



I L L I N O I S
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Pushing the boundaries of fluid dynamics

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**Symposium on Contemporary QCD Physics and
Relativistic Nuclear Collisions**



College of Physical Science and Technology
Central China Normal University, Wuhan, China
November 9 – 11, 2019

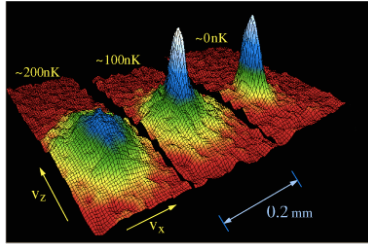
A depiction of how the seminars went at Columbia University ...



“Progress appears when theory is pushed to its limits”

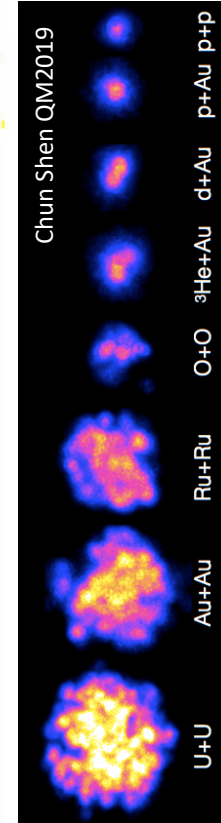
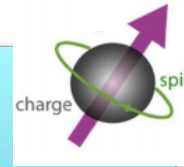
Emergence of fluid dynamics: A bird's-eye view

2 D velocity distributions



Bose-Einstein condensate

\hbar
Quantumness



Chun Shen QM2019

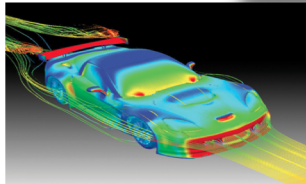
small systems

HIC

Large systems

Velocity

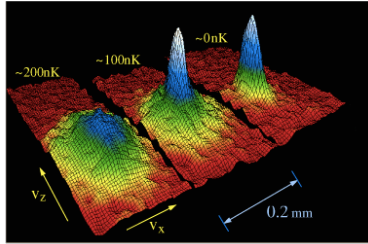
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Daily life

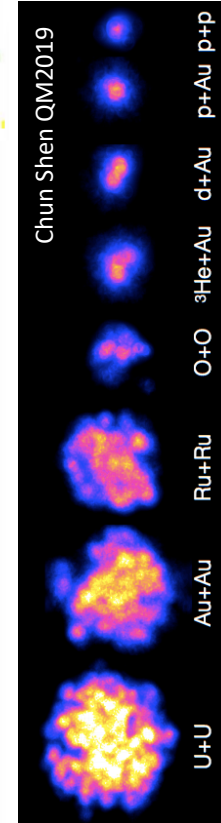
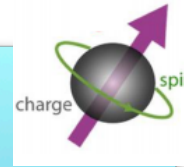
Emergence of fluid dynamics: A bird's-eye view

2 D velocity distributions



Bose-Einstein condensate

\hbar
Quantumness



Chun Shen QM2019

small systems

HIC

Large systems

0

Velocity

c

Gravity

Neutron star mergers
Black holes

How does a lump of baryon rich neutron star matter behave under strong gravitational fields?

Neutron Star Mergers

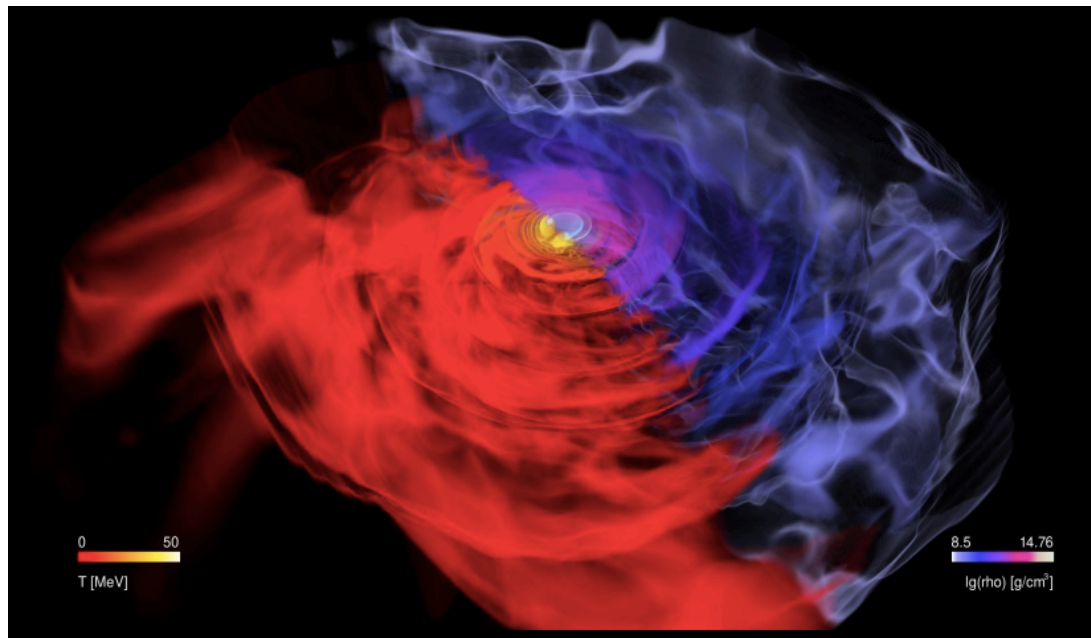


Fig. from L. Rezzolla's talk at QM2019

High Density QCD Matter: From the Lab to the Sky

Neutron Star Mergers

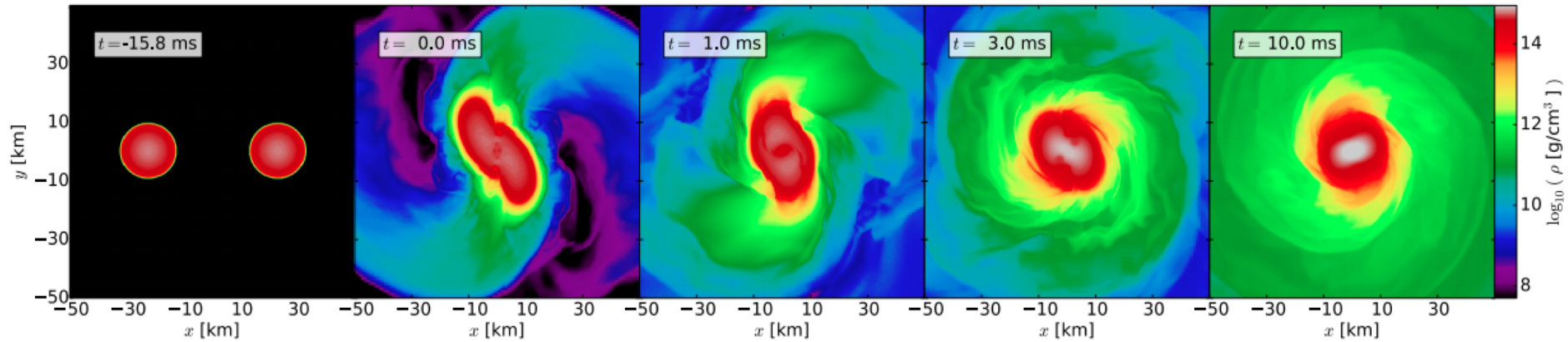


Fig. by L. Rezzolla

New signatures for deconfinement/phase transitions?

e.g. Most et al., PRL (2019)

Viscous fluid dynamics + strong gravitational fields?

