Third order correction to the muon lifetime

16th International Workshop on Tau Lepton Physics

Matteo Fael | Sept. 27, 2021

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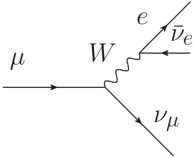
in collaboration with K. Schönwald, M. Steinhauser, Phys.Rev.D 104 (2021) 016003

Muon Decay



- Fundamental process to study weak interaction.
- Fermi Constant G_F extracted from muon lifetime

$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} F(\rho) \left[1 + H_1(\rho) \frac{\hat{\alpha}(m_\mu)}{\pi} + H_2(\rho) \left(\frac{\hat{\alpha}(m_\mu)}{\pi} \right)^2 + H_3(\rho) \left(\frac{\hat{\alpha}(m_\mu)}{\pi} \right)^3 \right]$$

with $\rho = m_e/m_\mu \simeq 1/210$.

- $O(\alpha)$ Behrends, Finkelstein, Sirlin Phys.Rev.101 (1956) 866.
- $O(\alpha^2)$
  van Ritbergen, Stuart Phys.Rev.Lett. 82 (1999) 488 $(m_e = 0)$
  Czarnecki, Pak, Phys.Rev.Lett. 100 (2008) 241807 $(m_e \ll m_\mu)$
- $O(\alpha^3)$ **NEW**
  MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 016003
  $(m_e \sim m_\mu)$
  parts of the calculation also confirmed in
  Czakon, Czarnecki, Dowling Phys.Rev.D 103 (2021) L111301.

Input Parameters of SM

- Fine structure constant CODATA

$$1/\alpha = 137.035\,999\,084\,(21)\ (0.15\ \text{ppb})$$

from a_e: 0.35 ppb Aoyama, Hayakawa, Kinoshita, Nio, PRL 109 (11) 111807
from h/m in ^{87}Rb: (0.08ppb) Morel, Yao, Cladé, Guellati-Khélifa, Nature 588 61

- Fermi Constant

$$G_F = 1.166\,378\,7\,(6) \times 10^{-5}\ \text{GeV}^{-2}\ (0.5\ \text{ppm})$$

Webber,et al., 2011, Phys.Rev.Lett.106, 041803; Phys.Rev.Lett.106, 079901

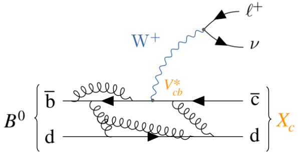
- *Z*-boson mass

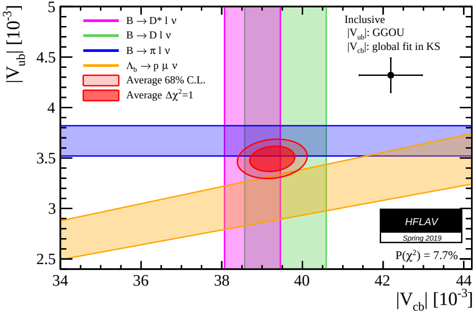
$$M_Z = 91.1876 \pm 0.0021\ \text{GeV}\ (23\ \text{ppm})$$

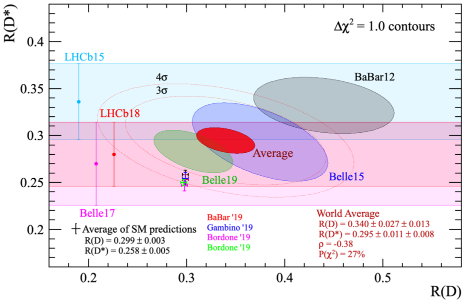
P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020)
FCC-ee aims at improving M_Z by factor 400 (stat).

