New τ-based evaluation of the hadronic contribution to the vacuum polarization piece of the muon anomalous magnetic moment

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Outline

Motivation

- The fermionic magnetic moment is given by

$$\vec{\mu} = g\frac{Qe}{2m}\vec{S}, \text{ where } Q = \pm 1 \text{ and } e > 0. \quad (1)$$

One of the biggest successes of the Dirac theory was the prediction of $g \equiv 2$.

Motivation

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One of the biggest successes of the Dirac theory was the prediction of $g \equiv 2$.

- For a few years, this situation was kept. The electron had g=2 and the Dirac eq. seems to describe nature.
- Motivated by an excess in the measurements of the hyperfine structure of hydrogen atom, in 1937 J. Schwinger showed that these discrepancies can be explained by an additional contribution from QED,

$$\frac{\delta\mu}{\mu} = \frac{\alpha}{2\pi} \simeq 0.001162 \quad (2)$$

- This prediction was confirmed by the Kush and Foley experiments. Since the electron anomaly has been measured up to a few ppb (10^{-9}), the QED calculation has to be extended to tenth-order (5 loops).

- It is useful to split the magnetic moment into two terms:

$$\mu_\ell = (1 + a_\ell)\frac{e\hbar}{2m}, \text{ where } a_\ell \equiv \frac{g_\ell - 2}{2}. \tag{3}$$

- The anomalous magnetic moment of electron (a_e) and muon (a_μ) have been measured with a precision of a few ppb (10^{-9}) and ppm (10^{-6}), respectively.
- A deviation of a_ℓ^{exp} concerning the SM theoretical value would be a signal of NP.
- Since the contributions from heavier NP to a_ℓ are proportional to $\Delta a_\ell \sim \frac{m_\ell^2}{M_{NP}^2}$, the NP effects in a_μ are magnified by a factor $(m_\mu/m_e)^2 \sim 4 \times 10^4$. For the a_τ these effects would be better, however the short lifetime of the τ makes it harder to measure.

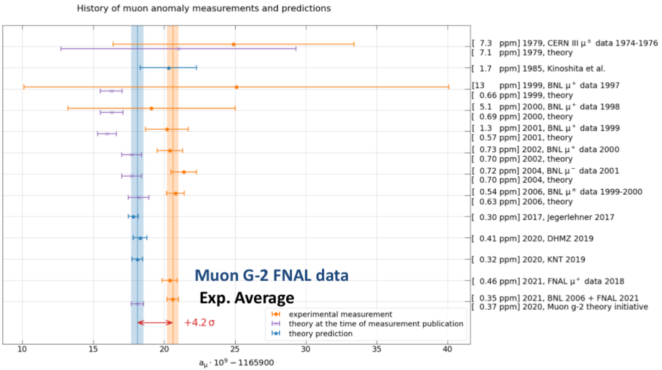
$$\tau_\mu = (2.1969811 \pm 0.0000022) \times 10^{-6} s, \qquad \tau_\tau = (290.3 \pm 0.5) \times 10^{-15} s. \tag{4}$$

- Nowadays, there is a discrepancy between the theoretical prediction and the experimental value for both, $\ell = e, \mu$,

$$\Delta a_\mu \equiv a_\mu^{exp} - a_\mu^{SM} = (2.51 \pm 0.59) \times 10^{-9}, \quad \Rightarrow +4.2\,\sigma$$
$$\Delta a_e \equiv a_e^{exp} - a_e^{SM} = (4.8 \pm 3.0) \times 10^{-13}, \quad \Rightarrow +1.6\,\sigma$$

The Anomalous Magnetic Moment of the Muon

Updated g-2 history (April 8 2021)



$$a_\mu(\text{AVG}) = 116\,592\,061(41) \times 10^{-11} \quad (0.35\,\text{ppm}).$$

G. Venanzoni, CERN Seminar, 8 April 2021

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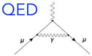
The Anomalous Magnetic Moment of the Muon

Contributions from known particles: The Standard Model

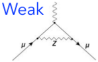
$$a_\mu(\text{SM}) = a_\mu(\text{QED}) + a_\mu(\text{Weak}) + a_\mu(\text{Hadronic})$$

QED



$+\ldots$ $116\,584\,718.9\,(1) \times 10^{-11}$ 0.001 ppm

Weak

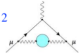


$+\ldots$ $153.6\,(1.0) \times 10^{-11}$ 0.01 ppm

Hadronic…

…Vacuum Polarization (HVP)

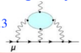
α^2



$+\ldots$ $6845\,(40) \times 10^{-11}$ [0.6%] 0.37 ppm

…Light-by-Light (HLbL)

α^3



$+\ldots$ $92\,(18) \times 10^{-11}$ [20%] 0.15 ppm

Numbers from Theory Initiative Whitepaper

Uncertainty dominated by hadronic contributions

C. Lehner. CERN EP Seminar, 8 April 2021

HVP

- Based on analyticity and unitarity, loop integrals containing HVP insertions in photon propagators can be expressed as dispersive integrals over the cross-section of a virtual photon decaying into hadrons:

$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{3\pi^2}\int_{m_\pi^2}^{\infty} ds\,\frac{K(s)}{s}R(s), \tag{5}$$

where $K(s)$ is a Kernel function $\Rightarrow\ K(s) \sim 1/s$,

$$R(s) = \frac{\sigma^0(e^+e^- \to \text{hadrons}(+\gamma))}{\sigma_{pt}}, \quad \sigma_{pt} = \frac{4\pi\alpha^2}{3s} \tag{6}$$

- An evaluation of the $a_\mu^{\text{HVP, LO}}$ can be obtained from the measurements of $\sigma(e^+e^- \to \text{hadrons})$ or the $\tau \to \nu_\tau + \text{hadrons}$ decays which can be related to the isovector component of the $e^+e^- \to$ hadrons cross-section through isospin-symmetry.

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- Since both are subject to theoretical uncertainties, it is a good strategy to keep using e^+e^- and τ data.

