Sine/Cosine
  - $g(t) = A\sin(\omega t)$ $\qquad \dfrac{dg}{dt} = ?$
  - $h(\omega) = B\cos(\omega t)$ $\qquad \dfrac{dh}{d\omega} = ?$
- Exponential
  - $l(T) = l_0 e^{aT}$ $\qquad \dfrac{dl}{dT} = ?$
- Logarithms
  - $k(s) = k_0 \ln(s)$ $\qquad \dfrac{dk}{ds} = ?$

Derivative to remember

- Polynomials
  - $f(x) = ax^n$ $\qquad \dfrac{df}{dx} = anx^{n-1}$
- Sine/Cosine
  - $g(t) = A\sin(\omega t)$ $\qquad \dfrac{dg}{dt} = A\omega\cos(\omega t)$
  - $h(\omega) = B\cos(\omega t)$ $\qquad \dfrac{dh}{d\omega} = -Bt\sin(\omega t)$
- Exponential
  - $l(T) = l_0 e^{aT}$ $\qquad \dfrac{dl}{dT} = l_0 a e^{aT}$
- Logarithms
  - $k(s) = k_0 \ln(s)$ $\qquad \dfrac{dk}{ds} = \dfrac{k_0}{s}$

$$f(x) = \sqrt{x}$$

$$g(t) = \left(\frac{1}{2}\right)^t$$

$$h(\omega) = \frac{10}{\omega^3}$$

$$L(I) = 10\log\left(\frac{I}{I_0}\right)$$

Chain rule

- Peel away the functions like layers on an onion.
- $f(x) = g(h(x));\qquad \frac{df}{dx} = \frac{dg}{dh}\frac{dh}{dx}$

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- Peel away the functions like layers on an onion.
  - $f(x) = g(h(x));$ $\quad \dfrac{df}{dx} = \dfrac{dg}{dh}\dfrac{dh}{dx}$
- $f(t) = \sin\left(e^{At}\right)$
  - What are *g* and *h*? Compute the derivative.
- Got time? $h(x) = \ln(\sqrt{x^2 + 1})$

Product rule

- $h(x) = f(x)g(x); \quad \frac{dh}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$
- Explain that in words to your partner.

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- Let's say that $h(x) = \dfrac{f(x)}{g(x)}$; use the product rule to derive the "quotient rule".
    - Hint: you need the chain rule too.

Product rule

- $h(x) = f(x)g(x); \qquad \dfrac{dh}{dx} = \dfrac{df}{dx}g + f\dfrac{dg}{dx}$
- Take turns working through these with a partner.
    - $h(x) = (x^2 - 4)(x^3 + x)$
    - $y = y_0 \tan x$
    - $P(t) = \dfrac{Ae^{-at}}{t^2 - 3}$

Product rule

- $h(x) = f(x)g(x)k(x);\quad \frac{dh}{dx} = \frac{df}{dx}gk + f\frac{dg}{dx}k + fg\frac{dk}{dx}$
- Explain that in words again.

Product rule

- $h(x) = f(x)g(x)k(x); \quad \frac{dh}{dx} = \frac{df}{dx}gk + f\frac{dg}{dx}k + fg\frac{dk}{dx}$
- Take turns with a partner.
  - $f(x) = (x^2 - 3)(x + 2)(x + 4)$
  - $g(t) = e^{at}\frac{\sin \omega t}{\cos 2\omega t}$

Example: Delta Functions

- $$\delta\left(g(x)\right) = \sum_{x_i \in \{x_i | g\left(x_i\right)=0\}} \frac{1}{|\,g'\left(x_i\right)\,|}\delta\left(x - x_i\right)$$

- Compute $\delta\left(g(x)\right)$ when $g(x) = (x-3)(x+1)(x+2)$.

Implicit differentiation

- If one cannot solve for a single variable, one can still find the derivative by differentiating each term and solving for the derivative you want.
- For example, what is $\dfrac{dy}{dx}$ when $x^2 - xy - y^2 - 2y = 0$.

Parametric derivatives

- Let's say we know position as a function of time.
- $\vec{r}(t) = (x(t), y(t))$, how do find $\dfrac{dy}{dx}$?

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- Let’s say we know position as a function of time.
    - $\vec{r}(t) = (x(t), y(t))$, how do find $\dfrac{dy}{dx}$?
- $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- $\vec{r}(t) = \left(v_0 t, -\dfrac{g}{2}t^2\right)$; find $\dfrac{dy}{dx}$

Euler-Langrange Equation

- $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0$
- Where $L = T - U$
- Simple pendulum
    - What are *T* and *U*?
    - Evaluate Euler-Langrange Equation