Semileptonic B decays

- $b \to c\ell\bar{\nu}_\ell$ ($\ell = e, \mu$) sensitive to $|V_{cb}|$.
- **Inclusive decays**
  - $\overline{B} \to X_c\ell\bar{\nu}$, with $X_c = D, D^*, D\pi, \ldots$

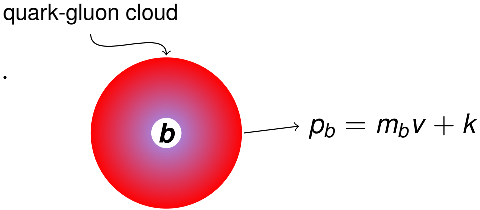






The Heavy-Quark Expansion

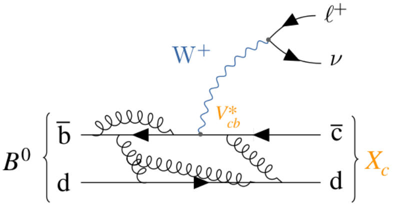
$$\Gamma_{\rm sl} = \Gamma_0 + \Gamma_{\mu_\pi}\frac{\mu_\pi^2}{m_b^2} + \Gamma_{\mu_G}\frac{\mu_G^2}{m_b^2} + \Gamma_{\rho_D}\frac{\rho_D^3}{m_b^3} + \Gamma_{\rho_{LS}}\frac{\rho_{LS}^3}{m_b^3} + \dots$$



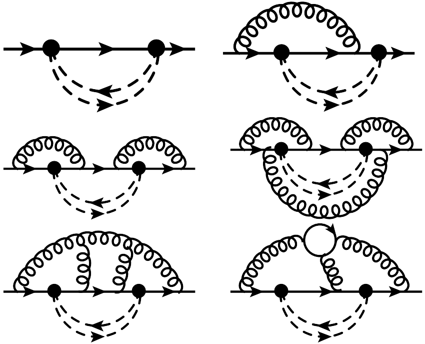
Reviews:

Benson, Bigi, Mannel, Uraltsev, Nucl.Phys. B665 (2003) 367;
Dingfelder, Mannel, Rev.Mod.Phys. 88 (2016) 035008.

- Γ_i are computed in **perturbative QCD**.
- **Γ_0: free-quark decay $b \to c\ell\nu$!**
- The HQE parameters:
  $\mu_\pi, \mu_G, \rho_D, \rho_{LS} \sim \langle B|\ \mathcal{O}_i^{\bar{b}b}\ |B\rangle$
- HQE parameters are
  extracted from data or lattice.



$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[X_0(\rho) + \sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n X_n(\rho)\right]$$



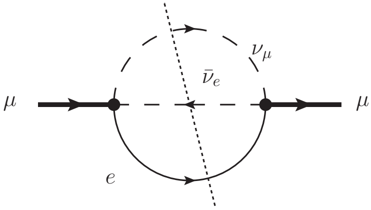
Possible strategies

- Exact result with m_μ and m_e only at $O(\alpha)$.
  Nir, Phys.Lett.B 221 (1989) 184
- Set $m_e = 0$ ($m_c = 0$).
  van Ritbergen, Stuart
- **Approximation exploiting $m_e < m_\mu$ ($m_c < m_b$).**
  Czarnecki, Pak

Computational Method

$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 - 8\rho^2 - 12\rho^4\log(\rho^2) + 8\rho^6 - \rho^8\right]$$

where $\rho = m_e/m_\mu$.



- Optical theorem.
- Multi-loop diagrams with **two scales m_e and m_μ.**
- Use **method of regions:**
  Beneke, Smirnov, NPB 522 (1998) 321; Smirnov, Springer Tracts Mod. Phys. 177
  - Expansion in $\rho = m_e/m_\mu$ (or $\rho = m_c/m_b$).
    Obvious choice, too difficult to extend at $O(\alpha^3)$.
  - Expansion in $\delta = 1 - m_e/m_\mu$ (or $\delta = 1 - m_c/m_b$)
    Crucial decoupling and simplifications of loop integrals.