- About 73% of the contributions to the a_μ^{HVP} and 58% of the total uncertainty correspond to the $\pi^+\pi^-(\gamma)$ final state at low-energies ($4m_\pi^2 \leq s \leq 0.8\,\mathrm{GeV}^2$).
- For the two-pion final state,

$$\sigma_{\pi^+\pi^-}(s) = \frac{\pi\alpha^2\beta^3_{\pi^-\pi^+}(s)}{3s}\,|F_V(s)|^2\,, \tag{7}$$

[1] Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)

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- Including isospin-breaking corrections at LO, we have

$$\sigma_{\pi^+\pi^-}(s) = \left[\frac{K_\sigma(s)}{K_\Gamma(s)}\frac{d\Gamma_{\pi\pi[\gamma]}}{ds}\right]\frac{R_{IB}(s)}{S_{EW}}, \quad s \equiv (p_{\pi^-}+p_{\pi^0})^2 \tag{8}$$

where $R_{IB}(s) = \frac{FSR(s)}{G_{EM}(s)}\frac{\beta^3_{\pi^+\pi^-}}{\beta^3_{\pi^0\pi^-}}\left|\frac{F_V(s)}{f_+(s)}\right|^2$,

$$\begin{aligned} K_\Gamma(s) &= \frac{G_F^2|V_{ud}|^2 m_\tau^3}{384\pi^3}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right), \\ K_\sigma(s) &= \frac{\pi\alpha^2}{3s}, \end{aligned} \tag{9}$$

- $G_{EM}(s)$ receives contributions from real and virtual photons [1].

[1] Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)

- There is a discrepancy between the values of $a_\mu^{HVP,LO}[\pi\pi]$ obtained through e^+e^- and τ decays. According to Cirigliano et al. [Phys.Rev.Lett. 122 (2019)] this could be a NP effect,

$$\frac{a_\mu^\tau - a_\mu^{ee}}{2a_\mu^{ee}} = \epsilon_L^\tau - \epsilon_L^e + \epsilon_R^\tau - \epsilon_R^e + 1.7\epsilon_T^\tau. \quad (10)$$

- A global fit using hadronic tau decays to set bounds on NP effective couplings at the low-energy limit of SMEFT was studied in Gonzàlez-Solís et al. [Phys.Lett.B 804 (2020)]. **See P. Roig talk**
- NP effects in $\tau^- \to \pi^-\pi^0\nu_\tau$ decays have been studied using an effective field theory setup for some observables in RM, JHEP 1811 (2018).
- Although the determinations of $a_\mu^{\text{HVP, LO}}$ using lattice QCD are still not competitive with respect to the e^+e^- based evaluation, there is one lattice calculation by the BMW collaboration [Nature **593** (2021) 51] with an error of $\pm 55 \cdot 10^{-11}$ in which the difference concerning the experimental value is reduced to $\sim 1\sigma$. **See A. El-Khadra talk**
- There is a solution given by Jegerlehner & Szafron that induces an additional correction due to the $\rho - \gamma$ mixing in which ρ^0 is regarded as a gauge boson.

$\tau^-(P) \to \pi^-(p_-)\pi^0(p_0)\gamma(k)\nu_\tau(q)$

- The matrix element for these decays has the following structure: **See Z.H. Guo talk**

$$T = e\, G_F V_{ud}^* \epsilon^\mu(k)^* \left\{ F_\nu \bar{u}(q)\gamma^\nu(1-\gamma_5)(m_\tau + \not{P} - \not{k})\gamma_\mu u(P) + (V_{\mu\nu} - A_{\mu\nu})\bar{u}(q)\gamma^\nu(1-\gamma_5)u(P) \right\} \quad (11)$$

  where $F_\nu = (p_0 - p_-)_\nu f_+(s)/2p \cdot k$ with $s \equiv (p_- + p_0)^2$.

- The $V_{\mu\nu} - A_{\mu\nu}$ term can be split into two parts, structure-independent (SI) and structure-dependent (SD), according to the Low and Burnett-Kroll theorems.

$$V^{\mu\nu} = V_{SI}^{\mu\nu} + V_{SD}^{\mu\nu}$$
$$A^{\mu\nu} = A_{SD}^{\mu\nu}$$

[2] Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)

[3] Flores-Tlalpa et al. '06, '07

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- At low-energies, the SM of electroweak and strong interactions is described by an effective field theory known as Chiral Perturbation Theory (χPT).

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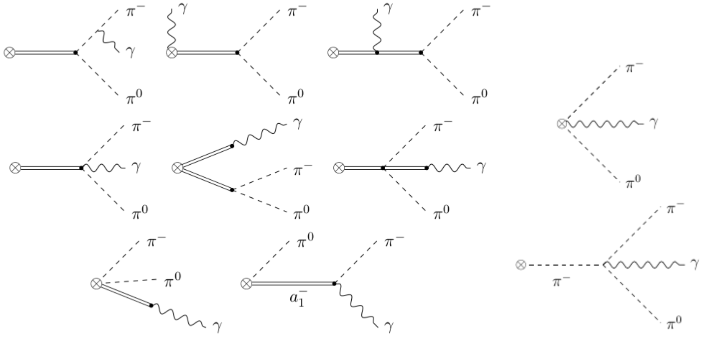
- At low-energies, the SM of electroweak and strong interactions is described by an effective field theory known as Chiral Perturbation Theory (χPT).
- There is a difference for the effects of the SD part when they are evaluated using χPT with resonances [2] and VMD [3].

[2] Cirigliano et al. Phys.Lett. B513 (2001). JHEP 0208 (2002)

[3] Flores-Tlalpa et al. '06, '07

Contributions at $\mathcal{O}\left(p^4\right)$

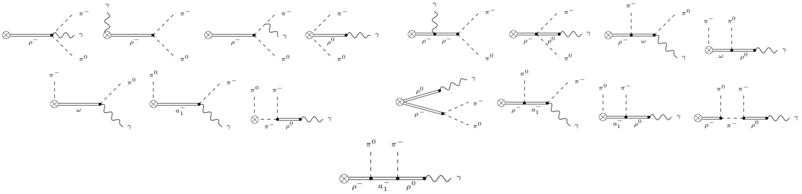
- At $\mathcal{O}\left(p^4\right)$ in χPT with resonances ($R\chi T$), the diagrams that contribute to these decays are [4]:



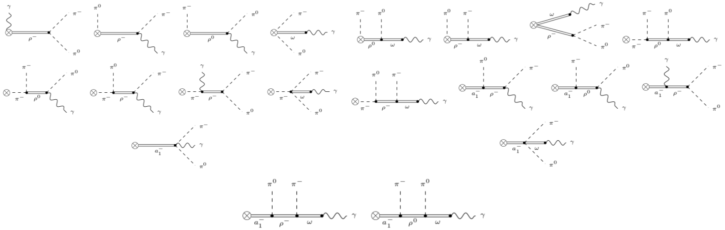
[4] Cirigliano, Ecker & Neufeld. JHEP 0208 (2002)

Contributions at $\mathcal{O}\left(p^6\right)$

- Using the basis given by Cirigliano et al. Nucl. Phys. B753 (2006) and Kampf & Novotný, Phys. Rev. D84 (2011), we get the following contributions at $\mathcal{O}\left(p^6\right)$:



for $V_{\mu\nu}$, and



for $A_{\mu\nu}$.

