Calculus Fundamentals

Derivative

$$L = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Total Differential

$$\begin{aligned} dF(x,y) &\equiv F(x+dx, y+dy) - F(x,y) \\ &= [F(x+dx, y+dy) - F(x, y+dy)] + [F(x, y+dy) - F(x,y)] \\ &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy, \end{aligned}$$

Note: x derivative taken at **y + dy**, not **y**. The derivative amount is altered by on order dy. Negligible in the limit **dy → 0**

Chain Rule

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx},$$

Fundamental Theorem of Calculus

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Integration by Parts

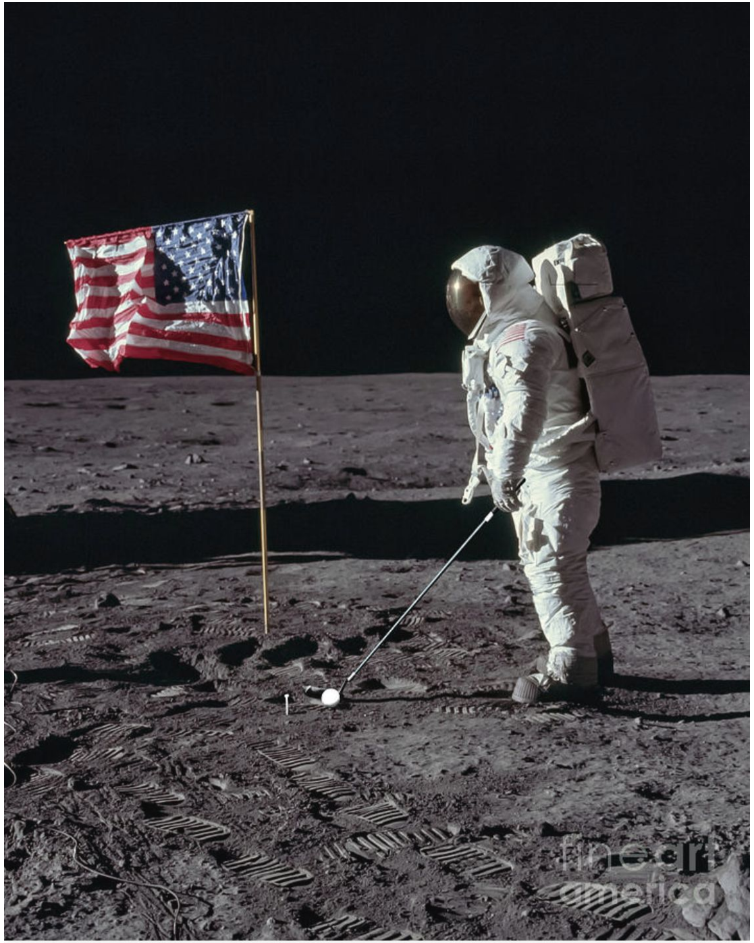
$$\int u dv = uv - \int v du$$

Kinematic Example (1 of 2)

A projectile has velocity parametrized by time as **v = (0, 5.3, 6.1-8.9*t)** m/s.

If the position of projectile at t=1s is **r**(t=1s) **= (-10, 5.3, 1.5)** m, what is the position as a function of time?

What is acceleration vector? (Guess which planet this is.)



Golf on the moon is a photograph by Delphimages Photo Creations which was uploaded on July 17th, 2020.

