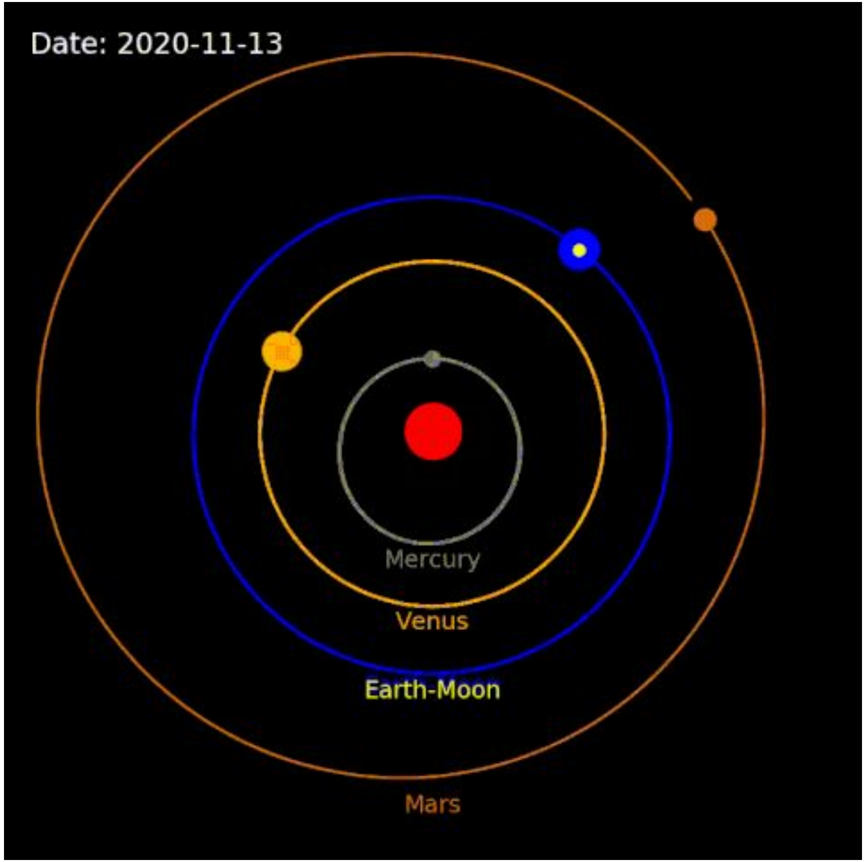
Vector Analysis and Calculus

July 10th, 2023

Geometry

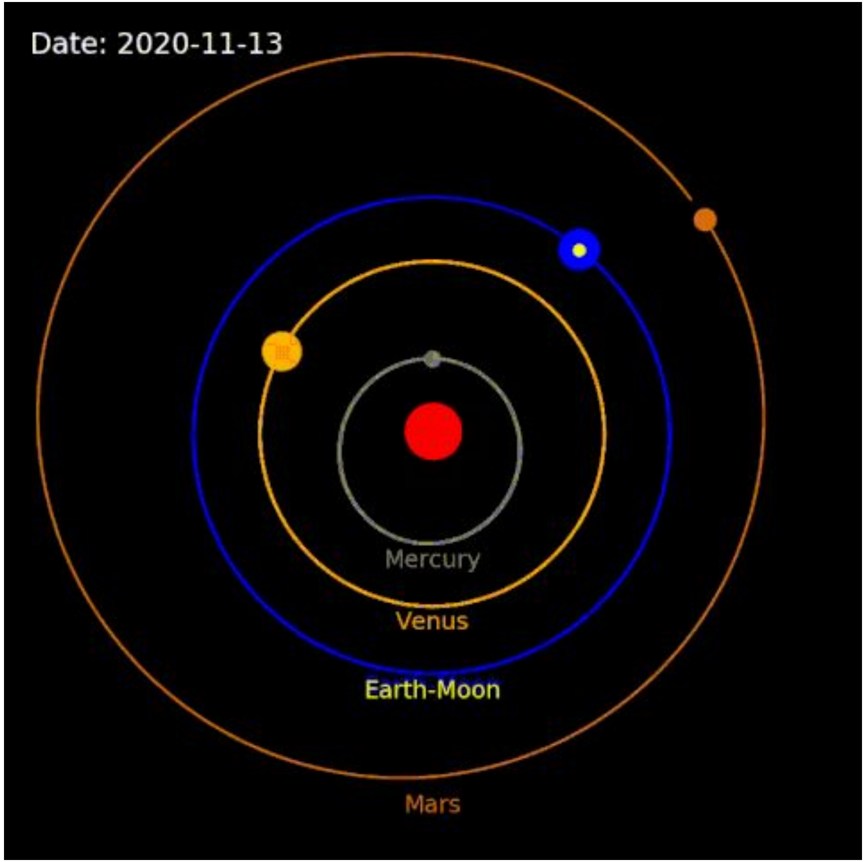
Distance to Venus at Quarter Phase

- The earth is 1 AU from the sun.
- Venus is 0.72 AU from the sun.
- When venus is in the quarter phase as viewed from the earth, what is the distance between venus and the earth?
    - First draw a picture of the three-body system.
    - Do you know any of the angles?
    - After drawing the picture, what is your strategy for solving?