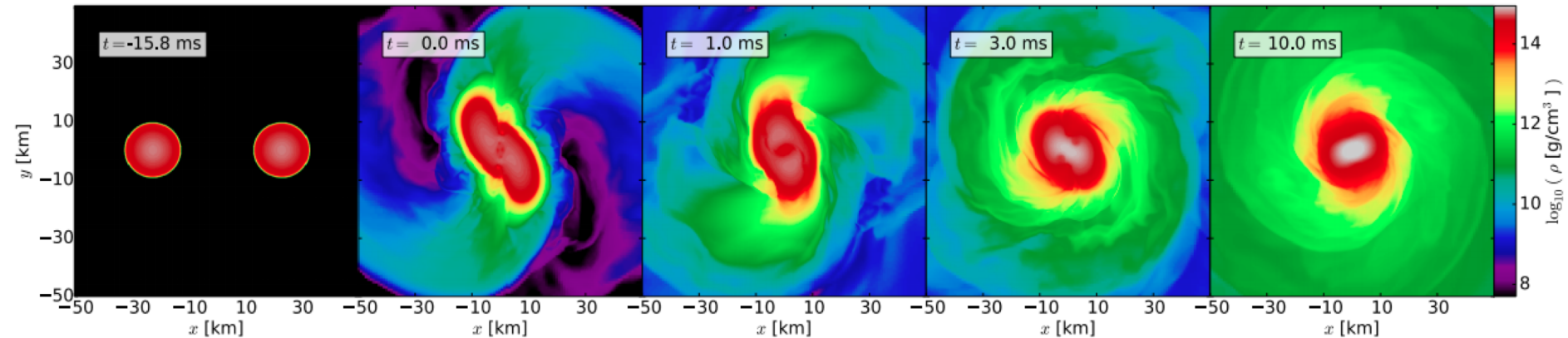
Viscous effects in neutron star mergers?

Duez et al PRD (2004), Shibata et al. PRD (2017), Alford et al. PRL (2018)

Viscous effects in binary neutron-star mergers?

Previous assumption ([since 1992](#)): viscous effects do not matter

Bildsten and Cutler, *Astrophys. J.* (1992)



Why? Based on the simulations/knowledge at that time:

- Transport time scales estimated to be far from \sim microseconds
- Temperatures not so large, system very smooth, gradients too small

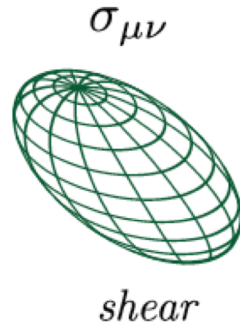
Viscous effects in binary neutron-star mergers

Previous exceptions: Duez et al PRD (2004), Shibata et al. PRD (2017)

Alford, Bovard, Hanauske, Rezzolla, Schwenzer, PRL (2018):

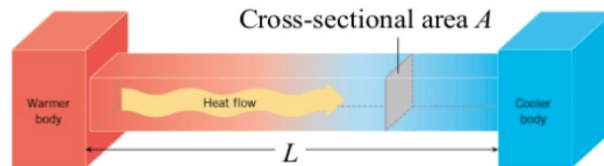
Post-merger phase

Shear dissipation:



Relevant for trapped neutrinos if $T > 10$ MeV and gradients at small scales ~ 0.01 km (e.g, turbulence).

Thermal transport:
“heat conductivity”



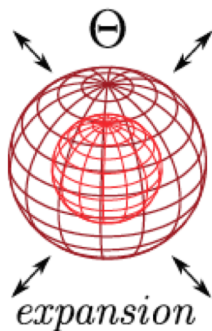
Relevant for trapped electron neutrinos if $T > 10$ MeV and gradients ~ 0.1 km

Viscous effects in binary neutron-star mergers

Alford et al. PRL (2018)

~~$B_1 \rightarrow B_2 + \ell + \bar{\nu}_\ell$;
 $B_2 + \ell \rightarrow B_1 + \nu_\ell$~~ If suppressed

Bulk viscosity:



Should affect density oscillations after merger!!!

See also Alford, Harris, PRC (2019)

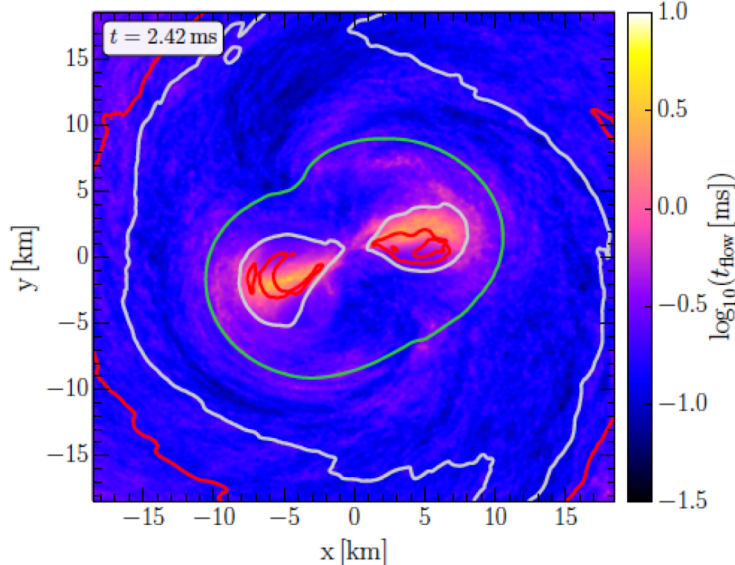
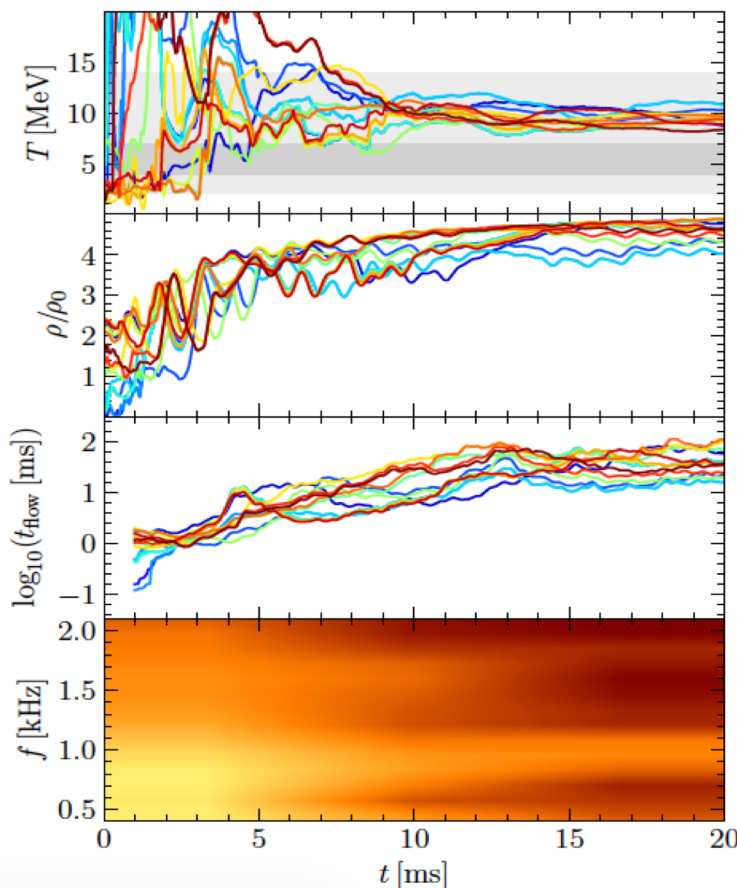


Figure 3: The flow timescale t_{flow} obtained from a numerical-relativity simulation of two $1.35 M_\odot$ neutron stars [40]. The red (4 MeV) and gray (7 MeV) contours show the boundaries of the temperature range in which the bulk viscosity roughly takes its maximum value, while the green contour shows the inner region where the rest-mass density exceeds nuclear saturation density.



Viscous effects in binary neutron-star mergers

Alford et al. PRL (2018)

“The effects of bulk viscosity should be consistently included in future merger simulations. This has not been attempted before and requires a formulation of the relativistic-hydrodynamic equations that is hyperbolic and stable”.