A toy example

Karlsruhe Institute of Technology

$$F(m,M,\varepsilon) = \int_0^\infty dk\, \frac{k^{-\varepsilon}}{(k+m)(k+M)} \overset{\varepsilon\to 0}{=} \frac{\log(M/m)}{M-m} = \frac{\log(M/m)}{M} \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n$$

When k is a *hard* scale $O(M)$

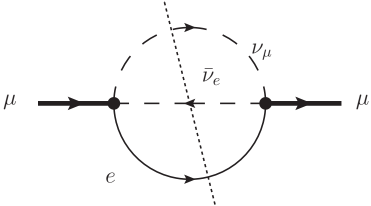
$$F_{\mathrm{h}}(m,M,\varepsilon) = \int_0^\infty dk\, \frac{k^{-\varepsilon}}{(k+M)} \left[\frac{1}{k} - \frac{m}{k^2} + \dots\right] \overset{\varepsilon\to 0}{=} \left[-\frac{1}{\varepsilon M} + \frac{\log M}{M}\right] \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n$$

k is a *soft* scale $O(m)$

$$F_{\mathrm{s}}(m,M,\varepsilon) = \int_0^\infty dk\, \frac{k^{-\varepsilon}}{(k+m)} \left[\frac{1}{M} - \frac{k}{M^2} + \dots\right] \overset{\varepsilon\to 0}{=} \left[\frac{1}{\varepsilon M} - \frac{\log m}{M}\right] \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n$$

Muon lifetime at tree level

$$\Gamma_\mu = \Gamma^{(\mathrm{hh})} + \Gamma^{(\mathrm{ss})} + \Gamma^{(\mathrm{hs})} + \Gamma^{(\mathrm{sh})}$$
$$= \Gamma_0 \left[1 - 8\rho^2 - 12\rho^4 \log(\rho^2) + 8\rho^6 - \rho^8 \right]$$



where $\rho = m_e/m_\mu$ and $\Gamma_0 = \dfrac{G_F^2 m_\mu^5}{192\pi^3}$

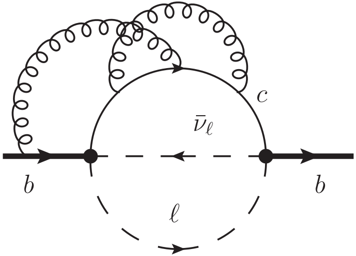
$$\Gamma^{(\mathrm{hh})} \sim -\frac{\rho^4}{12\varepsilon} - 24\rho^4 \log\left(\frac{\mu^2}{m_\mu^2}\right) + 1 - 8\rho^2 - 24\rho^4 + 16\rho^6 - 2\rho^8$$

$$\Gamma^{(\mathrm{hs})} \sim +\frac{\rho^4}{12\varepsilon} + 24\rho^4 \log\left(\frac{\mu^2}{m_\mu^2}\right) - 12\rho^4 \log(\rho^2) + 24\rho^4 - 8\rho^6 + \rho^8$$

