Beyond second order relativistic fluid dynamics

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Based on recent work with my postdoc Dr Mohammed Younus
in *Third order viscous hydrodynamics in Bjorken scenario*,
https://arxiv.org/abs/1910.11735

Nelson Mandela University, Summerstrand Campus, Port Elizabeth

Personal connections

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- Alice persuaded me to apply for my PhD studies in Minnesota.
- Then I met the rest of you and a whole lot of other people

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A celebration of life and science: Jean-Paul, Miklos, and Larry

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References

- M. Younus & A. Muronga *Third order viscous hydrodynamics in Bjorken scenario*, https://arxiv.org/abs/1910.11735
- A. Muronga, *New developments in relativistic dissipative fluid dynamics* , Journal of Physics G: Nuclear and Particle Physics, Volume 37, Number 9 (2010)
- A. El, Z. Xu, C. Greiner *Third-order relativistic dissipative hydrodynamics* Phys. Rev. C81 (2010) 041901
- Amaresh Jaiswal, *Relativistic third-order dissipative fluid dynamics from kinetic theory*. Phys. Rev. C88 : 021903, 2013
- A. Muronga, *Relaxation and Coupling Coefficients in Third Order Relativistic Fluid Dynamics* , Acta Phys. Polon. Supp. 7 (2014) 197

Motivation and Conclusions

- Relativistic *second order* dissipative fluid dynamics is a very important scientific achievement of the last two decades, and has inspired many authors to apply its methodology to the study of heavy ion collisions and astrophysics. In short it furnishes equations which are closed by imposing the entropy principle up to second order, with respect to equilibrium.
- We have so far refrained from exploiting subsequent orders because that requires long and cumbersome calculations
- However, the exploitation of subsequent orders with respect to equilibrium is desirable, for the following selected reasons
  (i) a second order approach is necessary to link more closely the relativistic case with the classical one
  (ii) the higher order terms depends also on the lower order terms; so it might happen that the existing condition of the solution imposes further conditions on the lower order terms and it may also happen that the further conditions affect the equilibrium expressions which, however, are already known.
  (iii) the couplings between the three primary processes in dissipative fluids are only realized by going up to third order

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- Working towards an exact solution and comparison with microscopic approach
- Theoretical understanding of macroscopic and miscroscopic approaches for new generation facilities. We will also know why hydrodynamics approach works for HIC description

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The objective of relativistic dissipative fluid dynamics for one component fluid is the determination of the 14 fields of

$N^{\mu}(x^{\beta})$	net charge density — net charge flux vector
$T^{\mu\nu}(x^{\beta})$	stress — energy — momentum tensor

$T^{\mu\nu}$ is assumed symmetric so that it has 10 independent components.

The 14 fields are determined from the field equations (fluid dynamical equations)

$\partial_{\mu} N^{\mu} = 0$	net charge (e.g., baryon, strangeness, etc) conservation
$\partial_{\nu} T^{\mu\nu} = 0$	energy – momentum conservation
$\partial_{\lambda} F^{\mu\nu\lambda} = P^{\mu\nu}$	balance law of fluxes

$F^{\mu\nu\lambda}$ is completely symmetric tensor of fluxes and $P^{\mu\nu}$ is its production density such that

$$F^{\mu\nu}_{\ \ \nu} = m^2 N^{\mu} \qquad \text{and} \qquad P^{\nu}_{\nu} = 0$$

We then have a set of 14 independent equations (net charge conservation (1); energy-momentum conservation (4); balance of fluxes (9))

However, the dynamic equations cannot serve as the field equations for the thermodynamic fields N^μ and $T^{\mu\nu}$. Because the additional fields $F^{\mu\nu\lambda}$ and $P^{\mu\nu}$ have appeared.