Challenge: Prove that the solutions $\{g_{\mu\nu}, T_{\mu\nu}, J_B^\mu\}$ are **well posed** (existence, uniqueness) and **causal in the full nonlinear regime**.

Einstein's equations

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Conservation laws

$$\nabla_\mu T^{\mu\nu} = 0$$

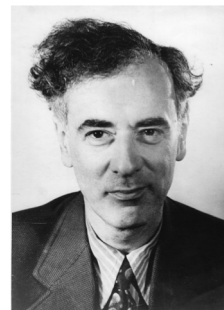
$$\nabla_\mu J_B^\mu = 0$$

+ Bulk Viscosity



Eckart
1940's

Relativistic Navier-Stokes equations



Landau
1950's

Energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu}$$

Bulk scalar

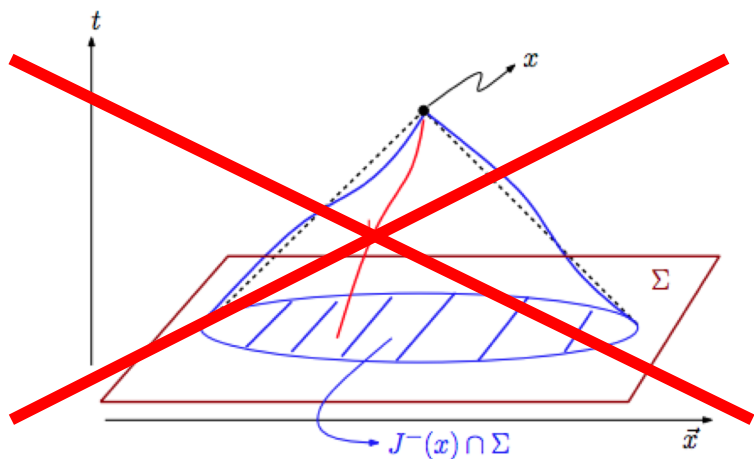
$$\Pi = -\zeta \nabla_\lambda u^\lambda$$

Conservation law

$$\nabla_\mu T^{\mu\nu} = 0$$

This theory is **acausal**

(mathematical proof by Pichon, 1965)



Nonlinear diffusion equations

$$\partial_t u = \frac{\zeta}{sT} \nabla^2 u + \dots$$

Also **unstable!!!**

(Hiscock, Lindblom, 1984)

How can one solve this problem?

- Heavy ion collisions define the state of the art of relativistic viscous fluid dynamics.
- Let us check how this is handled there ...

“Hydro” in our field is not simple textbook hydro

Israel-Stewart theory

Israel, Stewart, Ann. Phys. 118, 341 (1979)

Energy-momentum
tensor

$$T_{\mu\nu} \longrightarrow \varepsilon, u_{\mu}, \pi_{\mu\nu}, \Pi \quad \text{as dynamical variables}$$

Effective theory for hydrodynamic fields and non-hydrodynamic fields

Dynamics: $\nabla_{\mu} T^{\mu\nu} = 0$ (energy-momentum conservation)

$$u^{\lambda} \nabla_{\lambda} \Pi + F(\varepsilon, \nabla_{\alpha} u_{\beta}, \pi^{\alpha\beta}, \Pi) = 0 \quad (\text{bulk})$$

$$u^{\lambda} \nabla_{\lambda} \pi^{\mu\nu} + F^{\mu\nu}(\varepsilon, \nabla_{\alpha} u_{\beta}, \pi^{\alpha\beta}, \Pi) = 0 \quad (\text{shear})$$

Despite the impressive progress in recent years, outstanding questions remain in the nonlinear regime:

Open problems in physics and mathematics (since 1940)

- Can one formulate a theory of relativistic viscous fluids where causality holds in the full nonlinear regime?
- Is the mathematical problem well-posed?

Results are known only in the linearized regime around equilibrium

See, e.g, Hiscock, Lindblom, Ann. Phys. (1983)

Pu, Koide, Rischke, PRD (2010)

Nonlinearity is essential for mergers!!

Why is this so hard to do?

- 16 coupled nonlinear PDE's (Einstein-Israel-Stewart)

energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu}$$

conserved baryon current

$$J^\mu = n u^\mu$$

Israel-Stewart equation

$$\tau_\Pi u^\alpha \nabla_\alpha \Pi + \Pi + \zeta \nabla_\alpha u^\alpha + f(\varepsilon, n, \Pi) = 0$$

- Nonlinearity in hydrodynamics is notoriously hard to handle.
- Standard lore: Israel-Stewart equations are beyond current mathematical techniques.

Solution: Bemfica, Disconzi, JN, PRL 122, 221602 (2019)

Einstein-Israel-Stewart equations can be written as

$$A_0(\Phi) \nabla_t \Phi + A^i(\Phi) \nabla_i \Phi + B(\Phi) \Phi = 0$$

where $\Phi = (\varepsilon, n, u_\mu, \Pi, g_{\mu\nu}, \partial g_{\mu\nu})$

First mathematical proof of existence, uniqueness, and causality in the full nonlinear regime (fully embedded in GR)

Theorem 1. *Let $(\varepsilon, u^\alpha, \Pi, n, g_{\alpha\beta})$ be a solution to the generalized EIS equations defined on a globally hyperbolic spacetime M , and let Σ be a Cauchy surface. Suppose $\frac{\zeta}{\tau_\Pi(\varepsilon+P+\Pi)} + \alpha_1 + \frac{\alpha_2 n}{\varepsilon+P+\Pi} \geq 0$ and that (1) holds. Then, for any $p \in M$ in the future of Σ , $(\varepsilon(p), u^\alpha(p), \Pi(p), n(p), g_{\alpha\beta}(p))$ depends only on $(\varepsilon, u^\alpha, \Pi, n, g_{\alpha\beta}, \kappa_{\alpha\beta})|_{\Sigma \cap J^-(p)}$, where $J^-(p)$ is the causal past of p and κ is the extrinsic curvature of Σ in M .*

Theorem 2. *Let $\mathcal{I} = (\Sigma, \dot{\varepsilon}, \dot{u}^i, \dot{\Pi}, \dot{n}, \dot{g}_{ij}, \dot{\kappa}_{ij})$ be an initial data set for the generalized EIS system, with an equation of state $P = P(\varepsilon, n)$, a bulk viscosity $\zeta = \zeta(\varepsilon, n)$, and a relaxation time $\tau_\Pi = \tau_\Pi(\varepsilon, n)$. Assume that $\dot{\varepsilon} + P(\dot{\varepsilon}, \dot{n}) + \dot{\Pi}$, $\tau_\Pi(\dot{\varepsilon}, \dot{n})$, $\zeta(\dot{\varepsilon}, \dot{n}) > 0$, $\frac{\partial P}{\partial \varepsilon}(\dot{\varepsilon}, \dot{n}) + \frac{\partial P}{\partial n}(\dot{\varepsilon}, \dot{n})\dot{n}/(\dot{\varepsilon} + P(\dot{\varepsilon}, \dot{n}) + \dot{\Pi}) \geq 0$, and that \dot{n} , $\frac{\partial P}{\partial \varepsilon}(\dot{\varepsilon}, \dot{n})$, and $\frac{\partial P}{\partial n}(\dot{\varepsilon}, \dot{n})$ are nonzero. Suppose that $\dot{g}_{ij} \in H_{ul}^{s+1}(\Sigma)$, $\dot{\varepsilon}, \dot{u}^i, \dot{\Pi}, \dot{n}, \dot{\kappa}_{ij} \in H_{ul}^s(\Sigma)$, and that $P, \zeta, \lambda, \tau_\Pi \in C^s(\mathbb{R}^2)$, where $s \geq 3$. Suppose that (1) holds for \mathcal{I} . Then, there exists a globally hyperbolic development of \mathcal{I} .*

