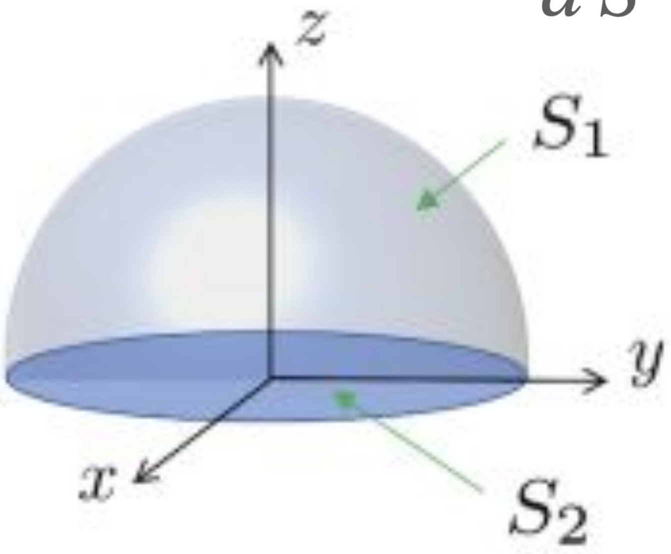
The Integral Theorems: Divergence/ Gauss' Theorem

$$\iiint \overrightarrow{\nabla} \cdot \overrightarrow{F} \, dV = \iint_{\text{boundary}} \overrightarrow{F} \cdot d\overrightarrow{S}$$

$$d\overrightarrow{S} = \hat{n} dA$$

- A relation between a surface integral of a vector and the volume integral of the divergence of that vector.
- Normal vector, $\hat{n}$, always points OUTSIDE the volume.



$$V : x^2 + y^2 + z^2 \leq R^2; \; z \geq 0$$

Evaluate the left- and right-hand side of the Divergence Theorem for this vector:

$$\overrightarrow{F} = (z + R)\,\hat{z}$$

$$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\text{boundary}} \vec{F} \cdot d\vec{S}$$

Uniformly charged sphere

$$d\vec{S} = \hat{n} dA$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

A uniformly charged completely hollow sphere of radius *R* has total charge *Q*. Use Gauss's Law to determine the field inside and outside the sphere.

1) What should the direction of the field be?
2) On which variables should the field depend?
3) Which coordinate system is best to use?
4) What is the shape of your volume to integrate?
5) Evaluate the volume integral.
6) Evaluate the different pieces of the surface integral.

Uniformly charged ball

A ball is uniformly charged and has total charge Q. The outer radius of the shell is r_o.

Use Gauss's Law to determine the field:

1. Inside the shell ($r<r_o$), and
2. Outside the shell ($r>r_o$).

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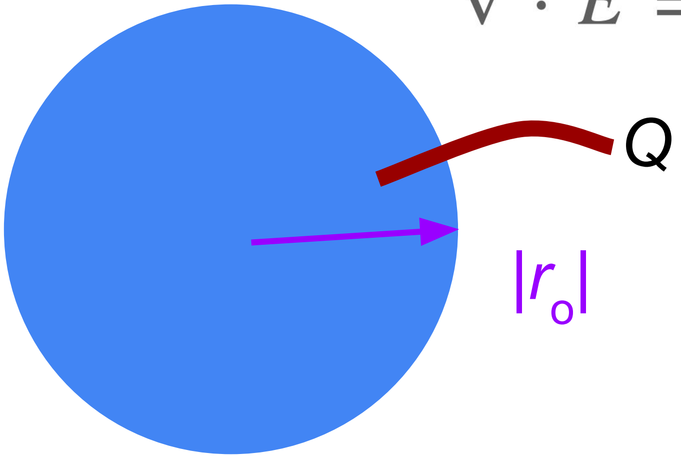

Uniformly charged ball with hollow interior

A thick spherical shell is uniformly charged and has total charge Q. The inner radius of the hollow region is r_i, and the outer radius of the shell is r_o.

Use Gauss's Law to determine the field:

1. Inside the hollow region ($r<r_i$),
2. Inside the shell ($r_i<r<r_o$), and
3. Outside the shell ($r>r_o$).

Draw a graph of the electric field as a function of r.

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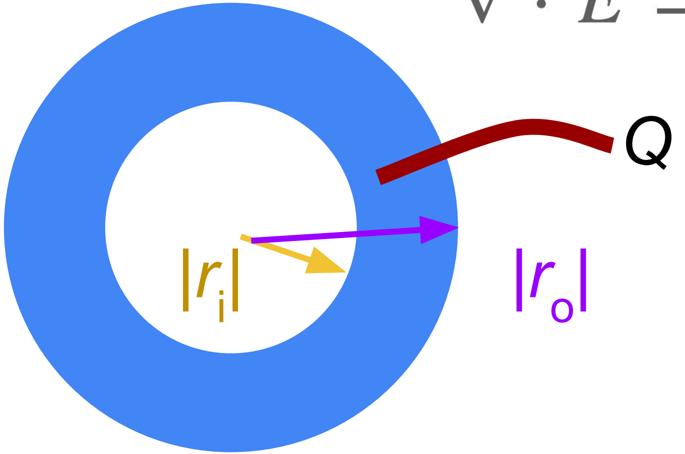

Electric field of an infinite plane of charge (pillbox)

A plane with uniform charge density, σ, produces an electric field. Use Gauss's law and Gauss's theorem to determine the field everywhere in space.