Restriction on the general form of the constitutive functions $F^{\mu\nu\lambda}(N^\alpha, T^{\alpha\beta})$ and $P^{\mu\nu}(N^\alpha, T^{\alpha\beta})$ is imposed by

- entropy principle —the entropy density–entropy flux vector $S^\mu(N^\alpha, T^{\alpha\beta})$ is a constitutive quantity which obeys the inequality

$$\partial_\mu S^\mu \geq 0 \qquad \text{for all thermodynamic process}$$

$$\text{i.e., instead of } \frac{\eta}{s} \quad \text{use} \quad \frac{\eta + \zeta + \kappa T}{s}$$

- requirement of hyperbolicity — ensures that Cauchy problems of our field equations are well-posed and all wave speeds are finite $\Longrightarrow$ our set of field equations should be symmetric hyperbolic

The 14 Fields and Tensor decomposition

Net charge 4-current $N^\mu = n u^\mu$

$$n \equiv \sqrt{N^\mu N_\mu} = u_\mu N^\mu$$ net charge density in fluid rest frame,

$$u^\mu \equiv \frac{N^\mu}{\sqrt{N^\nu N_\nu}}$$ the fluid 4-velocity,

$u^\nu u_\nu = 1 \implies u^\mu$ has 3 independent components

Stress–energy–momentum tensor $T^{\mu\nu} = \varepsilon u^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + 2q^{(\mu}u^{\nu)} + \pi^{\langle\mu\nu\rangle}$

$\varepsilon \equiv u_\mu u_\nu T^{\mu\nu}$ energy density in fluid rest frame,

$p \equiv p(\varepsilon\,, n)$ pressure in fluid rest frame,

Π bulk viscous pressure, $(p+\Pi) \equiv -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$

$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ projection tensor onto 3-space, $\Delta^{\mu\nu}u_\nu = \Delta^{\mu\nu}u_\mu = 0$

$g^{\mu\nu} \equiv \mathrm{diag}(+1, -1, -1, -1)$ metric tensor

$q^\mu \equiv \Delta^\mu_\alpha u_\beta T^{\alpha\beta}$ heat flux 4-current,

$q^\mu u_\mu = 0 \implies q^\mu$ has 3 independent components

$\pi^{\langle\mu\nu\rangle} \equiv T^{\langle\mu\nu\rangle}$ shear stress tensor

$\pi^{\langle\mu\nu\rangle}u_\mu = \pi^{\langle\mu\nu\rangle}u_\nu = 0, \quad \pi^{\langle\nu}_{\nu\rangle} = 0 \implies \pi^{\langle\mu\nu\rangle}$ has 5 independent components

Production densities tensor $P^{\mu\nu} = \mathcal{P}_\Pi \Pi \left(\Delta^{\mu\nu} - 3u^\mu u^\nu\right) + 2\mathcal{P}_q q^{(\mu} u^{\nu)} + \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}$

The functions $\mathcal{P}_\Pi$, $\mathcal{P}_q$, $\mathcal{P}_\pi$ are related to the bulk viscosity, heat conductivity and shear viscosity and thus may be determined from measurements of these coefficients

Tensor of fluxes (up to 2nd order)