Nonlinear causality constraint

$$\left[\frac{\zeta}{\tau_{\Pi}} + n \left(\frac{\partial P}{\partial n} \right)_{\varepsilon} \right] \frac{1}{\varepsilon + P + \Pi} < 1 - \left(\frac{\partial P}{\partial \varepsilon} \right)_n$$

- The initial value problem of the full nonlinear set of Einstein-Israel-Stewart equations can now be solved.
- Valid for arbitrary EOS.
- No symmetry assumptions made.
- No fundamental issues appear if $P + \Pi < 0$ as long as $\varepsilon + P + \Pi > 0$

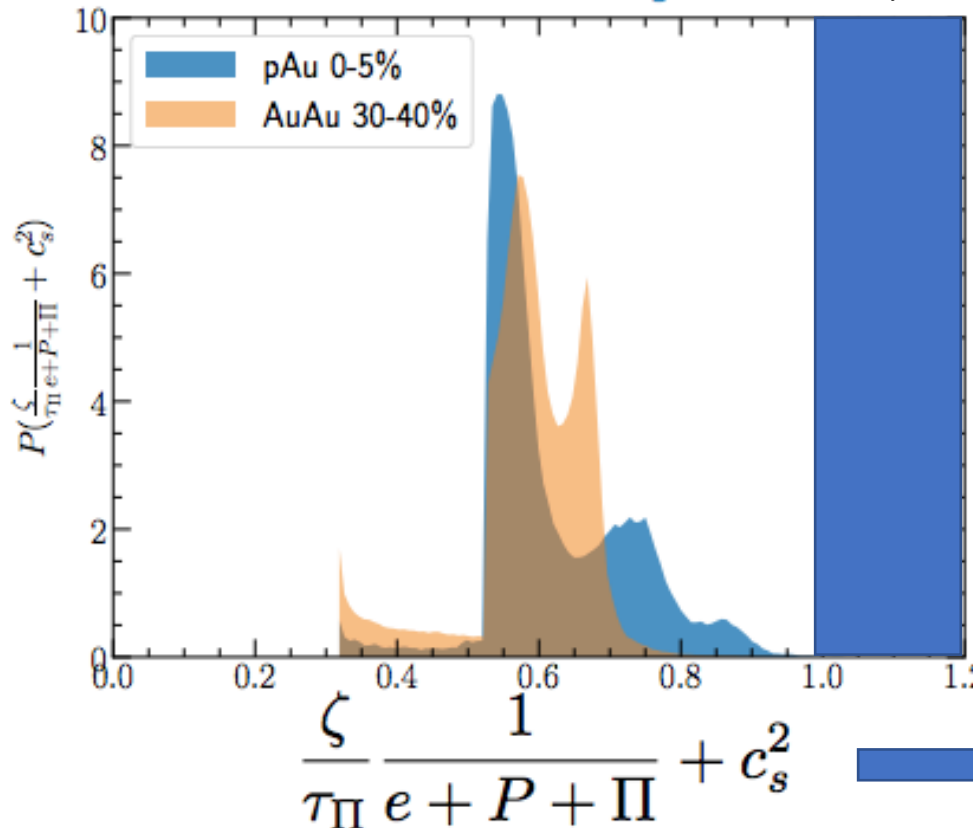
Causality constraint and small systems in heavy ions

$$\frac{\zeta}{\tau_{\Pi}(\varepsilon + P + \Pi)} < 1 - c_s^2$$

Zero baryon density

Causality

C. Shen, QM2019



Further constraints will appear when shear effects are included in the proof!

Dependence on transport coeff. !!!

Is the description of relativistic fluid dynamics unique?

- Israel-Stewart theory not unique (eg. rBRSSS vs. DNMR).
- Transient behavior of strongly coupled (holographic) liquids not described within Israel-Stewart approach.

Denicol, Niemi, JN, Rischke, PRD (2011).

Is there another approach that describes the motion of relativistic viscous fluids in a way compatible with (general) relativity?

Revisiting the gradient expansion

Hydrodynamics: Simplest effective theory for $\{\varepsilon, u_\mu\}$

$$\mu_B = 0$$

$$T^{\mu\nu} = T_{eq}^{\mu\nu}(\varepsilon, u^\lambda) + T_{viscous}^{\mu\nu}$$

Viscous correction? Gradient expansion

Chapman-Enskog

$$T_{viscous}^{\mu\nu} = \underbrace{T_{viscous}^{\mu\nu}(\nabla\varepsilon, \nabla u)}_{\text{All possible 1st order terms}} + \underbrace{\mathcal{O}(\nabla^2)}_{\text{higher orders}}$$

All possible 1st order terms
with no time derivatives in the local rest frame

Revisiting the gradient expansion


IMPORTANT: If Landau definition is assumed to be valid throughout $\longrightarrow u_\mu T^{\mu\nu} = -\varepsilon u^\nu$

- First order truncation leads to relativistic Navier-Stokes theory, which is acausal and unstable.
- This is not fixed by going to 2nd order in gradients (note this is NOT Israel-Stewart theory).
- But why doesn't the gradient expansion work?

A new approach to relativistic viscous fluid dynamics

Based on Bemfica, Disconzi, JN, PRD (2017) and PRD (2019)
See also P. Kovtun, JHEP (2019)

Effective theory: Space-time derivative expansion

$T_{viscous}^{\mu\nu}(\nabla\varepsilon, \nabla u)$  Most general derivative expansion compatible with symmetries

- Definition of ε and u^μ not unique out of equilibrium.
- Time derivatives should appear even in the rest frame.
- No reason to expect a priori that Landau's definition is ok.

Tsumura, Kunihiro, PLB (2008)
Van, Biro, EPJ ST (2008)

A new approach to relativistic viscous fluid dynamics

Based on Bemfica, Disconzi, JN, PRD (2017) and PRD (2019)
See also P. Kovtun, JHEP (2019)

Most general derivative expansion compatible with symmetries

$$T^{\mu\nu} = (\varepsilon + \mathcal{A}_1)u^\mu u^\nu + (P(\varepsilon) + \mathcal{A}_2)\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu} + u^\mu Q^\nu + u^\nu Q^\mu$$

where to 1st order in derivatives

$$\mathcal{A}_1 = \chi_1 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_2 \nabla_\alpha u^\alpha, \quad \mathcal{A}_2 = \chi_3 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_4 \nabla_\alpha u^\alpha, \quad Q_\mu = \lambda \left(\frac{c_s^2 \Delta_\mu^\nu \nabla_\nu \varepsilon}{\varepsilon + P} + u^\alpha \nabla_\alpha u_\mu \right)$$

Energy density correction

Pressure correction

Heat flow

Equations of motion: $\nabla_\mu T^{\mu\nu} = 0$

Causality and well-posedness are valid in the full nonlinear regime, also including Einstein's equations, when

$$\lambda, \chi_1 > 0, \eta \geq 0$$

Rigorous theorems in Bemfica, Disconzi, JN, PRD (2019)

$$\lambda \geq \eta,$$

Arbitrary EOS

$$3\chi_4 \geq 4\eta,$$

$$\lambda\chi_1 + c_s^2\lambda \left(\chi_4 - \frac{4\eta}{3} \right) \geq c_s^2\lambda\chi_2 + \lambda\chi_3 + \chi_2\chi_3 - \chi_1 \left(\chi_4 - \frac{4}{3}\eta \right) > 0.$$

Linear stability also holds: Bemfica, Disconzi, JN, PRD (2019).
See also P. Kovtun, JHEP (2019).

- Heat flow coefficient nonzero if shear is nonzero.
- Only 6 transport coefficients (Israel-Stewart > 10).
- No additional fields besides $\{\varepsilon, u_\mu\}$.
- Ready for use in heavy ion collisions.