Short-distance constraints

- Including the contributions up to $\mathcal{O}\left(p^6\right)$, we have now too many parameters allowed by the discrete symmetries of QCD and chiral symmetry that prevent making phenomenological predictions.
- It is possible to reduce the number of couplings using the SD properties of QCD and its OPE.
- To estimate the remaining parameters, we rely on chiral counting and estimation of the LECs C_i^R of the $\mathcal{O}(p^6)$ χPT Lagrangian [5].
- Since the κ_i^V couplings, which are related to the ω-exchange, contribute significantly to the radiative decays, we perform a global fit using the relations for the resonance saturation of the anomalous sector LECs at $\mathcal{O}(p^6)$ [6] [7].

[5] Cirigliano et al., Nucl.Phys.B 753 (2006) 139-177

[6] Kampf & Novotný, Phys.Rev.D 84 (2011) 014036

[7] Shao-Zhou Jiang et al., Phys.Rev.D 92 (2015) 025014

Decay spectrum

- In this work, we use the dispersive representation of the vector form factor [8] instead of the exponential parameterization [9] used by Cirigliano, Ecker & Neufeld, JHEP 0208 (2002).

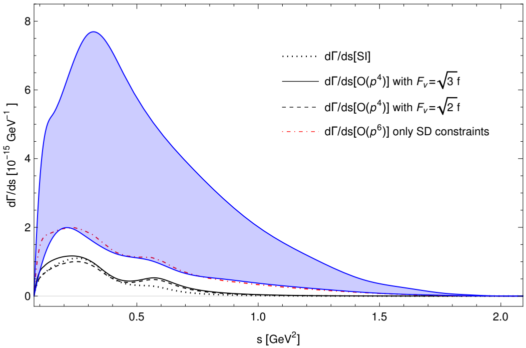
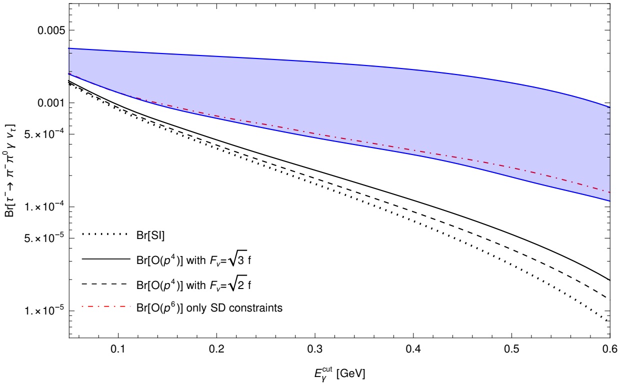


Figure: The $\pi^-\pi^0$ hadronic invariant mass distributions for $E_\gamma^{cut} = 300\,\mathrm{MeV}$.

[8] Gómez Dumm & Roig, Eur.Phys.J. C73 (2013)

[9] Guerrero & Pich, Phys. Lett. B412 (1997)

Branching ratio

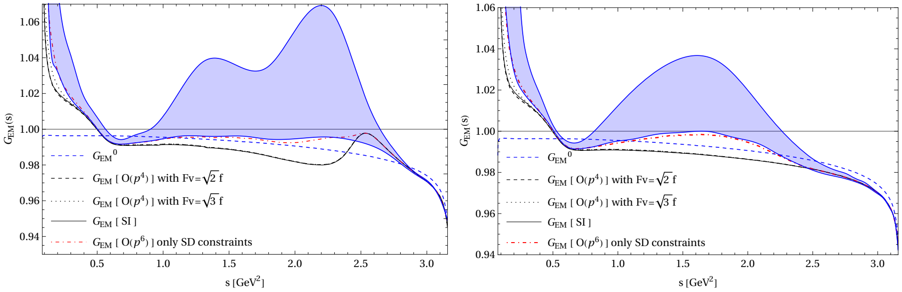


E_γ^{cut}	BR[$\mathcal{O}(p^4)$]	BR [$\mathcal{O}(p^6)$]
100 MeV	$(9.5^{+3.5}_{-0.5}) \cdot 10^{-4}$	$(1.9 \pm 0.3) \cdot 10^{-3}$
300 MeV	$(2.3^{+2.8}_{-0.4}) \cdot 10^{-4}$	$(1.1 \pm 0.3) \cdot 10^{-3}$
500 MeV	$(0.5^{+1.9}_{-0.2}) \cdot 10^{-4}$	$(0.6 \pm 0.2) \cdot 10^{-3}$

$G_{EM}(s)$

- Adding the contributions due to virtual and real photons and integrating over $u \equiv (P - p_1)^2$, we get the $G_{EM}(s)$ function,

$$\left.\frac{d\Gamma}{ds}\right|_{\pi\pi[\gamma]} = \frac{G_F^2 |V_{ud}|^2 m_\tau^3 S_{EW}}{384\pi^3} |f_+(s)|^2 \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \left(1 + \frac{2s}{m_\tau^2}\right) G_{EM}(s). \quad (12)$$



$\Delta a_\mu^{\text{HVP,LO}}$

We can estimate the effect of each IB correction through $\Delta a_\mu^{\text{HVP, LO}}$

$$\Delta a_\mu^{\text{HVP,LO}} = \frac{1}{4\pi^3}\int_{s_1}^{s_2} ds\, K(s)\left[\frac{K_\sigma(s)}{K_\Gamma(s)}\frac{d\Gamma_{\pi\pi[\gamma]}}{ds}\right]\left(\frac{R_{IB}(s)}{S_{EW}} - 1\right), \qquad (13)$$

Source	$\Delta a_\mu^{\text{HVP,LO}}[\pi\pi,\tau](\times 10^{11})$ FF1	FF2
S_{EW}	-103.1	
PS	-74.5	
FSR	$+45.5 \pm 4.6$	
FF	$+40.9 \pm 48.9$	$+77.6 \pm 24.0$
EM	$-15.9^{+5.7}_{-16.0}$	
Total	$-107.1^{+49.4}_{-51.7}$	$-70.4^{+25.1}_{-29.2}$

