Perturbative heavy quark contributions to the anomalous magnetic moment of the muon

J. Erler, P. D. Kennedy*, H. Spiesberger

30.09.21

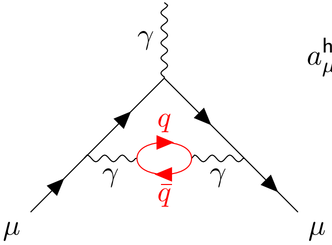
Speaker denoted by *

Overview

- Our method is based on the work of Erler and Luo[1]
- Calculation is expanded to $\mathcal{O}(\alpha_s^3)$
- Updated to the latest theory inputs for $\hat{m}_q$ and α_s
- Results available as both an explicit analytic formula and a numerical result
- Outline of the uncertainties
- Comparison with other results
- Aside on $(g-2)_e$

[1] [Erler and Luo, 2001]

Outline



$$a_\mu^{\text{had}}\Big|_{\text{2-loop}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{th}}^{\infty} \frac{ds}{s^2}\hat{K}(s)\cdot R(s)$$

$$\hat{K}(s) = \int_0^1 dx \frac{3x^2(1-x)}{\frac{m_\mu^2}{s}\cdot x^2 + (1-x)}$$

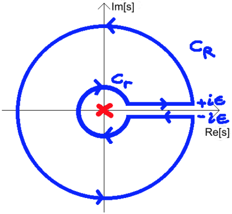
$$R(s) = 12\pi\,\mathrm{Im}\{\Pi(s+i\epsilon)\}$$

This is commonly calculated using e^+e^- cross-section data for $R(s) = \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-}$.[2] We want to make our calculation for heavy quarks independent of this.

[2] [Davier et al., 2020, Keshavarzi et al., 2020]

Application of Cauchy's theorem

$$\oint ds\, f(s) = 0$$



which leads the relation

$$2i \lim_{R\to\infty} \int_{s_0}^{R} \frac{ds}{s^2} \hat{K}(s)\, \mathrm{Im}\{\Pi(s)\} = -\int_{C_r} \frac{ds}{s^2} \hat{K}(s)\Pi(s)$$

$$a_\mu^{\text{had}} = \frac{1}{2i} \left(\frac{\alpha m_\mu}{3\pi}\right)^2 12\pi \int_{\bar{C}_r} \frac{ds}{s^2} \hat{K}(s)\Pi(s)$$

This holds if and only if the functions being integrated are analytically continuous.

Problem

The Kernel

$$\hat{K}(s) = \int_0^1 dx \frac{3x^2(1-x)}{\frac{m_\mu^2}{s}\cdot x^2 + (1-x)}$$

is not in a convenient form for integration. Therefore, it is useful to expand it

$$\hat{K}(s)\Big|_{\text{Exp.}} = 1 + \left(\frac{m_\mu^2}{s}\right)\cdot\left[\frac{25}{4} + 3\ln\left(\frac{m_\mu^2}{s}\right)\right] + \mathcal{O}\left(\frac{m_\mu^4}{s^2}\right)$$

which simplifies the calculation.[3] The logarithmic term here is not analytically continuous and cannot be integrated using Cauchy's theorem.

[3] [Erler and Luo, 2001]

Solution

To solve this, we integrate the terms separately in a form that will allow comparison with the previous result.[4] The separation is

$$\hat{K}(s)\Big|_{\text{Exp.}} = \hat{K}^a(s)\Big|_{\text{Exp.}} + \hat{K}^b(s)\Big|_{\text{Exp.}} + \hat{K}^c(s)\Big|_{\text{Exp.}}$$

where

$$\hat{K}^a(s)\Big|_{\text{Exp.}} = 1;\ \hat{K}^b(s)\Big|_{\text{Exp.}} = \frac{m_\mu^2}{s}\left(\frac{25}{4} + 3\ln\left(\frac{s_0}{s}\right)\right);\ \hat{K}^c(s)\Big|_{\text{Exp.}} = \frac{m_\mu^2}{s}\left(3\ln\left(\frac{m_\mu^2}{s_0}\right)\right)$$

and s_0 is chosen to be $\hat{m}_q$, for convenient comparison.
Only $\hat{K}^b(s)\Big|_{\text{Exp.}}$ has terms including $\ln(s)$ and can be integrated numerically along the real axis separately.