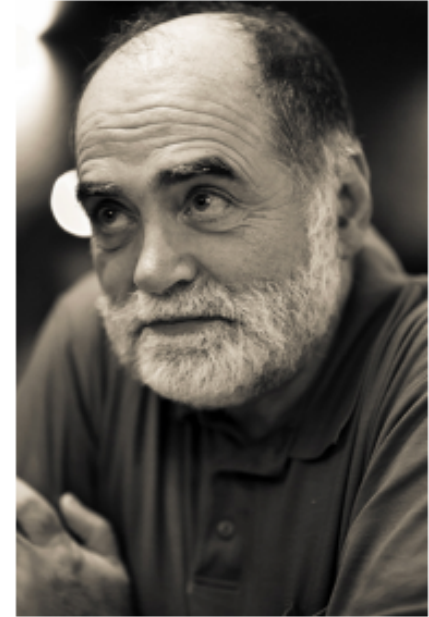
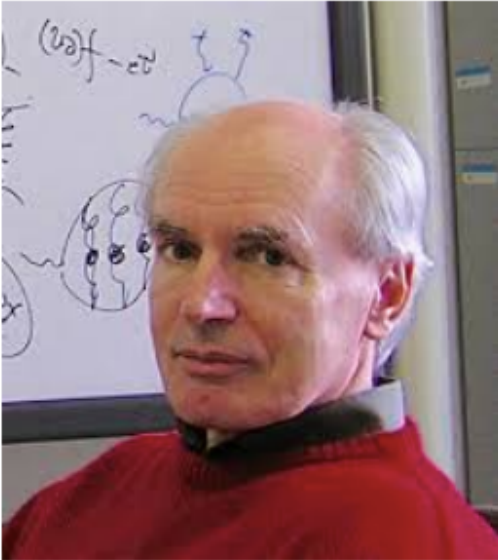
Conclusions

- Neutron star mergers can also be used to obtain essential info about viscous properties of ultradense matter.
- New results allow the investigation of dissipative phenomena in general relativistic fluids for the first time.
- Nonlinear causality may impose severe new constraints on the hydro description of small heavy ion collision systems.
- New approach to relativistic viscous hydrodynamics fixes decades-long problems.

Next steps

- Extend proof in Israel-Stewart theory to include shear viscous effects (to appear soon).
- Implement new viscous hydro equations in a code and investigate phenomenological consequences in heavy ions.
- Stochastic formulation of new viscous hydro equations (applications in heavy ions and cosmology).
- Inclusion of critical phenomena.

Many congratulations to



for their highly successful careers!

ADDITIONAL SLIDES

Bemfica, Disconzi, JN, PRD (2019)

Theorem I. Consider the energy-momentum tensor (1) and assume that λ , χ_a , $a = 1, \dots, 4$, η , and P are given real valued functions with domain $(0, \infty)$, where we recall that in (1) these quantities are functions of ε , i.e., $\lambda = \lambda(\varepsilon)$, $\chi_a = \chi_a(\varepsilon)$, $\eta = \eta(\varepsilon)$, and $P = P(\varepsilon)$. Suppose that λ , χ_a , η , and P are $G^{(s)}$ regular. Let $\mathcal{I} = (\Sigma, \hat{\varepsilon}, \hat{\mathcal{E}}, \hat{u}, \hat{U})$ be an initial data set for Einstein's equations coupled to (1). Assume that the initial data belongs to $G^{(s)}(\Sigma)$. Suppose that Σ is compact and that $\hat{\varepsilon} > 0$. Suppose that $P' \geq 0$, that $\lambda > 0$, $\chi_1 > 0$, $\eta \geq 0$, and that conditions (6) and (7) hold. Finally, assume that $1 < s < 20/19$. Then, there exist a four-dimensional Lorentzian manifold (M, g) , a vector field u and a real valued function ε , both defined on M , such that:

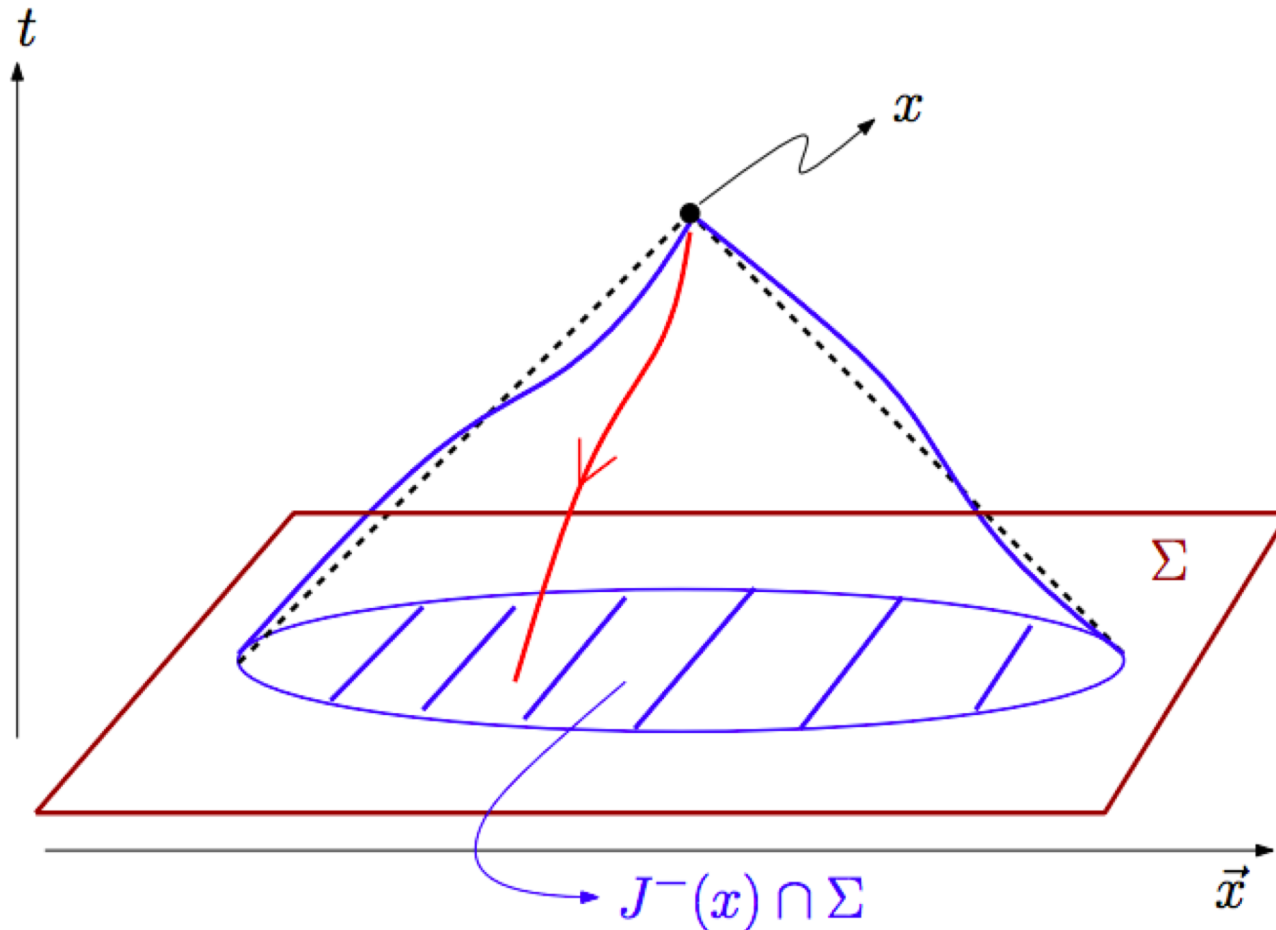
- (1) Einstein's equations coupled to (1) hold in M .
- (2) There exists an isometric embedding $i : (\Sigma, \hat{g}) \rightarrow (M, g)$ with second fundamental form κ .
- (3) Identifying Σ with its image $i(\Sigma)$ in M , we have $\varepsilon|_{\Sigma} = \hat{\varepsilon}$ and $\Pi_{\Sigma}(u) = \hat{u}$, where $\Pi_{\Sigma} : \mathcal{I}|_{\Sigma} \rightarrow \Sigma$ is the projection map. If $\{x^i\}_{i=1}^3$ are coordinates on Σ , then $\partial_0 \varepsilon|_{\Sigma} = \hat{\mathcal{E}}$ and $\partial_0 u^i|_{\Sigma} = \hat{U}^i$.
- (4) (M, g) is globally hyperbolic with Cauchy surface $i(\Sigma)$.
- (5) (M, g) is causal, in the following sense: for any x in the future⁵ of $i(M)$, $(g(x), u(x), \varepsilon(x))$ depends only on $\mathcal{I}|_{i(\Sigma) \cap J^-(x)}$, where $J^-(x)$ is the causal past of x (with respect to the metric g).
- (6) (M, g) is unique up to actions of diffeomorphisms of M .