Source	$\Delta a_\mu^{\text{HVP,LO}}[\pi\pi,\tau](\times 10^{11})$ FF1	FF2
S_{EW}	-103.1	
PS	-74.5	
FSR	$+45.5 \pm 4.6$	
FF	$+40.9 \pm 48.9$	$+77.6 \pm 24.0$
EM	$-75.9^{+65.7}_{-45.6}$	
Total	$-167.1^{+82.0}_{-67.0}$	$-130.4^{+70.1}_{-51.7}$

- It is possible to estimate the branching ratio $B_{\pi\pi^0} = \Gamma(\tau \to \pi\pi^0\nu_\tau)/\Gamma_\tau$ using e^+e^- data

$$B_{\pi\pi^0}^{CVC} = B_e \int_{4m_\pi^2}^{m_\tau^2} ds\, \sigma_{\pi^+\pi^-(\gamma)}(s)\mathcal{N}(s)\frac{S_{EW}}{R_{IB}(s)}, \tag{14}$$

where $\mathcal{N}(s) = \frac{3|V_{ud}|^2}{2\pi\alpha_0^2 m_\tau^2} s\left(1 - \frac{s}{m_\tau^2}\right)^2\left(1 + \frac{2s}{m_\tau^2}\right)$.
Using the most recent data obtained from BaBar for the $e^+e^- \to \pi^+\pi^-(\gamma)$ cross section, we get

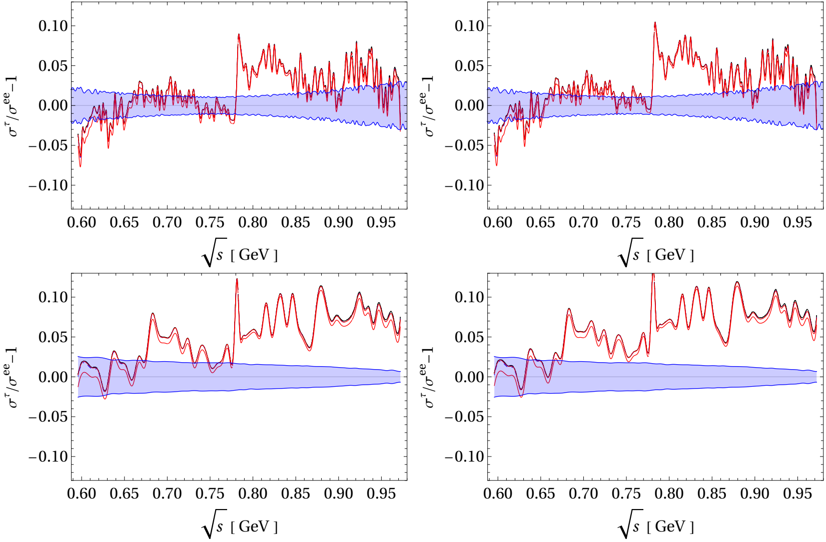
$$B_{\pi\pi^0}^{CVC} = (24.68 \pm 0.11 \pm 0.10 \pm 0.01 \pm 0.01 \pm 0.02^{+0.03}_{-0.00})\%, \text{ at } \mathcal{O}(p^4), \tag{15}$$

and

$$B_{\pi\pi^0}^{CVC} = (24.70 \pm 0.11 \pm 0.10 \pm 0.01 \pm 0.01 \pm 0.02^{+0.21}_{-0.01})\%, \text{ at } \mathcal{O}(p^6). \tag{16}$$

- These results are in good agreement with the value reported by the Belle collaboration, $B^\tau_{\pi\pi^0} = (25.24 \pm 0.01 \pm 0.39)\%$ at $1.3\,\sigma$ $(1.2\,\sigma)$,
and ALEPH, $\mathrm{B}^\tau_{\pi\pi^0} = (25.471 \pm 0.097 \pm 0.085)\%$ at $4.0\,\sigma$ $(1.2\,\sigma)$.

Comparison between the different data sets from BaBar (above) and KLOE (below) with $\Delta\Gamma_{\pi\pi\gamma} = 1.5$ MeV (left-hand) and $\Delta\Gamma_{\pi\pi\gamma} = 0.45$ MeV (right-hand) using the Belle spectrum.



Taking into account all di-pion tau decay data from ALEPH, Belle, CLEO and OPAL Colls., we get

$$10^{10} \cdot a_\mu^{HVP,LO}|_{\pi\pi,\tau\ \mathrm{data}} = 519.6 \pm 2.8_{spectra+BRs}{}^{+1.9}_{-2.1}{}_{IB}\ ,\ \text{at } \mathcal{O}(p^4), \tag{17}$$

and

$$10^{10} \cdot a_\mu^{HVP,LO}|_{\pi\pi,\tau\ \mathrm{data}} = 514.6 \pm 2.8_{spectra+BRs}{}^{+5.0}_{-3.9}{}_{IB}\ ,\ \text{at } \mathcal{O}(p^6). \tag{18}$$

When these results are supplemented with the four-pion tau decays measurements and with e^+e^- data [10], we find the overall HVP LO contribution

$$10^{10} \cdot a_\mu^{HVP,LO}|_{\tau\ \mathrm{data}} = 705.7^{+4.0}_{-4.1}\ ,\ \text{at } \mathcal{O}(p^4), \tag{19}$$

and

$$10^{10} \cdot a_\mu^{HVP,LO}|_{\tau\ \mathrm{data}} = 700.7^{+6.1}_{-5.2}\ ,\ \text{at } \mathcal{O}(p^6). \tag{20}$$

[10] Eur. Phys. J., C80(3):241, 2020, Eur. Phys. J. C, 74(3):2803, 2014.

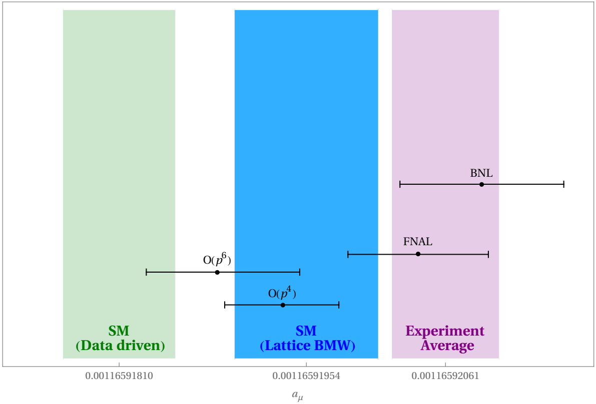
When all other (QED, EW and subleading hadronic) contributions are added, the 4.2σ deficit of the SM prediction with respect to the FNAL+BNL average is reduced to

$$\Delta a_\mu \equiv a_\mu^{exp} - a_\mu^{SM} = (12.5 \pm 6.0) \cdot 10^{-10}\,, \tag{21}$$

at $\mathcal{O}(p^4)$, and

$$\Delta a_\mu \equiv a_\mu^{exp} - a_\mu^{SM} = (17.5^{+6.8}_{-7.5}) \cdot 10^{-10}\,, \tag{22}$$

at $\mathcal{O}(p^6)$, which are 2.1 and 2.3 $\boldsymbol{\sigma}$, respectively.