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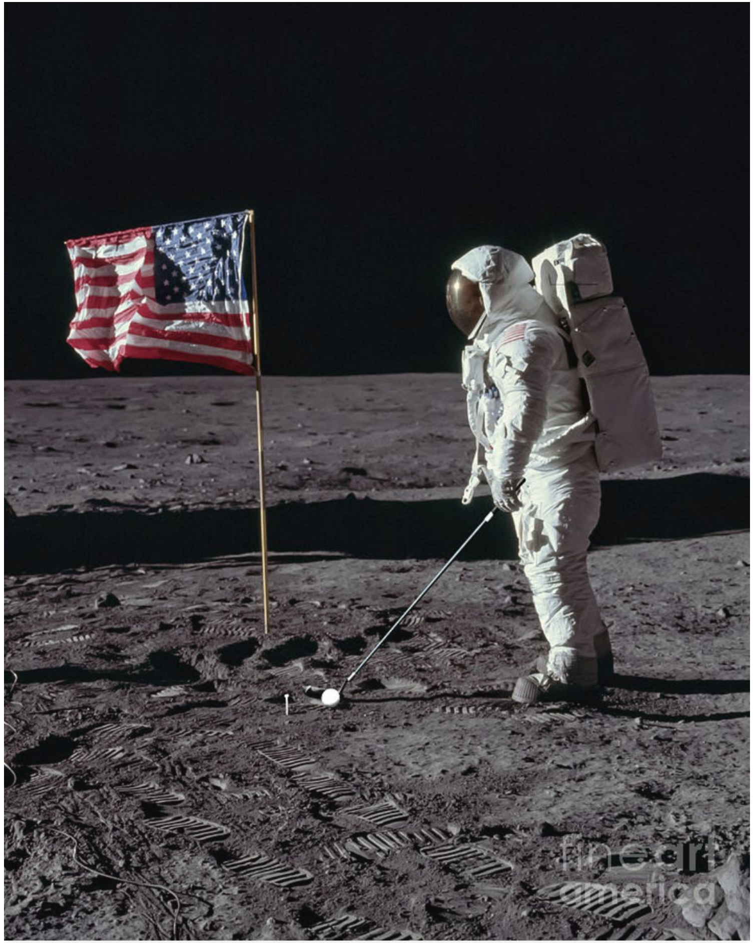
Kinematic Example (2 of 2)

A projectile has velocity parametrized by time as **$v = (0, 5.3, 6.1-8.9*t)$** m/s.

What is the displacement of the projectile from t=0 to t=9 s?

What is the distance traveled from t=0 to t=9 s? (Write the integral out and estimate it with a sum.)

Why are the distance and displacement so different?



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Chain rule reminder

Chain Rule $$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx},$$

This is a way of taking the derivatives of nested functions.

$$f(x) = k\left(h\left(g\left(x\right)\right)\right)$$

$$\frac{df}{dx} = \frac{dk}{dh}\frac{dh}{dg}\frac{dg}{dx}$$

Chain rule reminder

Chain Rule $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx},$

$$f(x) = A \cdot e^{Bx^2} \quad \frac{df}{dx} = ?$$

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$$\frac{df}{dx} = \frac{dk}{dh}\frac{dh}{dg}\frac{dg}{dx}$$

$$f(x) = A \cdot e^{Bx^2} \quad \frac{df}{dx} = ?$$

$$x\left(t\right) = C \sin \omega t \quad \frac{df}{dt} = ?$$

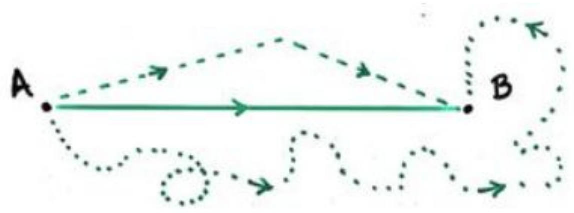
Differential Operators: Gradient

Gradient: $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$ —Application→ $\nabla\varphi = \hat{\mathbf{x}}\frac{\partial\varphi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\varphi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\varphi}{\partial z}$

- If the change in energy depends only upon the endpoints, the force is **conservative,** and a potential energy exists.