Theorem II. The same conclusions of Theorem I hold if one assumes that the initial data belongs to the Sobolev space H^s for sufficiently large s .

See also Bemfica, Disconzi, Rodriguez, Shao, arXiv:1911.02504 [math.AP].

Why is this so hard to do?

Causality in the nonlinear regime in curved spacetime



The ubiquitousness of fluid dynamics

Based on conservations laws + large separation of length scales

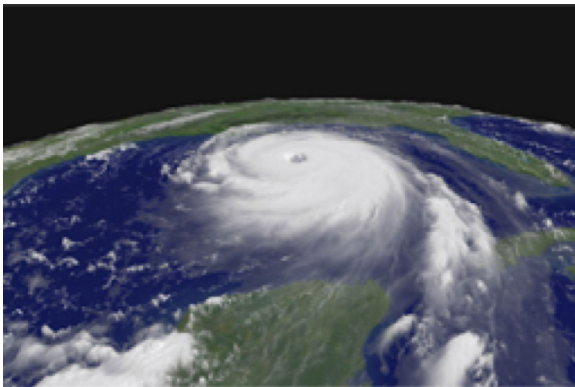
Separation of scales → macroscopic: L microscopic: ℓ

Knudsen number

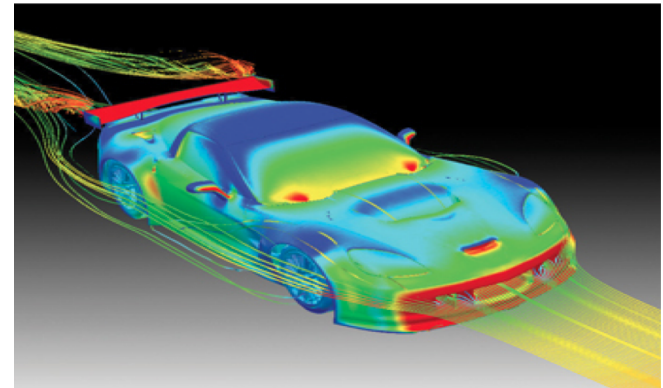
$$K_N \sim \frac{\ell}{L} \ll 1$$



FLUID



$$L \sim 10^5 \text{ m} \quad \ell \sim 1 \text{ m}$$



$$L \sim 1 \text{ m} \quad \ell \sim 10^{-7} \text{ m}$$

How can a pp system behave like a fluid?

Proton at high energies: average shape but with strong color fluctuations

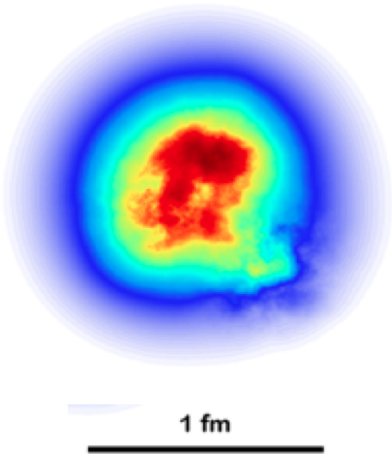


Fig. from Mantysaari, Schenke, PRL (2016)

A simple uncertainty principle estimate

$$\Delta x \sim 0.1 \text{ fm}$$

$$\Delta p \sim \frac{1}{\Delta x} \sim 2 \text{ GeV} > \langle p_T \rangle$$

Quantum correlations should be important !!

$$\langle \hat{T}^{\mu\nu} \rangle, \langle \hat{T}^{\mu\nu} \hat{T}^{\alpha\beta} \rangle, \dots$$

- **Opportunity:** Investigate quantum entanglement in a non-Abelian theory.
- Here we should really go beyond the “everything is hydro approach” ...

Ideal (Euler) relativistic fluid dynamics

Energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P(\epsilon, n) \Delta^{\mu\nu}$$

Conserved charge

$$J^\mu = n u^\mu$$

$$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

$$u_\mu u^\mu = -1$$

Assuming that speed of sound and $\epsilon + P > 0$

- System is **well-posed** (Sobolev space H^s with $s > 5/2$)
- - **Existence, uniqueness, continuous dependence on initial data**
- **Causal** (domain of dependence property in the sense of relativity)

Remember that a system of PDE's is well-posed (Hadamard) when:

1) A solution exists.

<http://www.math.ucla.edu/~tao/Dispersive/>

2) The solution is unique.

3) The solution depends continuously on initial data (e.g., initial conditions, boundary conditions).

Ex.: $u_{xx} + u = 0$

a) $u(0) = 0, u(\frac{\pi}{2}) = 1 \Rightarrow$ unique solution $u(x) = \sin(x)$

b) $u(0) = 0, u(\pi) = 1 \Rightarrow$ no solution

c) $u(0) = 0, u(\pi) = 0 \Rightarrow$ infinitely many solutions: $u(x) = A \sin(x)$

Ex.:

$\left\{ \begin{array}{ll} u_t = u_{xx} & \text{heat equation} \\ u(0, t) = u(1, t) = 0 & \text{boundary conditions} \\ u(x, 0) = u_0(x) & \text{initial conditions} \end{array} \right\}$ well-posed

$\left\{ \begin{array}{ll} u_t = -u_{xx} & \text{backwards heat equation} \\ u(0, t) = u(1, t) & \text{boundary conditions} \\ u(x, 0) = u_0(x) & \text{initial conditions} \end{array} \right\}$ no continuous dependence on initial data

Mathematical definition of causality (relativity)

See, e.g., Choquet-Bruhat, Wald

Consider a system of (linear or nonlinear) PDE's

$$P_K^I \varphi^K = 0.$$

N unknowns

$$\{\phi^K\}_{K=1}^N$$

The system is **causal** if for any point x in the future of Σ

$\varphi^K(x)$ depends only on $J^-(x) \cap \Sigma$

causal past

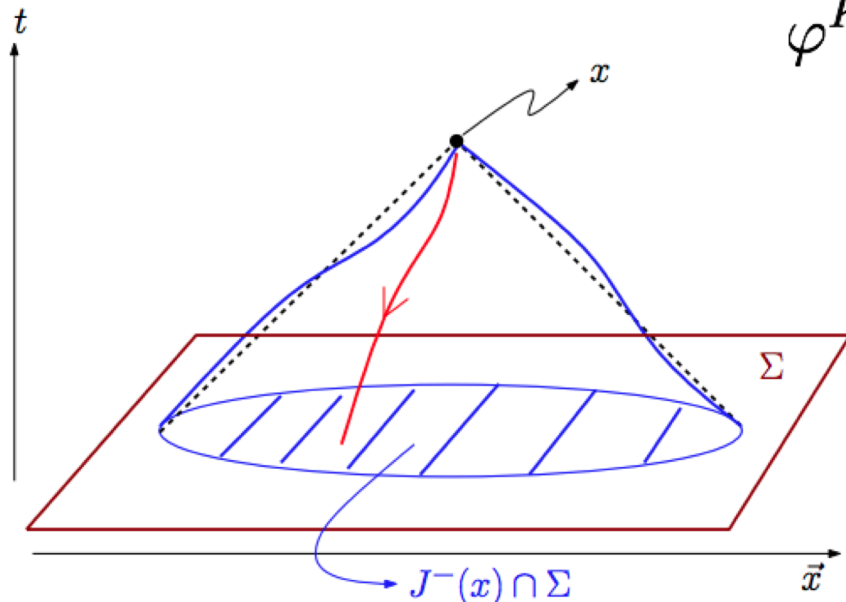


FIG. 1: (color online) Illustration of causality. In curved spacetime $J^-(x)$ looks like a distorted light-cone opening to the past (blue region); in flat spacetime the cone would be straight (dotted line). Points inside $J^-(x)$ can be joined to a point x in spacetime by a causal past directed curve (e.g. the red line). The Cauchy surface Σ supports the initial data and the value of the field $\varphi(x)$ depends only on the initial data on $J^-(x) \cap \Sigma$.