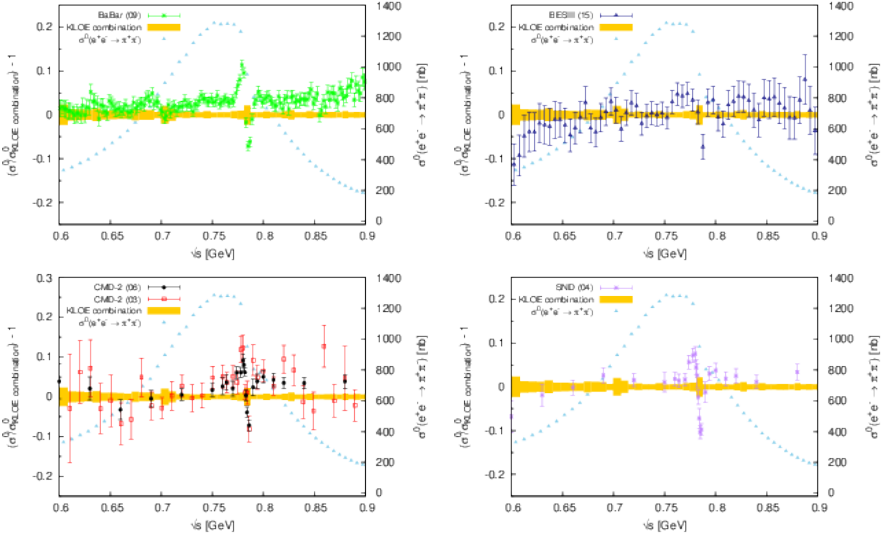
Conclusions

- There is a global effort in improving the hadronic contributions to a_μ. Specifically, dedicated studies to improve the HVP part from lattice, dispersion relations and improved e^+e^- data and Monte Carlos are being undertaken.
- The observables for the $\tau \to \pi\pi\gamma\nu_\tau$ decays have the potential to reduce drastically the errors in our estimation.
- Our IB corrections improve the agreement between e^+e^- and tau data, on the spectrum and the branching ratio.
- Evaluating the HVP, LO contributions from tau data, we get $a_\mu^{HVP,LO|_{\tau\ \mathrm{data}}} = (705.7^{+4.0}_{-4.1}) \cdot 10^{-10}$ at $\mathcal{O}(p^4)$, and $a_\mu^{HVP,LO|_{\tau\ \mathrm{data}}} = (700.7^{+6.1}_{-5.2}) \cdot 10^{-10}$ at $\mathcal{O}(p^6)$. This reduces the anomaly $\Delta a_\mu \equiv a_\mu^{exp} - a_\mu^{SM}$ to 2.1 and 2.3 σ, respectively.

References

- V. Cirigliano, G. Ecker and H. Neufeld, "Radiative tau decay and the magnetic moment of the muon," JHEP **0208**, 002 (2002)
- V. Cirigliano, G. Ecker and H. Neufeld, "Isospin violation and the magnetic moment of the muon," Phys. Lett. B **513**, 361 (2001)
- J. Miranda and P. Roig. "New τ-based evaluation of the hadronic contribution to the vacuum polarization piece of the muon anomalous magnetic moment", Phys.Rev.D 102 (2020) 114017

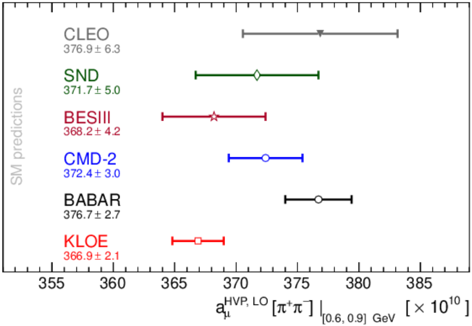
The $\pi^+\pi^-$ cross section from the KLOE combination compared to the BABAR, CMD-2, SND, and BESIII data points in the 0.6–0.9 GeV range [11].



[11] JHEP 03 (2018) 173

$a_\mu^{\mathrm{HVP,LO}}[\pi\pi]$ from e^+e^- data

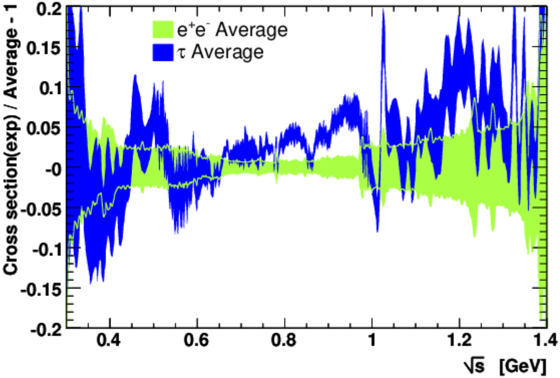
Comparison of results for $a_\mu^{\mathrm{HVP,LO}}[\pi\pi]$, evaluated between 0.6 GeV and 0.9 GeV for the various experiments [12].



[12] [illegible]

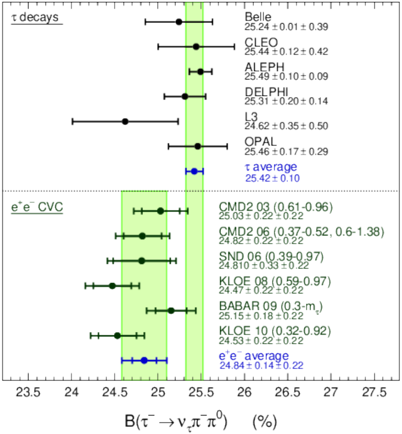
e^+e^- vs τ data

Relative comparison between the combined τ (after the IB corrections) and $e^+e^- \to \pi^+\pi^-$ spectral function contributions [13].