1) What should the direction of the field be?
2) On which variables should the field depend?
3) Which coordinate system is best to use?
4) What is the shape of your volume to integrate?
5) Evaluate the volume integral.
6) Evaluate the different pieces of the surface integral.

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Electric field of an infinite line of charge

A line with uniform charge density, λ, produces an electric field. Use Gauss's law and Gauss's theorem to determine the field everywhere in space.

$$\iiint \overrightarrow{\nabla} \cdot \overrightarrow{F}\, dV = \iint_{\text{boundary}} \overrightarrow{F} \cdot d\overrightarrow{S}$$

$$d\overrightarrow{S} = \hat{n} dA$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\varepsilon_0}$$

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2) On which variables should the field depend?
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4) What is the shape of your volume to integrate?
5) Evaluate the volume integral.
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The Integral Theorems: Stokes' Theorem

$$\iint_S \left(\vec{\nabla} \times \vec{V}\right) \cdot d\vec{S} = \oint_{\partial S} \vec{V} \cdot d\vec{l}$$

- The open surface integral on the left is equivalent to the close line integral on the right.
- The open surface can be ANY surface with the boundary ∂S.
- The orientation of the line integral and the normal vector will follow the right hand rule.
    - Your thumb is the normal vector and your fingers point in the direction of the path.

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- Compute the left- and right-hand side of Stokes' Theorem when the vector is:

$$\vec{V} = \left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}$$

- Over the path
  - (0,0,0)->(0,1,0)->(0,1,1)->(0,0,1)->(0,0,0)

Computing magnetic fields with Stokes' theorem

$$\iint_S \left(\vec{\nabla} \times \vec{V}\right) \cdot d\vec{S} = \oint_{\partial S} \vec{V} \cdot d\vec{l}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

In a very similar fashion to Gauss' law with Gauss' theorem, we can use Ampere's law with Stokes' theorem to also find magnetic fields when you can find a surface where the magnetic field is constant or 0.

This is a much easier method than integrating over all moving charges to compute magnetic fields (Biot-Savart).

Again, one needs to be able to determine the direction of the magnetic field before evaluating the line integral.

Magnetic field from an infinite wire with current, *I*

What is the magnetic field produced by a THIN infinite line with total current, *I*?

1) What should the direction of the field be?
2) On which variables should the field depend?
3) Which coordinate system is best to use?
4) What is the shape of your volume to integrate?
5) Evaluate the surface area integral.
6) Evaluate the different pieces of the line integral.

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Magnetic field inside a thick wire

What is the magnetic field produced by a THICK infinite line with total current, *I*, and radius, *R*, INSIDE and OUTSIDE the wire?

$$\iint_S \left(\vec{\nabla} \times \vec{V}\right) \cdot d\vec{S} = \oint_{\partial S} \vec{V} \cdot d\vec{l}$$

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1) What should the direction of the field be?
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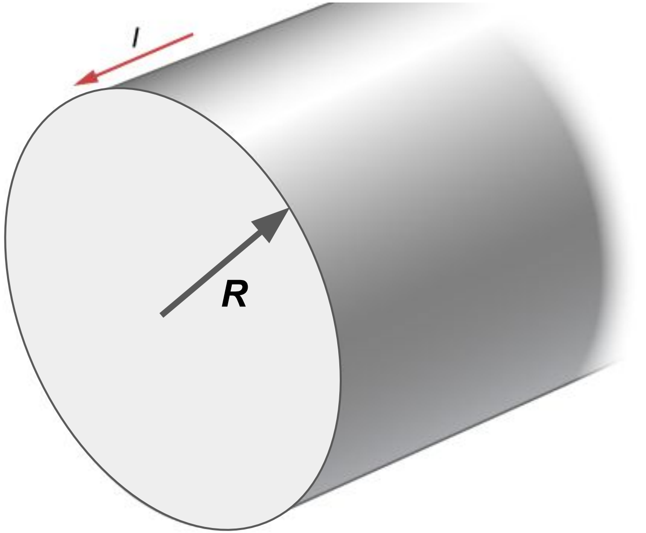

Magnetic field inside a coaxial cable

What is the magnetic field produced in each region of the coaxial cable shown in the figure?

1) What should the direction of the field be?
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3) Which coordinate system is best to use?
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5) Evaluate the surface area integral.
6) Evaluate the different pieces of the line integral.

$$\iint_S \left(\vec{\nabla} \times \vec{V}\right) \cdot d\vec{S} = \oint_{\partial S} \vec{V} \cdot d\vec{l}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$