$$\Gamma^{(\mathrm{ss})} = \Gamma^{(\mathrm{sh})} = 0$$

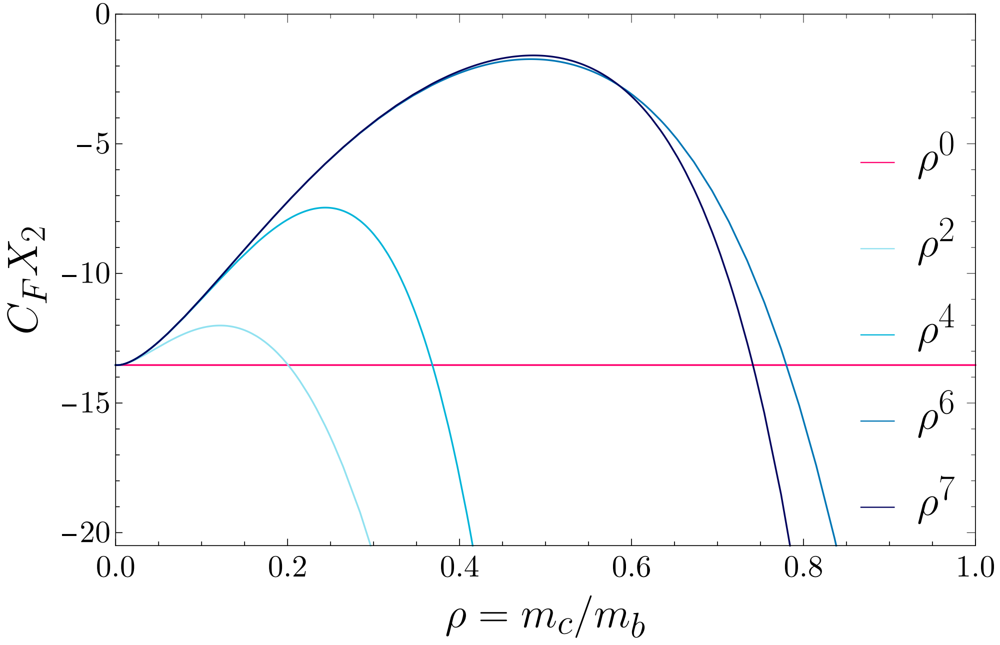
Second order corrections

$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[X_0 + \frac{\alpha}{\pi}X_1 + \left(\frac{\alpha}{\pi}\right)^2 X_2 + \dots\right]$$



Czarnecki, Pak, PRD 78 (2008) 114015; PRL 100 (2008) 241807.

- Four-loop diagrams, all loop momenta can scale **hard** (m_μ) or **soft** (m_e).
- Each diagram considered up to **11 different regions**.
- The *all-hard* region reduces to **33 four-loop master integrals**.
- Expansion depth: $O(\rho^7)$ ($\rho = m_e/m_\mu$).



$C_F X_2$
0
−5
−10
−15
−20
0.0
0.2
0.4
0.6
0.8
1.0
$\rho = m_c/m_b$
ρ^0
ρ^2
ρ^4
ρ^6
ρ^7

Towards the third order corrections

	α_s^2		α_s^3
n. diagrams	62	$\to$	1450
n. loops	4	$\to$	5
regions	11	$\to$	O(20)
expansion depth	7	$\to$	?
master integrals	33	$\to$	?


RUSSIA
North Pole
Svalbard (Norway)
NORWAY
Bering Sea
ARCTIC OCEAN
Ellesmere Island
ICELAND
Greenland (Denmark)
Alaska (U.S.)
Banks Island
Victoria Island
Baffin Bay
Baffin Island
ATLANTIC OCEAN
PACIFIC OCEAN
CANADA
Hudson Bay
Labrador
W N S E
0 500 1,000 miles
0 500 1,000 kilometers
Northwest Passage
© 2012 Encyclopædia Britannica, Inc.

The heavy daughter limit

Dowling, Piclum, Czarnecki, PRD 78 (2008) 074024

- Is the most natural expansion parameter sometimes also the best one?