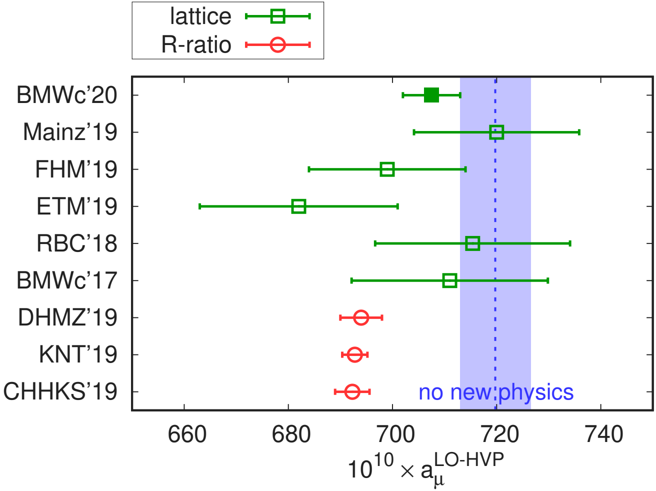
[13] Eur.Phys.J.C66:1-9,2010

The measured branching fractions for $\tau^- \to \pi^-\pi^0\nu_\tau$ compared to the predictions from the $e^+e^- \to \pi^+\pi^-$ spectral functions, applying the IB corrections [14].



[14] Eur.Phys.J.C66:127-136,2010

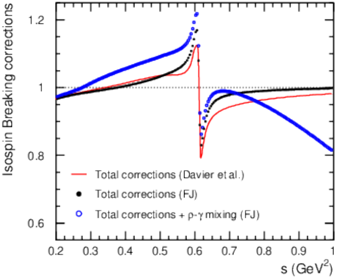
Comparison of recent results for the leading-order, hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon [15].

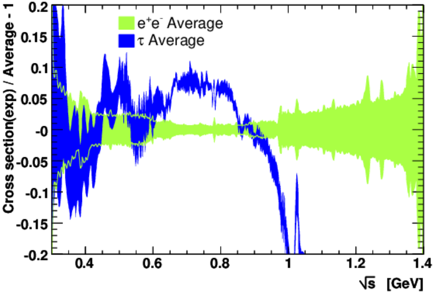


[15] Nature (2021)

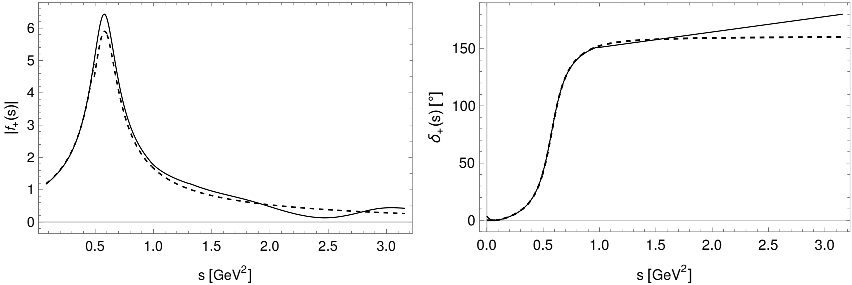
$\rho - \omega$ mixing

$\rho - \gamma$ mixing corrections proposed in Eur.Phys.J.C71:1632,2011.





Form Factor



Short-distance constraints

Using the relations for 2-point Green functions at $\mathcal{O}(p^4)$, we have:

$$F_V = \sqrt{2}F \quad G_V = \frac{F}{\sqrt{2}} \quad F_A = F. \tag{23}$$

Using the relations for 2 and 3-point Green functions at $\mathcal{O}(p^6)$, we have

$$F_V = \sqrt{3}F \quad G_V = \frac{F}{\sqrt{3}} \quad F_A = \sqrt{2}F. \tag{24}$$

Short-distance constraints

For the parameters contributing to the leading-order chiral LECs:

$$\begin{aligned} F_V G_V &= F^2, \qquad F_V^2 - F_A^2 = F^2, \\ F_V^2 M_V^2 &= F_A^2 M_A^2, \qquad 4 c_d c_m = F^2, \\ 8\left(c_m^2 - d_m^2\right) &= F^2, \qquad c_m = c_d = \sqrt{2} d_m = F/2. \end{aligned} \tag{25}$$

For the even-intrinsic parity sector:

$$\begin{aligned} \lambda_{13}^P &= 0, \quad \lambda_{17}^S = \lambda_{18}^S = 0, \\ \lambda_{17}^A &= 0, \quad \lambda_{21}^V = \lambda_{22}^V = 0. \end{aligned} \tag{26}$$

The analysis of the $\langle VAS \rangle$ Green function yields:

$$\begin{aligned} \kappa_2^S = \kappa_{14}^A = 0, \quad \kappa_4^V = 2\kappa_{15}^V, \quad \kappa_6^{VA} = \frac{F^2}{32 F_A F_V}, \\ F_V\left(2\kappa_1^{SV} + \kappa_2^{SV}\right) = 2 F_A \kappa_1^{SA} = \frac{F^2}{16\sqrt{2} c_m}. \end{aligned} \tag{27}$$

Short-distance constraints

The study of the $\langle VAP \rangle$ and $\langle SPP \rangle$ Green functions yield the following restrictions on the resonance couplings:

$$\begin{aligned}
\sqrt{2}\lambda_0 &= -4\lambda_1^{VA} - \lambda_2^{VA} - \frac{\lambda_4^{VA}}{2} - \lambda_5^{VA} = \frac{1}{2\sqrt{2}}\left(\lambda' + \lambda''\right), \\
\sqrt{2}\lambda' &= \lambda_2^{VA} - \lambda_3^{VA} + \frac{\lambda_4^{VA}}{2} + \lambda_5^{VA} = \frac{M_A}{2M_V}, \\
\sqrt{2}\lambda'' &= \lambda_2^{VA} - \frac{\lambda_4^{VA}}{2} - \lambda_5^{VA} = \frac{M_A^2 - 2M_V^2}{2M_V M_A}, \\
\lambda_1^{PV} &= -4\lambda_2^{PV} = -\frac{F\sqrt{M_A^2 - M_V^2}}{4\sqrt{2}d_m M_A}, \quad \lambda_1^{PA} = \frac{F\sqrt{M_A^2 - M_V^2}}{16\sqrt{2}d_m M_V}.
\end{aligned} \tag{28}$$