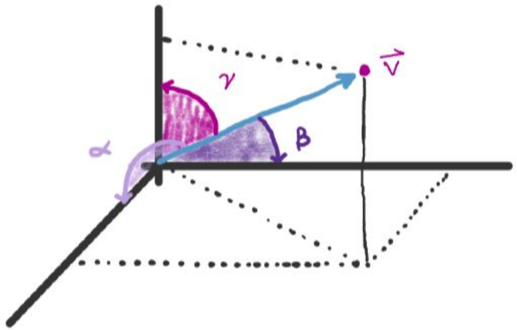


Distance to Venus

- The earth is 1 AU from the sun.
- Venus is 0.72 AU from the sun.
- The angle between the sun and venus is 15˚.
- What could the distance between venus and the earth?
    - First draw a picture of the three-body system.
    - After drawing the picture, what is your strategy for solving?



Vectors Inner Product (Dot Product)



A **direction cosine** of the vector **v** is the angle between the **vector and the coordinate axis**: cos(α), cos(β), cos(γ)

Dot Product:

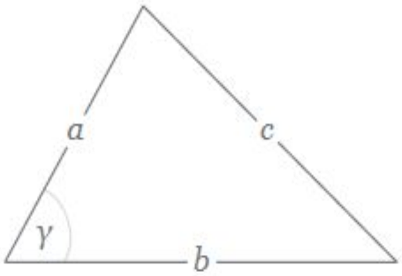
Geometric: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$

Algebraic: $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$

These are equivalent definitions in Cartesian Coordinates. In a different basis, the Algebraic form may be something else.

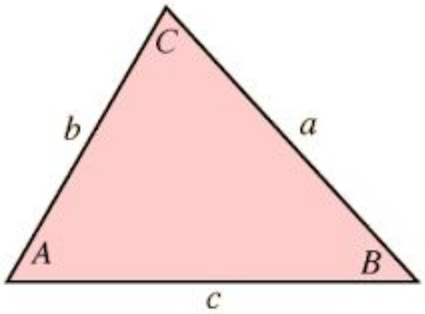
Law of Cosines

$$c = \sqrt{a^2 + b^2 - 2ab \cdot \cos\gamma}$$



Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Three Stars - Vectors

Alpha Centauri 4.7 Light Years: **V** = (3.62, 3, 0)

Polaris 433 Light Years: **V** = (155, 58, 400)

What is the angle between the two stars from the earth's point of view?

What is the vector from Polaris to Alpha Centauri?

What is the distance between Polaris and Alpha Centauri?

What is the angle between earth and Alpha Centauri from Polaris?

Dot Product Examples

Dot Product with Unit Basis Vector

$$\vec{A}\cdot\hat{e}_x = |\vec{A}||\hat{e}_x|\cos\alpha = A\cos\alpha = A_x$$

Algebraic vs Geometric: Cylindrical Coordinates

$$\vec{A}\cdot\vec{A} = |A||A|\cos(0) = |A|^2 = r^2 + z^2$$

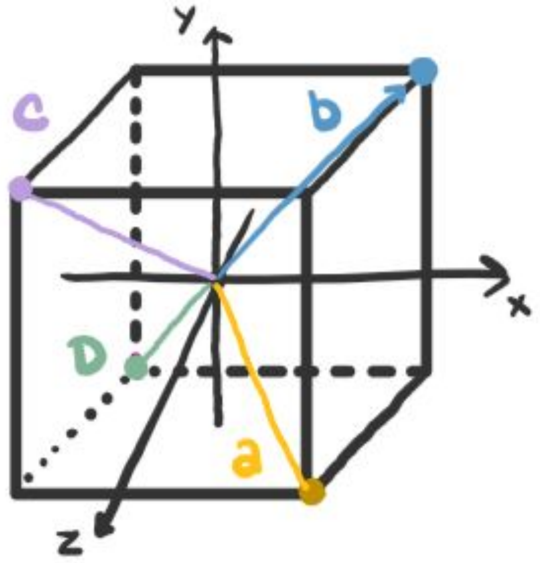
$$\vec{A}\cdot\vec{A} \neq \rho^2 + \theta^2 + z^2$$

Prove the Schwarz Inequality

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Methane Bond Angle

- Methane can be inscribed in a cube with the carbon at the center and the four hydrogens in at four corners of the cube.
- What are the vectors for the positions of the hydrogens relative to the carbon at the origin?
- Use two of these vectors to determine the bond angle between two hydrogen atoms.



The Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

Compute determinants:

$$\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix} \qquad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$$

The Determinant and The Vector Product

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

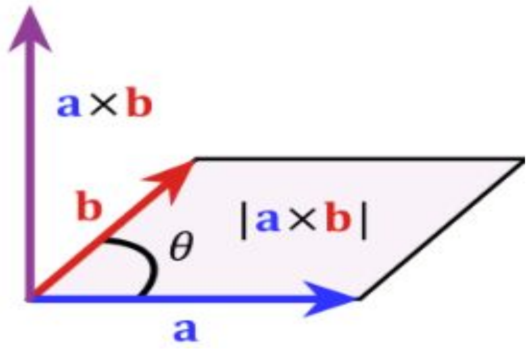
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

Vector or Cross Product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Area of parallelogram base:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, |\sin\theta|$$



Vector Product Exercises (1 of 4)

Exercise 1:

Given three vectors**:** **P = (3, 2, -1), Q = (-6, -4, 2),** and **R = (1, -2, -1)**
Using the cross product. Find the two that are perpendicular and two that are parallel/antiparallel.

Exercise 2:

Starting with **C = A + B,** show that **C x C = 0** leads to **A x B = - B x A**

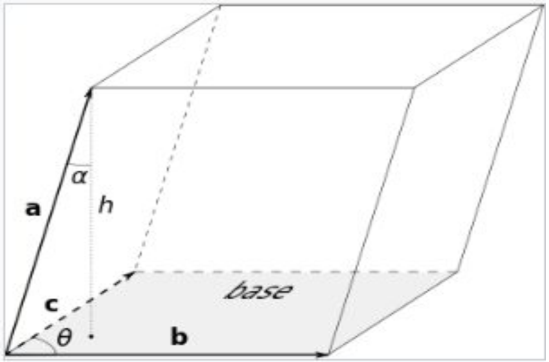
Triple Scalar Product; Triple Vector Product

Triple Scalar Product:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

Volume of parallelepiped

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$



More Vector Products → **Vector Triple Product:**

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}(\mathbf{c}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{c}\cdot\mathbf{a}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

Vector Product Exercises (2 of 4)

Exercise 3:

Given three vectors**:** **P = (3, 2, -1), Q = (-6, -4, 2),** and **R = (1, -2, -1)**, what is the triple scalar product? What is the geometric interpretation?

Exercise 4:

Derive the triple vector product.

Step 1) in the plane of which two vectors will **A x (B x C)** lie?

Step 2) write out the generic form of a vector in this plane.

Step 3) use a triple scalar product to simplify the general form of the vector in step 2.

Step 4) There should be only 1 free parameter left in the derivation of the formula. Assume **A = (1, 0, 0), B = (0, 1, 0),** and **C = (1, 0, 0)**. Determine the remaining constant by inputting these vectors into the expression from step 3.

Vector Product Exercises (3 of 4)

Exercise 5:

The orbital angular momentum $\mathbf{L}$ of a particle is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$, where $\mathbf{p}$ is the linear momentum. With linear and angular velocity related by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, show that

$$\mathbf{L} = mr^2\left[\boldsymbol{\omega} - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \boldsymbol{\omega})\right].$$

Here $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{r}$-direction. For $\mathbf{r} \cdot \boldsymbol{\omega} = 0$ this reduces to $\mathbf{L} = I\boldsymbol{\omega}$, with the moment of inertia I given by mr^2. In Section 3.5 this result is generalized to form an inertia tensor.

Vector Product Exercises (4 of 4)

Exercise 6:

Given

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}},$$

and $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} \neq 0$, show that →

(a) $\mathbf{x}\cdot\mathbf{y}' = \delta_{xy}, (\mathbf{x}, \mathbf{y} = \mathbf{a}, \mathbf{b}, \mathbf{c})$,

(b) $\mathbf{a}'\cdot\mathbf{b}'\times\mathbf{c}' = (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})^{-1}$,

(c) $\mathbf{a} = \dfrac{\mathbf{b}'\times\mathbf{c}'}{\mathbf{a}'\cdot\mathbf{b}'\times\mathbf{c}'}$.

Calculus Fundamentals

Derivative

$$L = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Total Differential

$$\begin{aligned} dF(x,y) &\equiv F(x+dx, y+dy) - F(x,y) \\ &= [F(x+dx, y+dy) - F(x, y+dy)] + [F(x, y+dy) - F(x,y)] \\ &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy, \end{aligned}$$

Note: x derivative taken at **y + dy**, not **y**. The derivative amount is altered by on order dy. Negligible in the limit **dy → 0**

Chain Rule

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx},$$

Fundamental Theorem of Calculus

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Integration by Parts

$$\int u dv = uv - \int v du$$