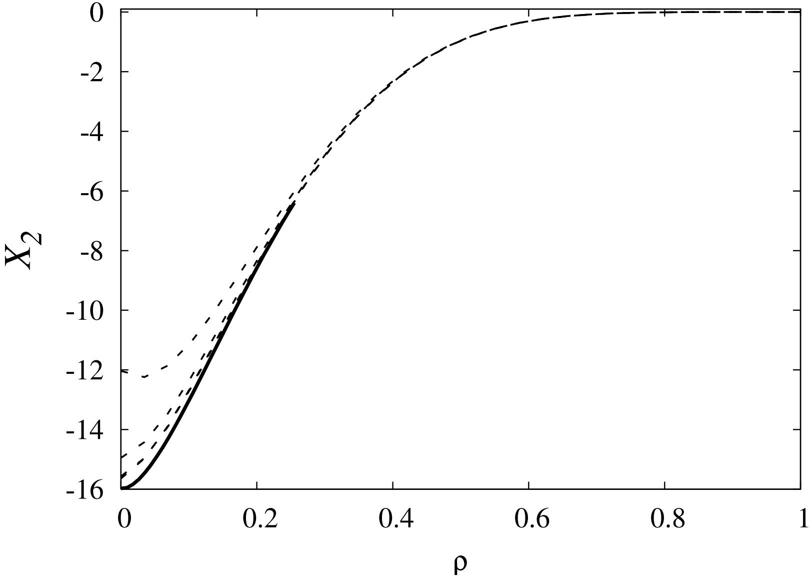
$$\frac{m_e}{m_\mu} \sim \frac{1}{210} \qquad \frac{m_c}{m_b} \sim 0.3$$

- Perform the expansion in the limit $m_e \sim m_\mu$ ($m_c \sim m_b$):

$$\delta = 1 - \rho = 1 - \frac{m_e}{m_\mu} \ll 1$$

- The width must behave in the the $m_e \to m_\mu$ limit as:

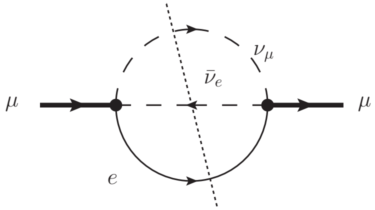
$$\Gamma_\mu \overset{m_e \to m_\mu}{\simeq} \frac{G_F^2}{192\pi^3}(m_\mu - m_e)^5 = \frac{G_F^2 m_\mu^5}{192\pi^3}\delta^5$$



Dowling, Piclum, Czarnecki, PRD 78 (2008) 074024

Muon Lifetime Reloaded

$$\Gamma_\mu = \Gamma^{(\mathrm{hh})} + \Gamma^{(\mathrm{ss})} + \Gamma^{(\mathrm{sh})} + \Gamma^{(\mathrm{hs})}$$
$$= \Gamma_0 \left[\frac{64}{5}\delta^5 - \frac{96}{5}\delta^6 + \frac{288}{35}\delta^7 + \dots \right]$$



where $\rho = m_e/m_\mu$ and $\Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3}$

- **Crucial simplifications** in the heavy daughter limit!
- At least one electron's propagator must scale *soft* to generate an imaginary part $\log(-\delta)$.
- **Much smaller number of regions**, e.g. $\Gamma^{(\mathrm{hh})} = \Gamma^{(\mathrm{sh})} = \Gamma^{(\mathrm{hs})} = 0$

Divide et Impera

- The heavy daughter limit converts **5 loops** $\longrightarrow$ **3 loops**!
- Loop integrals associated to ν and $\bar{\nu}$ momenta decouple and are integrated analytically at once.
- Explanation is rather technical, connected to the appearance of linear propagators:

$$\frac{1}{(p+k)^2 - m_c^2} \to \frac{1}{2p \cdot k - \delta}$$

	scaling	n. regions
$\mathcal{O}(\alpha)$	h, u	2
$\mathcal{O}(\alpha^2)$	**hh**, hu, **uu**	4
$\mathcal{O}(\alpha^3)$	**hhh**, **uuu**, h**uu**, **hh**u	8

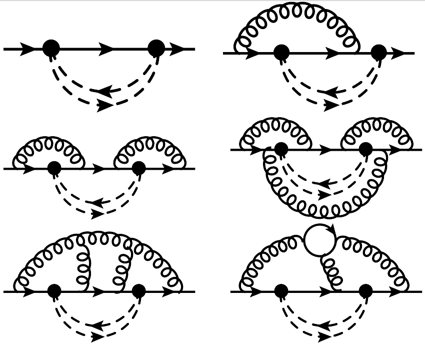
Computational Challenges

- 1450 five-loop diagrams.
- Several subtleties with `FORM`
  - Propagators expanded up to **10th - 12th order.**
  - Major obstacle is to keep as small as possible **the size of intermediate expressions**.
  - Efficient expansion of propagators and memory management.
- Intermediate `FORM` expressions up to $O(100)$ GB.
- Master integrals:
  - $O(\alpha^2)$: 3 (ss) and 3 (hh).
  - $O(\alpha^3)$: 20 (sss) and 19 (hhh).