Fluid dynamics in the relativistic regime

- Einstein + Euler equations: Locally well-posed and causal

Choquet-Bruhat 1958, 1966
Lichnerowicz, 1967

- Einstein + Euler equations: Not globally well-posed (shocks occur)

Christodoulou, 2007

What about **dissipative** fluids? $T^{\mu\nu} = T_{ideal}^{\mu\nu} + \pi^{\mu\nu}$

- (a) Einstein+viscous fluid admit **existence + uniqueness** of solutions?
- (b) **Causality**?
- (c) **Stability** (at least in the linear regime)?
- (d) Does the solution really describe the **physics** of the system?

Causality does not imply (linear) stability

$$T^{\alpha\beta} = T_{ideal}^{\alpha\beta} + \left(\lambda + \frac{\nu}{4}\right) g^{\alpha\beta} (\nabla_\lambda C^\lambda) - \frac{\nu}{2} (\nabla^\alpha C^\beta + \nabla^\beta C^\alpha)$$

Dynamic velocity $C^\alpha = (\epsilon + P)u^\alpha$

Choquet-Bruhat, 2006

$$\nabla_\mu T^{\mu\nu} = 0 \quad \rightarrow \text{Principal part: } \frac{\nu}{2} \{\nabla_\alpha \nabla^\alpha C^\beta + \nabla_\alpha \nabla^\beta C^\alpha\} + \left(\lambda - \frac{1}{3}\nu\right) \nabla^\beta \nabla_\alpha C^\alpha.$$

Characteristic matrix: $\frac{\nu}{2} X_\alpha X^\alpha C^\beta + a X_\alpha X^\beta C^\alpha \quad a := \frac{\nu}{6} + \lambda.$

Characteristic determinant: $\frac{\nu}{2} \left(\frac{\nu}{2} + a\right) (X^\alpha X_\alpha)^4 \quad \longrightarrow \quad \mathbf{CAUSAL}$

However, this system is linearly unstable around global equilibrium

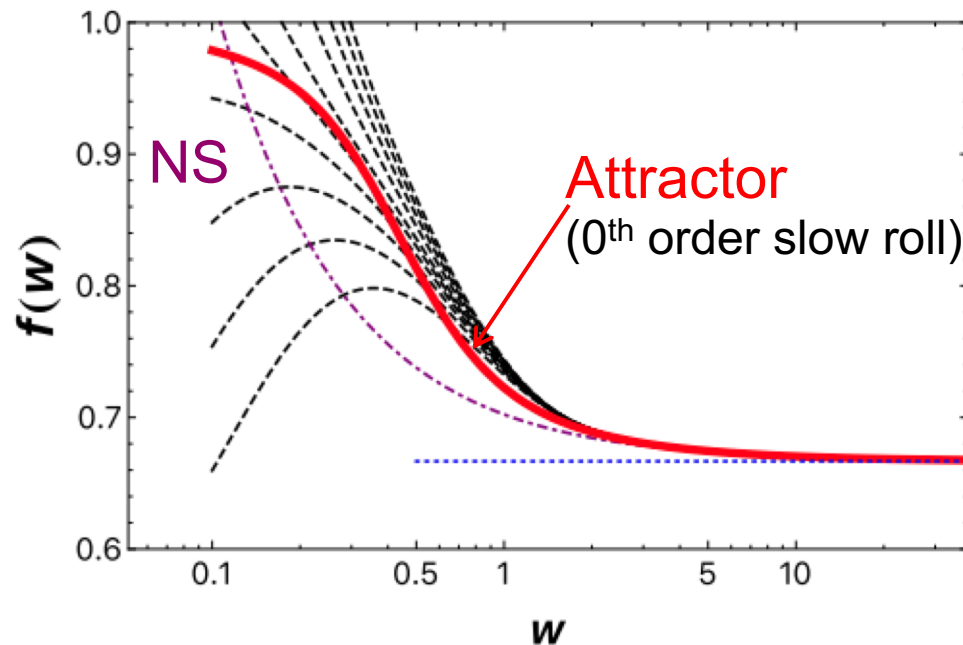
Applications: Bjorken flow

Milne coordinates $u_\mu = (-1, 0, 0, 0)$ $T \rightarrow T(\tau)$

Equation of motion: $w = \tau T$ and $f = 1 + \tau \partial_\tau T / T$

Heller-Spalinski form

$$\bar{\chi} w f(w) \frac{df(w)}{dw} + 3\bar{\chi} f(w)^2 + f(w) \left(w - \frac{14}{3} \bar{\chi} \right) + \frac{16\bar{\chi}}{9} - \frac{4\bar{\eta}}{9} - \frac{2w}{3} = 0$$



$$\eta/s = 0.08$$

$$\chi = 4\eta$$

Applications: Gubser flow

$dS_3 \otimes \mathbb{R}$ spacetime $u_\mu = (-1, 0, 0, 0)$ $T \rightarrow \hat{T}(\rho)$

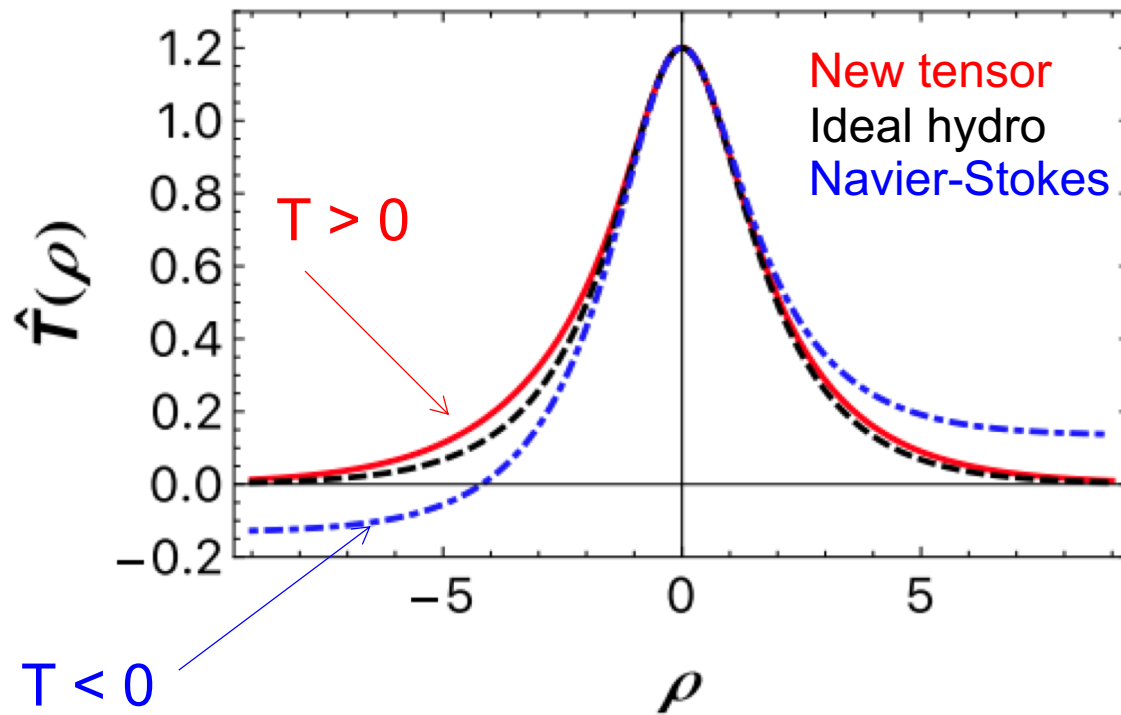
Equation of motion (written in 1st order form):