For the odd-intrinsic parity sector:

$$\begin{aligned}
&\kappa_{14}^V = \frac{N_C}{256\sqrt{2}\pi^2 F_V}, \quad 2\kappa_{12}^V + \kappa_{16}^V = -\frac{N_C}{32\sqrt{2}\pi^2 F_V}, \quad \kappa_{17}^V = -\frac{N_C}{64\sqrt{2}\pi^2 F_V}, \quad \kappa_5^P = 0, \\
&\kappa_2^{VV} = \frac{F^2 + 16\sqrt{2}d_m F_V \kappa_3^{PV}}{32F_V^2} - \frac{N_C M_V^2}{512\pi^2 F_V^2}, \quad 8\kappa_2^{VV} - \kappa_3^{VV} = \frac{F^2}{8F_V^2}.
\end{aligned} \tag{29}$$

Fit results

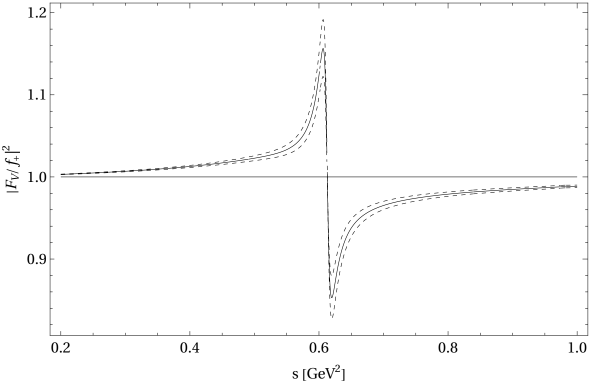
Since the κ_i^V couplings are related with the ω exchange which is known to give an important contribution to the $\tau \to \pi\pi\gamma\nu_\tau$ decays, we perform a global fit using the relations for the resonance saturation of the anomalous sector LECs [16]

$$\kappa_1^V = (-2.1 \pm 0.7) \cdot 10^{-2}\ \text{GeV}^{-1}, \tag{30a}$$
$$\kappa_2^V = (-8.8 \pm 9.1) \cdot 10^{-3}\ \text{GeV}^{-1}, \tag{30b}$$
$$\kappa_3^V = (2.2 \pm 5.8) \cdot 10^{-3}\ \text{GeV}^{-1}, \tag{30c}$$
$$\kappa_6^V = (-2.1 \pm 0.3) \cdot 10^{-2}\ \text{GeV}^{-1}, \tag{30d}$$
$$\kappa_7^V = (1.2 \pm 0.5) \cdot 10^{-2}\ \text{GeV}^{-1}, \tag{30e}$$
$$\kappa_8^V = (3.1 \pm 0.9) \cdot 10^{-2}\ \text{GeV}^{-1}, \tag{30f}$$
$$\kappa_9^V = (-0.1 \pm 5.9) \cdot 10^{-3}\ \text{GeV}^{-1}, \tag{30g}$$
$$\kappa_{10}^V = (-5.9 \pm 9.6) \cdot 10^{-3}\ \text{GeV}^{-1}, \tag{30h}$$
$$\kappa_{11}^V = (-3.0 \pm 0.6) \cdot 10^{-2}\ \text{GeV}^{-1}, \tag{30i}$$
$$\kappa_{12}^V = (1.0 \pm 0.8) \cdot 10^{-2}\ \text{GeV}^{-1}, \tag{30j}$$
$$\kappa_{13}^V = (-5.3 \pm 1.1) \cdot 10^{-3}\ \text{GeV}^{-1}, \tag{30k}$$
$$\kappa_{18}^V = (4.7 \pm 0.8) \cdot 10^{-3}\ \text{GeV}^{-1}. \tag{30l}$$

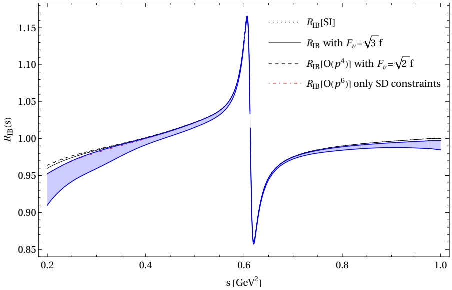
These values are in good agreement with our earlier estimation $|\kappa_i^V| \lesssim 0.025\,\text{GeV}^{-1}$.

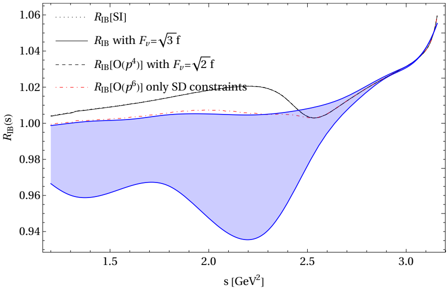
[16] Phys.Rev.D 92 (2015) 025014

$|F_0/F_-|$



Total corrections





$a_\mu^{HVP,LO}[\pi\pi,\tau]$ at $\mathcal{O}(p^4)$			
Experiment	$2m_{\pi^\pm} - 0.36\,\mathrm{GeV}$	$0.36 - 1.8\,\mathrm{GeV}$	TOTAL
Belle	$8.81 \pm 0.00 \pm 0.14^{+0.16}_{-0.34}$	$511.14 \pm 1.94 \pm 7.99^{+1.91}_{-2.09}$	$519.95 \pm 1.94 \pm 7.99^{+1.91}_{-2.12}$
ALEPH	$8.89 \pm 0.00 \pm 0.05^{+0.16}_{-0.34}$	$508.26 \pm 4.48 \pm 2.82^{+1.91}_{-2.09}$	$517.15 \pm 4.48 \pm 2.82^{+1.91}_{-2.12}$
CLEO	$8.85 \pm 0.00 \pm 0.15^{+0.16}_{-0.34}$	$510.63 \pm 3.40 \pm 8.93^{+1.90}_{-2.08}$	$519.48 \pm 3.40 \pm 8.93^{+1.90}_{-2.11}$
OPAL	$8.89 \pm 0.00 \pm 0.12^{+0.15}_{-0.34}$	$522.81 \pm 10.04 \pm 7.00^{+1.87}_{-2.12}$	$531.70 \pm 10.04 \pm 7.00^{+1.87}_{-2.15}$