4 [Erler and Luo, 2001]

Explicit result

$$
\begin{aligned}
a_\mu^q \quad = \quad & \frac{\alpha^2 Q_q^2}{4\pi^2}\Bigg[\frac{m_\mu^2}{4\hat{m}_q^2}\Big(\frac{16}{15}+\frac{3104}{1215}a_s \\
& \qquad +\Big(0.50988+\frac{2414}{3645}n_l\Big)a_s^2+\Big(1.87882-2.79492n_l+0.09610n_l^2\Big)a_s^3\Big) \\
& +\frac{m_\mu^4}{16\hat{m}_q^4}\Big(\frac{108}{1225}-0.194294\cdot a_s+(-15\pm 28-(1\mp 1)n_l)\cdot a_s^2\Big) \\
& +3\frac{m_\mu^4}{16\hat{m}_q^4}\ln\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)\Big(\frac{16}{35}+\frac{15728}{14175}a_s \\
& \qquad +\Big(1.41227+\frac{290179}{637875}n_l\Big)a_s^2 \\
& \qquad +\Big(-6.23488+0.96156n_l-0.01594n_l^2\Big)a_s^3\Big)\Bigg]
\end{aligned}
$$

Input values

The quark masses we use are

$$\hat{m}_c(\hat{m}_c) = 1.273 \pm 0.009\text{GeV; for } \alpha_s(M_Z) = 0.1185 \pm 0.0016$$
$$\hat{m}_b(\hat{m}_b) = 4.180 \pm 0.008\text{GeV; for } \alpha_s(M_Z) = 0.1185 \pm 0.0016$$

from[5].

5 [Erler et al., 2017, Erler et al., 202x]

6 [Zyla et al., 2020]

7 [Schmidt and Steinhauser, 2012]

Input values

The quark masses we use are

$$\begin{aligned}\hat{m}_c(\hat{m}_c) &= 1.273 \pm 0.009\text{GeV; for } \alpha_s(M_Z) = 0.1185 \pm 0.0016 \\ \hat{m}_b(\hat{m}_b) &= 4.180 \pm 0.008\text{GeV; for } \alpha_s(M_Z) = 0.1185 \pm 0.0016\end{aligned}$$

from[5]. $\alpha_s(\hat{m}_q)$ values are

$$\begin{aligned}\alpha_s(\hat{m}_c) &= 0.396 \pm 0.020 \\ \alpha_s(\hat{m}_b) &= 0.2267 \pm 0.0061\end{aligned}$$

given $\alpha_s(M_Z) = 0.1185 \pm 0.0016$ in the EW fit, [6] using CRunDec.[7]

[5] [Erler et al., 2017, Erler et al., 202x]

[6] [Zyla et al., 2020]

[7] [Schmidt and Steinhauser, 2012]

Numerical results (in unit of 10^{-10})

Our prediction:

$$a_\mu^c = 14.5 \pm 0.2$$
$$a_\mu^b = 0.302 \pm 0.002$$

Parametric error budget:

Quark	Charm	Bottom
$\Delta\hat{m}_q$	-0.18	-0.0011
$\Delta\alpha_s(\hat{m}_q)$	0.19	0.0013
(Anti-)Correlation on $\Delta\alpha_s(\hat{m}_q)$	-0.09	0.0001
Total	0.21	0.0018

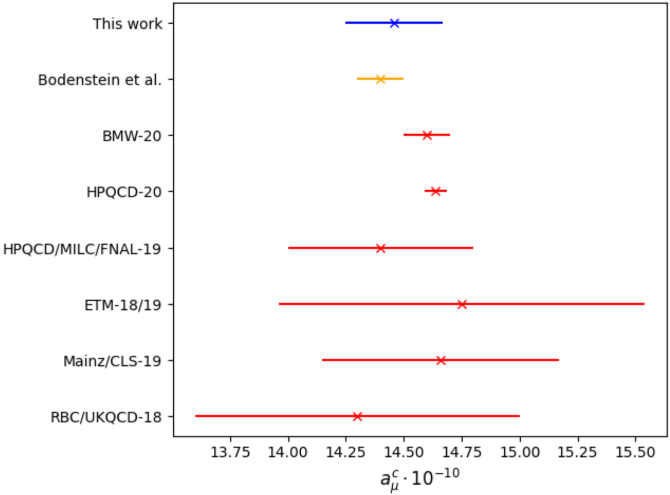
Charm theoretical uncertainties

$a_\mu^c \cdot 10^{10}$	$\mathcal{O}(\alpha_s^0)$	$\mathcal{O}(\alpha_s^1)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$
$\mathcal{O}\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)$	11.0131	3.3214	0.4087	0.1163	$< \pm 0.12$
$\mathcal{O}\left(\frac{m_\mu^4}{\hat{m}_q^4}\ln\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)\right)$	-0.1214	-0.0371	-0.0117	0.0019	$< \pm 0.002$
$\mathcal{O}\left(\frac{m_\mu^4}{\hat{m}_q^4}\right)$	0.0016	-0.00044	-0.0034	$0.00004 A_{32}^0$	$< \pm 0.00004 A_{32}^0$
$\mathcal{O}\left(\frac{m_\mu^6}{\hat{m}_q^6}\ln\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)\right)$	-0.00074	-0.00018	-0.000072	0.000016	$< \pm 0.00002$
$\mathcal{O}\left(\frac{m_\mu^6}{\hat{m}_q^6}\right)$	-0.000031	-0.000020	$5A_{23}^0 \cdot 10^{-7}$	$6A_{33}^0 \cdot 10^{-8}$	$< \pm 6A_{33}^0 \cdot 10^{-8}$