$$\frac{1}{\hat{T}} \frac{d\hat{T}}{d\rho} + \frac{2}{3} \tanh \rho = \hat{\mathcal{F}}(\rho) \quad \bar{\chi} \frac{d\hat{\mathcal{F}}}{d\rho} + 3\bar{\chi}\hat{\mathcal{F}}^2 + \frac{2}{3}\bar{\chi}\hat{\mathcal{F}} \tanh \rho + \hat{T}\hat{\mathcal{F}} - \frac{4}{9}\bar{\eta}(\tanh \rho)^2 = 0$$

$$\eta/s = 0.2$$

$$\chi = 4\eta$$

$$\hat{T}_0 = 1.2$$



Definition 8.6 (Strongly causal spacetime)

A spacetime \mathcal{M} is strongly causal if given an arbitrarily chosen event $\mathfrak{p} \in \mathcal{M}$ for each $U \subset \mathcal{M}$ open neighborhood of \mathfrak{p} there exist another open neighborhood of \mathfrak{p} , $V \subset U$, such that no casual curve intersects it more than once.

Definition 8.7 (Inextendible causal curve)

A causal curve γ_C is called future (resp. past) inextendible if it is impossible to find an event $\mathfrak{p} \in \mathcal{M}$ such that for all $U \subset \mathcal{M}$, U neighborhood of \mathfrak{p} , there exist a t' such that $\gamma_C(t) \in U$ for all $t > t'$ (resp $t < t'$)

In more concrete words, this means that γ_C has no *future* (resp. *past*) *endpoint*.

Definition 8.10 (Domains of dependence)

Let \mathcal{A} be a closed achronal set. The set $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$) of all spacetime events \mathfrak{p} such that every past (resp. future) inextendible causal curve passing through \mathfrak{p} intersects \mathcal{A} is called the future (resp. past) domain of dependence of \mathcal{A} . The set $D(\mathcal{A}) = D^+(\mathcal{A}) \cup D^-(\mathcal{A})$, union of the past and of the future domains of dependence is the domain of dependence of \mathcal{A} .

Definition 8.11 (Cauchy surface and global hyperbolicity)

Let $\mathcal{A} \subset \mathcal{M}$ be an achronal set such that $D(\mathcal{A}) = \mathcal{M}$. Then \mathcal{A} is called a Cauchy surface (we instead use the denomination partial Cauchy surface for a closed achronal set without edge). A spacetime \mathcal{M} which admits a Cauchy surface is called globally hyperbolic.

Analytic functions obey ($\alpha = \text{multi-index}$):

$$|\partial^\alpha f| \leq C^{|\alpha|+1} \alpha!$$

The **Gevrey** class $\gamma^{(\sigma)}$, $\sigma > 1$, consists of C^∞ functions that obey the **weaker** inequality:

$$|\partial^\alpha f| \leq C^{|\alpha|+1} (\alpha!)^\sigma.$$

Advantage: large class of functions, including compactly supported (not determined by values on an open set).

The larger the σ , the larger the space. Larger σ : more general results.

$\gamma^{(\infty)}$ = Sobolev space.

$\gamma^{(\sigma)}$: used in the study of non-relativistic viscous fluids; also have had applications in General Relativity (magneto-hydrodynamics).

Sobolev H_s

M. Disconzi

$$\|f\|_s^2 = \sum_{|j| \leq s} \|\partial^j u\|_{L^2}^2$$

Characteristics.

Consider the linear differential operator:

$$Lu = a^{\mu\nu}(x) \frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b(x, \partial u)$$

or, more generally (α =multi-index),

$$Lu = \sum_{|\alpha|=m} a^\alpha(x) \partial_\alpha^{|\alpha|} u + b(x, \partial^{m-1} u, \dots, \partial u, u).$$

We define the **characteristic cone** V_x of L at T_x^*M by

$$h(x, \xi) \equiv \sum_{|\alpha|=m} a^\alpha(x) \xi_\alpha = 0.$$

$h(x, \xi)$ (= **characteristic polynomial**) is a homogeneous polynomial of degree m .

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Hyperbolic polynomials (Leray).

$h(x, \xi)$ is called a **hyperbolic polynomial** (at x) if there exists $\zeta \in T_x^*M$ such that every line through ζ that does not pass through the origin intersects V_x at m real distinct points ($m = \text{degree of } h = \text{order of } L$).

In this case, the set of $\zeta \in T_x^*M$ with this property forms the interior of two opposite **convex half-cones** Γ_x^\pm .

The differential operator L is called **hyperbolic** (at x) if $h(x, \xi)$ is hyperbolic.

Dualizing, one obtains $C_x^\pm \subset T_xM$. For example

$$C_x^+ = \{v \in T_xM \mid \zeta(v) \geq 0 \text{ for all } \zeta \in \Gamma_x^+\}.$$

$\Sigma^n = \{\varphi(x) = 0\} \subset M^{n+1}$ is **characteristic for L** if

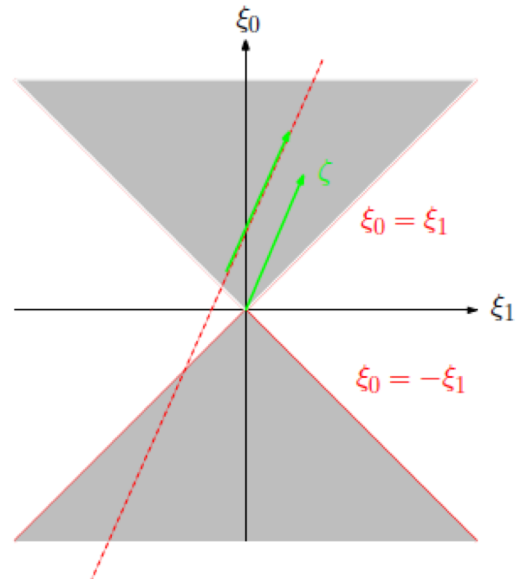
$$\sum_{|\alpha|=m} a^\alpha(x) \partial_\alpha \varphi = 0.$$

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Wave equation: characteristics.

Consider $Lu = u_{tt} - u_{xx}$, $\xi = (\xi_0, \xi_1)$. Then:

$$\xi_0^2 - \xi_1^2 = 0 \Rightarrow \xi_0 = \pm \xi_1.$$



Γ_x and C_x are both given by the “light-cone”.

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Hyperbolic and weakly hyperbolic operators.

Hyperbolic operators (sometimes called strictly hyperbolic) have a **Cauchy problem** that is well-posed in Sobolev spaces.

When the definition of a hyperbolic polynomial is weakened to:

there exists $\zeta \in T_x^*M$ such that every line through ζ that does not pass through the origin intersects V_x at m , **not necessarily distinct**, real points, we obtain **weakly** hyperbolic polynomials and operators ($m = \text{degree of } h = \text{order of } L$).

Weakly hyperbolic operators are well-posed in Gevrey spaces, but there are counter-examples to well-posed in Sobolev spaces.

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