$$
\begin{aligned}
F^{\mu\nu\lambda} = {} & \frac{1}{2}\mathcal{F}_1^0 g^{(\mu\nu}u^{\lambda)} + \frac{1}{2}\mathcal{F}_2^0\left(g^{(\mu\nu}u^{\lambda)} - 2u^\mu u^\nu u^\lambda\right) \\
& + \mathcal{F}_1^1 \Pi\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_2^1\left(\Delta^{(\mu\nu}q^{\lambda)} - 5u^{(\mu}u^\nu q^{\lambda)}\right) \\
& + \mathcal{F}_3^1 \pi^{(\langle\mu\nu\rangle}u^{\lambda)} \\
& + \mathcal{F}_1^2 \Pi^2\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_2^2\left(-q^\nu q_\nu \Delta^{(\mu\nu}u^{\lambda)} - 3u^{(\mu}q^\nu q^{\lambda)}\right) \\
& - \mathcal{F}_3^2 q^\alpha q_\alpha\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_4^2\left(3u^\mu \pi^{2\langle\nu\lambda\rangle} - \pi^{2\langle\alpha\alpha\rangle}u^\mu u^\nu u^\lambda\right) \\
& + \mathcal{F}_5^2 \pi^{2\langle\alpha\alpha\rangle}\left(\Delta^{(\mu\nu}u^{\lambda)} - u^\mu u^\nu u^\lambda\right) + \mathcal{F}_6^2\left(q^{(\mu}\pi^{\langle\nu\lambda\rangle)} - 2u^{(\mu}u^\nu \pi^{\langle\lambda)\nu\rangle}q_\nu\right) \\
& + \mathcal{F}_7^2\left(\Delta^{(\mu\nu}\pi^{\langle\lambda)\alpha\rangle)}q_\alpha - 5u^\mu u^\nu \pi^{\langle\lambda)\alpha\rangle}q_\alpha\right) + \mathcal{F}_8^2 \Pi u^{(\mu}\pi^{\langle\nu\lambda\rangle)} \\
& + \mathcal{F}_9^2 \Pi\left(\Delta^{(\mu\nu}q^{\lambda)} - 5q^{(\mu}u^\nu u^{\lambda)}\right)
\end{aligned}
$$

Zeroth order (Equilibrium) + First order + Second order

14 Fields and Tensor decomposition

Entropy 4-current (up to 3rd order)

$$
\begin{aligned}
S^{\mu} = \; & \mathcal{S}_1^0 u^{\mu} \\
& + \mathcal{S}_1^1 \Pi u^{\mu} + \mathcal{S}_2^1 q^{\mu} \\
& + \left(\mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^{\alpha} q_{\alpha} + \mathcal{S}_3^2 \pi^{2\langle\alpha\alpha\rangle} \right) u^{\mu} \\
& + \mathcal{S}_4^2 \Pi q^{\mu} + \mathcal{S}_5^2 \pi^{\langle\mu\alpha\rangle} q_{\alpha} \\
& + \left(\mathcal{S}_1^3 \Pi^3 - \mathcal{S}_2^3 \Pi q_{\alpha} q^{\alpha} + \mathcal{S}_3^3 \Pi \pi^{2\langle\alpha\alpha\rangle} + \mathcal{S}_4^3 q_{\alpha} q_{\beta} \pi^{\langle\alpha\beta\rangle} + \mathcal{S}_5^3 \pi^{3\langle\alpha\alpha\rangle} \right) u^{\mu} \\
& + \left(\mathcal{S}_6^3 \Pi^2 - \mathcal{S}_7^3 q_{\alpha} q^{\alpha} + \mathcal{S}_8^3 \pi^{2\langle\alpha\alpha\rangle} \right) q^{\mu} + \mathcal{S}_9^3 \Pi \pi^{\langle\mu\alpha\rangle} q_{\alpha} + \mathcal{S}_{10}^3 \pi^{2\langle\mu\alpha\rangle} q_{\alpha}
\end{aligned}
$$

Zeroth order (Equilibrium) + First order + Second order +Third order

Equilibrium

Equilibrium is defined as a process in which production densities vanish and/or the entropy production vanishes

$$\left.\begin{array}{rcl} P^{\mu\nu}_{Eq.} & = & 0 \\ \Xi_{Eq.} & = & 0 \end{array}\right\} \Longrightarrow \Pi_{Eq.} = 0\,, \qquad q^{\mu}_{Eq} = 0\,, \quad \pi^{\langle\mu\nu\rangle}_{Eq} = 0$$

$$\begin{array}{rcl} F^{\mu\nu\lambda}_{Eq} & = & \frac{1}{2}\mathcal{F}^{0}_{1} g^{(\mu\nu}u^{\lambda)} + \frac{1}{2}\mathcal{F}^{0}_{2}\left(g^{(\mu\nu}u^{\lambda)} - 2u^{\mu}u^{\nu}u^{\lambda}\right) \\ S^{\mu}_{Eq.} & = & s(\varepsilon, n)u^{\mu} \end{array}$$