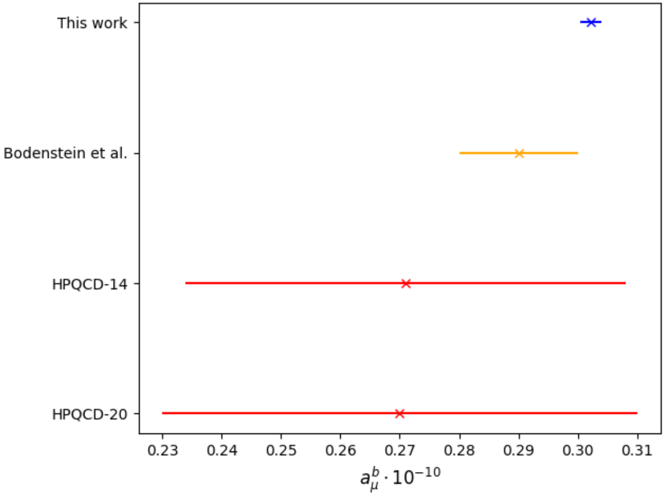
Bottom theoretical uncertainties

$a_\mu^b \cdot 10^{10}$	$\mathcal{O}(\alpha_s^0)$	$\mathcal{O}(\alpha_s^1)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^4)$
$\mathcal{O}\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)$	0.255335	0.044128	0.003937	-0.000698	$< \pm 0.0007$
$\mathcal{O}\left(\frac{m_\mu^4}{\hat{m}_q^4}\ln\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)\right)$	-0.00039	-0.000068	-0.000014	0.0000008	$< \pm 0.0000008$
$\mathcal{O}\left(\frac{m_\mu^4}{\hat{m}_q^4}\right)$	0.000003	$-5 \cdot 10^{-7}$	-0.000002	$A_{32}^0 \cdot 10^{-8}$	$< \pm A_{32}^0 \cdot 10^{-8}$
$\mathcal{O}\left(\frac{m_\mu^6}{\hat{m}_q^6}\ln\left(\frac{m_\mu^2}{\hat{m}_q^2}\right)\right)$	$-2 \cdot 10^{-7}$	$-3 \cdot 10^{-8}$	$-9 \cdot 10^{-9}$	$4 \cdot 10^{-10}$	$< \pm 4 \cdot 10^{-10}$
$\mathcal{O}\left(\frac{m_\mu^6}{\hat{m}_q^6}\right)$	$-6 \cdot 10^{-9}$	$-2 \cdot 10^{-9}$	$3A_{23}^0 \cdot 10^{-11}$	$2A_{33}^0 \cdot 10^{-12}$	$< \pm 2A_{33}^0 \cdot 10^{-12}$

Comparison of a_μ^c



Comparison of a_μ^b



Results for $(g-2)_e$ (in units of 10^{-15})

Our prediction:

$$a_e^c = 34.2 \pm 0.5$$
$$a_e^b = 0.708 \pm 0.004$$

Parametric error budget:

Quark	Charm	Bottom
$\Delta\hat{m}_q$	-0.43	-0.0027
$\Delta\alpha_s(\hat{m}_q)$	0.46	0.0031
(Anti-)Correlation on $\Delta\alpha_s(\hat{m}_q)$	-0.22	0.0002
Total	0.49	0.0042

Conclusions

- Both of our results are in good agreement with the literature
- Our result for a_μ^b currently highest precision in literature
- We have results independent of cross-section data and LQCD
- Explicit formula allows for increased precision in the future

Conclusions

- Both of our results are in good agreement with the literature
- Our result for a_μ^b currently highest precision in literature
- We have results independent of cross-section data and LQCD
- Explicit formula allows for increased precision in the future

Thank you for listening!

Bibliography I

Davier, M., Hoecker, A., Malaescu, B., and Zhang, Z. (2020).
A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $\boldsymbol{\alpha}(\mathbf{m_Z^2})$.
Eur. Phys. J. C, 80(3):241.
[Erratum: Eur.Phys.J.C 80, 410 (2020)].

Erler, J. and Luo, M.-x. (2001).
Hadronic loop corrections to the muon anomalous magnetic moment.
Phys. Rev. Lett., 87:071804.

Bibliography II

Erler, J., Masjuan, P., and Spiesberger, H. (2017).
Charm quark mass with calibrated uncertainty.
The European Physical Journal C, 77:99.

Erler, J. P. A., Masjuan Queralt, P., and Spiesberger, H. (202x).
Bottom quark mass with calibrated uncertainty.
In preparation.

Keshavarzi, A., Nomura, D., and Teubner, T. (2020).
$g-2$ of charged leptons, $\alpha(M_Z^2)$, and the hyperfine splitting of muonium.
Phys. Rev. D, 101(1):014029.

Bibliography III

Schmidt, B. and Steinhauser, M. (2012).
CRunDec: a C++ package for running and decoupling of the strong coupling and quark masses.
Comput. Phys. Commun., 183:1845–1848.

Zyla, P. et al. (2020).
Review of Particle Physics.
PTEP, 2020(8):083C01.