    Melnikov, van Ritbergen, Nucl.Phys.B 591 (2000) 515;
    MF, Schönwald, Steinhauser, Phys.Rev.Lett. 125 (2020) 5.
- Renormalization constants at 3 loops with two massive fermions.

  MF, Schönwald, Steinhauser, JHEP 10 (2020) 087.

$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[X_0 + \sum_{n\geq 1}\left(\frac{\hat{\alpha}(m_\mu)}{\pi}\right)^n X_n\right]$$

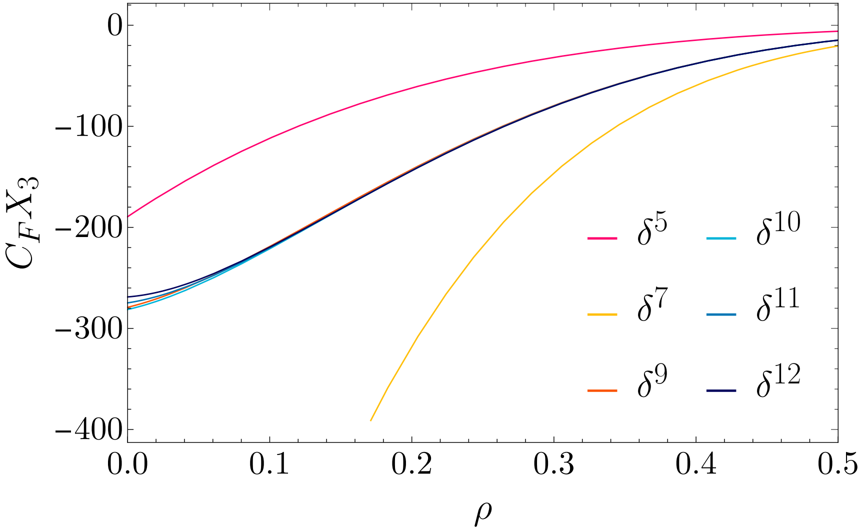


- **NEW**: X_3 up to δ^{12} (first 8 terms).

$$X_3 = \sum_{m\geq 5} x_{3,m}\delta^m = \delta^5\left[\frac{256}{3}\mathrm{a4} - 128\zeta(5) + \frac{56}{5}\pi^2\zeta_3 + \frac{1984}{45}\zeta_3 + \frac{452}{675}\pi^4 - \frac{3652}{405}\pi^2 - \frac{18451}{270} + \frac{32}{9}\log^4(2) - \frac{352}{45}\pi^2\log^2(2) - \frac{32}{45}\pi^2\log(2)\right] + O(\delta^6)$$

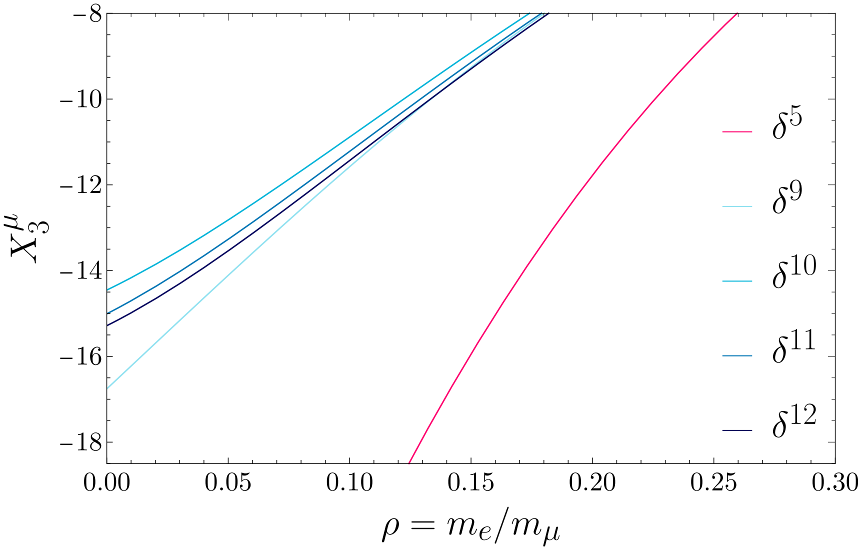
MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 016003