$a_\mu^{HVP,LO}[\pi\pi,\tau]$ at $\mathcal{O}(p^6)$			
Experiment	$2m_{\pi^\pm} - 0.36\,\mathrm{GeV}$	$0.36 - 1.8\,\mathrm{GeV}$	TOTAL
Belle	$7.77 \pm 0.00 \pm 0.12^{+1.20}_{-0.59}$	$507.18 \pm 1.91 \pm 7.88^{+4.72}_{-3.76}$	$514.95 \pm 1.91 \pm 7.88^{+4.87}_{-3.81}$
ALEPH	$7.84 \pm 0.00 \pm 0.04^{+1.21}_{-0.60}$	$504.37 \pm 4.35 \pm 2.79^{+4.63}_{-3.70}$	$512.21 \pm 4.35 \pm 2.79^{+4.78}_{-3.75}$
CLEO	$7.80 \pm 0.00 \pm 0.14^{+1.21}_{-0.59}$	$506.74 \pm 3.28 \pm 8.84^{+4.63}_{-3.71}$	$514.54 \pm 3.28 \pm 8.84^{+4.78}_{-3.76}$
OPAL	$7.84 \pm 0.00 \pm 0.10^{+1.20}_{-0.60}$	$518.32 \pm 9.69 \pm 6.92^{+5.25}_{-4.12}$	$526.16 \pm 9.69 \pm 6.92^{+5.39}_{-4.16}$

- The IR divergencies that appear at NLO for $\tau^- \to \pi^-\pi^0\nu_\tau$ decays are canceled out by the IR divergencies of the radiative decay.

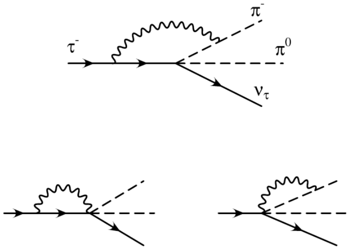
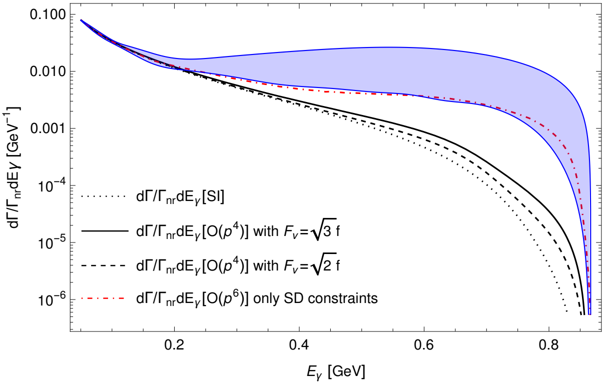


Figure: Contributions at NLO for $\tau^- \to \pi^-\pi^0\nu_\tau$ decays.[17]

[17] Cirigliano, Ecker & Neufeld. Phys.Lett.B 513 (2001)

Photon energy distribution



$\Delta a_{\mu}^{\mathrm{HVP,LO}}$

We can estimate the effect of each IB correction through $\Delta a_{\mu}^{\mathrm{HVP,\ LO}}$

$$\Delta a_{\mu}^{\mathrm{HVP,LO}} = \frac{1}{4\pi^3} \int_{s_1}^{s_2} ds\, K(s) \left[\frac{K_{\sigma}(s)}{K_{\Gamma}(s)} \frac{d\Gamma_{\pi\pi[\gamma]}}{ds} \right] \left(\frac{R_{IB}(s)}{S_{EW}} - 1 \right), \tag{31}$$

Contributions to $\Delta a_{\mu}^{HVP,LO}$ in units of 10^{-11} using the dispersive representation of the form factor.

$[s_1, s_2]$	$\Delta a_{\mu,\,\mathrm{G_{EM}^{(0)}}}^{\mathrm{HVP,LO}}$	$\Delta a_{\mu,\,\mathrm{SI}}^{\mathrm{HVP,LO}}$	$\Delta a_{\mu,\,[\mathcal{O}(p^4)]}^{\mathrm{HVP,LO}}$	$\Delta a_{\mu,\,[\mathcal{O}(p^4)]}^{\mathrm{HVP,LO}}$	$\Delta a_{\mu,\,[SD]}^{\mathrm{HVP,LO}}$	$\Delta a_{\mu,\,[\mathcal{O}(p^6)]}^{\mathrm{HVP,LO}}$
$[4m_\pi^2, 1\,\mathrm{GeV}^2]$	$+17.8$	-11.0	-11.3	-17.0	-32.4	-74.8 ± 44.0
$[4m_\pi^2, 2\,\mathrm{GeV}^2]$	$+18.3$	-10.1	-10.3	-16.0	-31.9	-75.9 ± 45.5
$[4m_\pi^2, 3\,\mathrm{GeV}^2]$	$+18.4$	-10.0	-10.2	-15.9	-31.9	-75.9 ± 45.6
$[4m_\pi^2, m_\tau^2]$	$+18.4$	-10.0	-10.2	-15.9	-31.9	-75.9 ± 45.6

$\Delta a_\mu^{\text{HVP,LO}}$

IB contributions to $a_\mu^{\text{HVP,LO}}[\pi\pi,\tau]$ at $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$.

- For FF1 we use the following numerical inputs: $\theta_{\rho\omega} = (-3.5 \pm 0.7) \times 10^{-3}\,\mathrm{GeV}^2$ [JHEP 08 (2002) 002], $\Gamma_{\rho^0} - \Gamma_{\rho^+} = 0.3 \pm 1.3$ MeV, $m_{\rho^\pm} - m_{\rho^0} = 0.7 \pm 0.8$ MeV and $m_{\rho^0} = 775.26 \pm 0.25$ MeV from PDG.
- For FF2 we use the same inputs as FF1 except by $\Gamma_{\rho^0\to\pi^+\pi^-\gamma} - \Gamma_{\rho^\pm\to\pi^\pm\pi^0\gamma} = 0.45 \pm 0.45$ MeV [JHEP 08 (2002) 002].

Source	$\Delta a_\mu^{\text{HVP,LO}}[\pi\pi,\tau](\times 10^{11})$ FF1	FF2
S_{EW}	-103.1	
PS	-74.5	
FSR	$+45.5 \pm 4.6$	
FF	$+40.9 \pm 48.9$	$+77.6 \pm 24.0$
EM	$-15.9^{+5.7}_{-16.0}$	
Total	$-107.1^{+49.4}_{-51.7}$	$-70.4^{+25.1}_{-29.2}$

Source	$\Delta a_\mu^{\text{HVP,LO}}[\pi\pi,\tau](\times 10^{11})$ FF1	FF2
S_{EW}	-103.1	
PS	-74.5	
FSR	$+45.5 \pm 4.6$	
FF	$+40.9 \pm 48.9$	$+77.6 \pm 24.0$
EM	$-75.9^{+65.7}_{-45.6}$	
Total	$-167.1^{+82.0}_{-67.0}$	$-130.4^{+70.1}_{-51.7}$