- Useful in expressing the relation between a force field and a potential field. The negative ensures the force is in the direction that minimizes the potential (see next slide)

$$\text{force } \mathbf{F} = -\nabla(\text{potential } V)$$



$$W = \int_C \mathbf{F}\cdot d\mathbf{x} = U(\mathbf{x}_A) - U(\mathbf{x}_B)$$

Gradient example

If $S(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}$, find

(a) ∇S at the point (1, 2, 3);
(b) the magnitude of the gradient of S, $|\nabla S|$ at $(1, 2, 3)$;

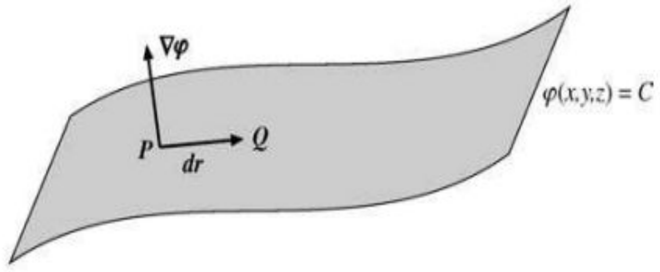
Gradient: A Geometrical Interpretation

An immediate application of the Gradient: The change in the scalar function corresponding to a change in position d**r**.

$$d\mathbf{r} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz. \quad \Longrightarrow \quad \nabla\varphi \cdot d\mathbf{r} = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = d\varphi,$$

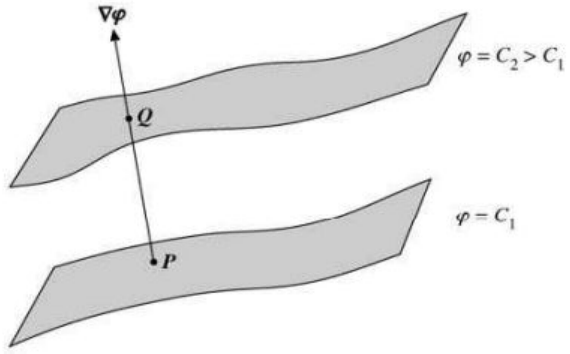
Take P and Q to be two points on a surface φ(x, y, z) = constant. Moving from P to Q, the change in φ(x, y, z) = constant is zero because we stay on the surface. Thus, $\nabla\varphi$ is perpendicular to d**r**.

$$d\varphi = (\nabla\varphi)\cdot d\mathbf{r} = 0$$



If we let d**r** be perpendicular to the surface, it takes us to an adjacent surface, from φ = C1 to φ = C2.

$$d\varphi = C_1 - C_2 = \Delta C = (\nabla\varphi)\cdot d\mathbf{r}$$



For a given |d**r**|, the change in the scalar function (dφ) is maximized by choosing d**r** parallel to $\nabla\varphi$.
Hence $\nabla\varphi$ is a vector having the direction of maximum space rate of change of φ.

Differential Operators: Divergence

The divergence is defined through the operation: $$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Exercise 1:

(1) For a general function f(r) and the position vector **r**, compute $\nabla \cdot (\mathbf{r} f(r))$

(2) Compute for the specific case: $f(r) = r^{n-1}$

Exercise 2:

Compute the divergence of: $$\vec{F} = x^2 y\,\vec{i} - \left(z^3 - 3x\right)\vec{j} + 4y^2\vec{k}.$$

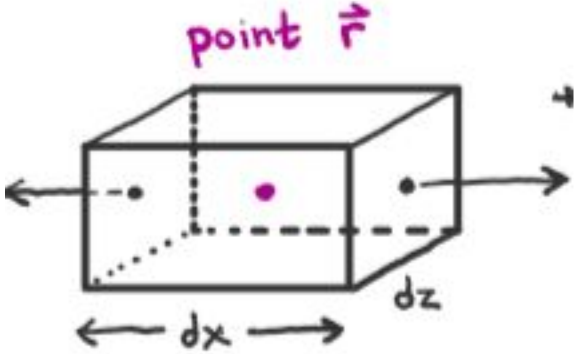
Divergence: A Physical Interpretation

Suppose a vector field **v**(**r**) is the velocity of a fluid and ρ(**r**) is the fluid density. Then the direction and magnitude of the flow rate at any point is **v**(**r**)ρ(**r**).