confirmed C_F^3, $C_F^2 N_H$ and $C_F N_H^2$ color factors up to δ^9 Czakon, Czarnecki, Dowling, Phys.Rev.D 103 (2021) L111301



$$C_F X_3(\rho = 0.28) = -91.2 \pm 0.4 \quad (0.4\%)$$

MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003



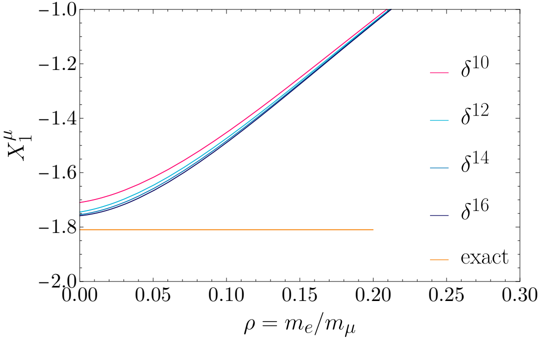
$$X_3^\mu = -15.3 \pm 2.3 \quad (15\%)$$

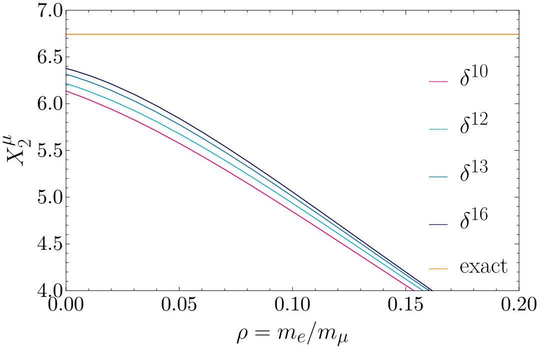
MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003

Previous estimate: $X_3^\mu \simeq -20$

Ferroglia, Ossola, Sirlin, Nucl.Phys.B 560 (1999) 23

Theoretical uncertainties





order	X_n/X_{exact}	$x_{n,12}/X_n$
α	0.96	0.6 %
α^2	0.92	1.0%
α^3	$\sim$ 0.85	1.8 %

Final Error Budget

$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^2} F(\rho)\,(1 + \Delta q)$$

$$\Delta q = -(4\,234\,530|_{\hat\alpha} + 36\,332|_{\hat\alpha^2} + 200|_{\hat\alpha^3} \pm 29|_{\delta\hat\alpha^3} \pm 11|_{\delta\mathrm{had}}) \times 10^{-9}$$

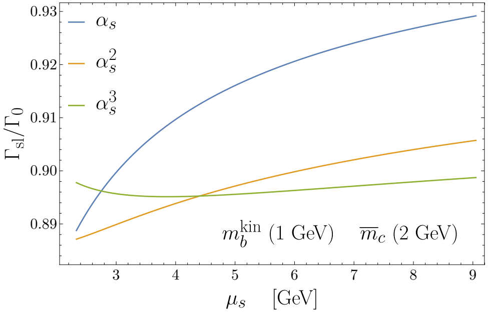
- $\delta(1 + \Delta q) = 0.031$ ppm
- $\frac{1}{2}\frac{\delta\tau_\mu}{\tau_\mu} = 0.5$ ppm
- $\frac{5}{2}\frac{\delta m_\mu}{m_\mu} = 0.05$ ppm