We seek the net rate of change fluid density in a volume element at about the point **r.**

Flow out face at $x + \frac{dx}{2}$: $\rho v_x \big|_{x+\frac{dx}{2}} \, dy\,dz$

Flow out face at $x - \frac{dx}{2}$: $-\rho v_x \big|_{x-\frac{dx}{2}} \, dy\,dz$



Divergence: A Physical Interpretation

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We seek the net rate of change fluid density in a volume element at about the point **r.**

Recognize this as the differential of **v**(**r**)ρ(**r**) in the x-direction

$$\left[-(\rho V_x)\Big|_{x-dx/2} + \rho V_x\Big|_{x+dx/2} \right] dy\,dz = \left(\frac{\partial(\rho V_x)}{\partial x} \right) dx\,dy\,dz$$

Adding corresponding contributions from other directions, we get the **net flow out per unit time**.

$$\left[\frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z) \right] = \nabla \cdot (\rho \vec{V})\, dx\,dy\,dz$$

Hence **∇** * (ϱ**V**) is the net outward flux per unit volume

Objectives this week

1. Complete the review of differential operators
2. Figure out how to define differential operators in arbitrary (curvilinear) coordinates
    a. Particularly going from cartesian to cylindrical or spherical coordinates.
3. Have a functional knowledge of:
    a. Generalized Stokes' theorem and applications of it.
    b. E.g. Gauss' theorem (3d->2d), Green's theorem, Stokes' theorem (2d->1d) (the specific one…)

Differential Operators: Curl

$$\vec{\nabla} \times \vec{V} = \quad ?$$

Review: Differential Operators: Gradient and Divergence

Gradient:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Divergence:

$$\vec{\nabla}\cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot$$

$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Differential Operators: Curl

$$\overrightarrow{\nabla} \times \overrightarrow{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Curl and divergence of electric field from a point charge

Exercise 3:
Compute the divergence **and** the curl of the field of an electric charge in cartesian coordinates.

What happens at the origin?

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Central field example, $\vec{r}f(r)$

Exercise 1:

(1) Why is this called a “central field”?

(2) For a general function f(r) and the position vector **r**, compute

$$\vec{\nabla} \cdot \left(\vec{r}f(r)\right)$$

(3) Compute for the specific case:

$$f(r) = r^{n-1}$$

(4) Compute the curl of an arbitrary central field.

Determining a magnetic from a force on test charge, *q*

The magnetic induction **B** is **defined** by the Lorentz force equation,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}).$$

Carrying out three experiments, we find that if

$$\mathbf{v} = \hat{\mathbf{x}}, \qquad \frac{\mathbf{F}}{q} = 2\hat{\mathbf{z}} - 4\hat{\mathbf{y}},$$

$$\mathbf{v} = \hat{\mathbf{y}}, \qquad \frac{\mathbf{F}}{q} = 4\hat{\mathbf{x}} - \hat{\mathbf{z}},$$

$$\mathbf{v} = \hat{\mathbf{z}}, \qquad \frac{\mathbf{F}}{q} = \hat{\mathbf{y}} - 2\hat{\mathbf{x}}.$$

From the results of these three separate experiments calculate the magnetic induction **B**.

Using differential operators in Maxwell's equation

This is just a starting point.

In the future, you'll be expected to derive the EM wave equation from these.

What's the first step?

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\varepsilon_0}$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} - \frac{1}{c^2}\frac{\partial \overrightarrow{E}}{\partial t} = \mu_0 \overrightarrow{J}$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0$$

Using differential operators in Maxwell's equation

This is just a starting point.

In the future, you'll be expected to derive the EM wave equation from these.

Get rid of sources, and curl one of the equations. Which one?

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} - \frac{1}{c^2}\frac{\partial \overrightarrow{E}}{\partial t} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0$$

Using differential operators in Maxwell's equation

This is just a starting point.

In the future, you'll be expected to derive the EM wave equation from these.

Get rid of sources, and curl one of the equations. Keep going!

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) = 0$$

Orthogonal Curvilinear Coordinates

Until now, we've relied on Cartesian coordinates to define our differential operators.

We can describe a general point **r** using some coordinates:

$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

The total differential is then

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3$$

We define how the point changes with respect this new coordinate system.

$$\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{q}_i$$

h_i = scale factor

$h_i > 0$

$$d\vec{r} = h_1 \hat{q}_1 dq_1 + h_2 \hat{q}_2 dq_2 + h_3 \hat{q}_3 dq_3$$

Orthogonal Curvilinear Coordinates Exercises

Exercise

(a) Find the scale factors, h_is, for cylindrical polar coordinates

(b) Find the scale factors, h_is, for spherical polar coordinates

Cylindrical Transformation

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$

Spherical Transformation

$$x = \rho \cos\phi \sin\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\theta$$

Scale Factor

$$\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{q}_i$$

Differential Operators in Curvilinear Coordinates

Gradient - Because the coordinates are orthogonal, the gradient takes the same form as for Cartesian coordinates:

$$\nabla = \hat{\mathbf{e}}_1 \frac{1}{h_1}\frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2}\frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3}\frac{\partial}{\partial q_3}.$$

Provided we use differential displacements:

$$\mathbf{dr_i = h_i\, dq_i}$$

Divergence

$$\nabla \cdot \mathbf{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right].$$

Curl

$$\nabla \times \mathbf{B} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{\mathbf{e}}_1 h_1 & \hat{\mathbf{e}}_2 h_2 & \hat{\mathbf{e}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

- These will be revisited later! They can be proved relatively easy with the appropriate integral theorems.

Exercise: The Radial Function

Note: the laplacian can be computed from the divergence and the gradient.

Compute

$\nabla f(r)$

$\nabla \cdot (\hat{e}_r f(r))$

$\nabla \times (\hat{e}_r f(r))$ and

$\nabla^2 f(r) = \nabla \cdot \nabla f(r)$

For $f(r) = r^n$

Useful equations

$$\nabla = \hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial q_3}.$$

$$\nabla \times \mathbf{B} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{e}}_1 h_1 & \hat{\mathbf{e}}_2 h_2 & \hat{\mathbf{e}}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

$$\nabla \cdot \mathbf{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2) \right].$$

Exercise: The Magnetic Vector Potential

A single current loop in the xy-plane has a vector potential **A** that is a function only of the radius and polar angle. It is related to the current density **J** by the equation:

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = \nabla \times \left[\nabla \times \hat{\mathbf{e}}_\varphi A_\varphi(r, \theta) \right]$$

Evaluate the cross products to get an expression for the current density.

Vector Integration Review

Line, Surface, and Volume Integrals

- We will be doing several types of integration.
- At this point we have essentially two forms of integrands (the thing you integrate):
  - Scalar functions
    - E.g. mass density, scalar potentials, dot products of
  - Vectors functions
    - E.g. forces, electric fields, cross products of other vectors.
- The integrand type (e.g. scalar or vector) is independent of differential.
- Up to now we have performed only 1-dimensional integration, but there's a lot more that you can do now.

- The differential tells you how many integrations you need to do.

- Line integrals, dx
- Surface integrals, dA=dx dy
- Volume integrals, dV=dx dy dz

- The differential can also be a scalar or a vector.
  - So it is possible for the integrand and the the differential to both be vectors, but the integral to a scalar. How?

Volume integral of uniform density planet

The density of a planet is uniform, ρ.

Setup the integral for the total mass of the planet, *M*. The planet has a finite radius, *R*.

What is dV in cartesian coordinates? What is dV in spherical coordinates? Is dV a scalar or a vector?

Evaluate the mass integral.

Volume integral of non-uniform density planet

The density of a planet is linearly declining with increasing radius in this way:

$$\rho(r) = \rho_0 \left(1 - 0.5r\right)$$

Setup the integral for the total mass of the planet, *M*. The planet has a finite radius, *R*.

What is dV in cartesian coordinates? What is dV in spherical coordinates? Is dV a scalar or a vector?

Evaluate the mass integral.