- at $O(\alpha^2)$: $\delta(1 + \Delta q) = 0.17$ ppm
  van Ritbergen, Stuart, Nucl.Phys.B 564 (2000) 343
  Sirlin, Ferroglia, Rev.Mod.Phys. 85 (2013) 1
- Precise $\overline{\mathrm{MS}}$-on shell conversion of α up to 4 loops

  Baikov, Chetyrkin, Kuhn, Sturm, Nucl.Phys.B 867 (2013) 182

Lessons for $B \to X_c \ell \nu$

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} F(\rho) \left[1 + \sum_n Y_n \left(\frac{\alpha_s(m_b)}{\pi} \right)^n \right]$$



n=1 Jezabek, Kühn, Jezabek, Kuhn, NPB 314 (1989) 1
n=2 Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015.
n=3 Fael, Schönwald, Steinhauser, hep-ph/2011.13654

$$m_b^{\rm OS} : m_c^{\rm OS} \quad 1 - 1.78 \left(\frac{\alpha_s}{\pi}\right) - 13.1 \left(\frac{\alpha_s}{\pi}\right)^2 - 163.3 \left(\frac{\alpha_s}{\pi}\right)^3$$

$$\overline{m}_b(\overline{m}_b) : \overline{m}_c(3\text{ GeV}) \quad 1 + 3.07 \left(\frac{\alpha_s}{\pi}\right) + 13.3 \left(\frac{\alpha_s}{\pi}\right)^2 + 62.7 \left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm kin}(1\text{ GeV}) : \overline{m}_c(2\text{ GeV}) \quad 1 - 1.24 \left(\frac{\alpha_s}{\pi}\right) - 3.65 \left(\frac{\alpha_s}{\pi}\right)^2 - 1.0 \left(\frac{\alpha_s}{\pi}\right)^3$$

Improvement in $|V_{cb}|$

- Fit BR and moments from B factories
- Global fit strategy from 2014
  Gambino, Schwanda, Phys.Rev.D 89 (2014) 014022
  Alberti, Gambino, Healey, Nandi, Phys.Rev.Lett. 114 (2015) 6, 061802
- $O(\alpha_s^3)$ semileptonic width
- $O(\alpha_s^3)$ relation between $\overline{m}_b - m_b^{\mathrm{kin}}$
  MF, Schönwald, Seinhauser, Phys.Rev.Lett. 125 (2020) 052003; Phys.Rev.D 103 (2021) 1, 014005

- Precise input from lattice

$$\overline{m}_c(3\ \mathrm{GeV}) = 0.988\,(7)\ \mathrm{GeV}$$
$$\overline{m}_b(\overline{m}_b) = 4.198\,(12)\ \mathrm{GeV}$$
$$\longrightarrow m_b^{\mathrm{kin}} = 4.56\,(19)\ \mathrm{GeV}$$

FLAG2019

$$|V_{cb}| = 42.16\,(30)_{\mathrm{th}}(32)_{\mathrm{exp}}(25)_{\Gamma} \times 10^{-3}$$
$$= 42.16\,(50) \times 10^{-3}$$

Bordone, Capdevila, Gambino, hep-ph/2107.00604
error improvement of 34% compared to 2014.

Conclusions

- QED $O(\alpha^3)$ to the muon lifetime
- QCD α_s^3 corrections to $\Gamma(b \to X_c \ell \nu)$.
- Heavy daughter limit: small parameter $\delta = 1 - m_e/m_\mu$ and $\delta = 1 - m_c/m_b$.
- $\Delta q^{(3)}$ with relative 15% uncertainty. Theory error on Δq reduced to 0.03 ppm.
- 1% theory uncertainty in semileptonic decays width.
- Improved extraction of $|V_{cb}|$ inclusive by about 34%.