The energy-momentum tensor reduces to

$$T^{\mu\nu}_{Eq.} = \varepsilon u^{\mu}u^{\nu} - p\Delta^{\mu\nu}$$

In the ideal “perfect” fluid limit one has 5 independent fields ($p(n, e)$(2), u^{μ}(3)) and 5 field equations

14-Fields Theory of Relativistic Dissipative Fluid Dynamics :

In dissipative(non-ideal) fluid dynamics one needs 9 additional equations for the dissipative fluxes. The 14 fields $p(n, \varepsilon)$, Π, u^α, q^α, $\pi^{\langle\alpha\beta\rangle}$ are governed by the following fields equations

$$
\begin{aligned}
\partial_\mu N^\mu &= 0 \\
\Delta_{\alpha\mu}\partial_\nu T^{\mu\nu} &= 0 \\
u_\mu \partial_\nu T^{\mu\nu} &= 0 \\
u_\mu u_\nu \partial_\lambda F^{\mu\nu\lambda} &= -\mathcal{P}_\Pi \Pi \\
\Delta^\mu_\alpha u_\nu \partial_\lambda F^{\alpha\nu\lambda} &= \mathcal{P}_q q^\mu \\
\left(\Delta^{(\mu}_\alpha \Delta^{\nu)}_\beta - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)\partial_\lambda F^{\alpha\beta\lambda} &= \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}
\end{aligned}
$$

For all thermodynamic processes the entropy principle holds

$$\partial_\mu S^\mu \geq 0$$

Dissipative fluxes: Zeroth order: Equilibrium

$$\begin{aligned} \Pi &= \Pi_{Eq.} = 0 \\ q^{\alpha} &= q^{\alpha}_{Eq.} = 0 \\ \pi^{\langle\alpha\beta\rangle} &= \pi^{\langle\alpha\beta\rangle}_{Eq.} = 0 \end{aligned}$$

$$
\begin{aligned}
\Pi^{(1)} &= \Pi_E = -\zeta \nabla_\alpha u^\alpha \\
q^{\alpha\,(1)} &= q^\alpha_E = \kappa T \Delta^{\alpha\mu}\left(\frac{\nabla_\alpha T}{T} - \dot{u}_\alpha\right) \\
\pi^{\langle\alpha\beta\rangle\,(1)} &= \pi^{\langle\alpha\beta\rangle}_E = 2\eta \Delta^{\alpha\mu}\Delta^{\beta\nu}\nabla_{\langle\alpha} u_{\beta\rangle}
\end{aligned}
$$

Relativistic versions of the laws of Navier-Stokes and Fourier first derived by Eckart, Landau-Lifshitz.

ζ is the bulk viscosity, κ is the thermal conductivity, η is the shear viscosity

- simple algebraic expressions of dissipative fluxes
- may lead to acausal and unstable equations of motion

Dissipative fluxes: Second order

Müller-Israel-Stewart (MIS) equations: $F^{\mu\nu\lambda}$ linear (first-order) in dissipative fluxes and S^{μ} quadratic (second-order) in dissipative fluxes
Resulting equations causal and hyperbolic

$$
\begin{aligned}
\Pi^{(2)} &= \Pi_{MIS} = -\zeta\left[2\mathcal{S}_1^2\dot{\Pi} + \mathcal{S}_4^2\nabla_\alpha q^\alpha\right] \\
&\quad -\zeta\left[\Pi(\dot{\mathcal{S}}_1^2 + \mathcal{S}_1^2\nabla_\alpha u^\alpha) + q^\alpha(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2\dot{u}_\alpha)\right] \\
q^{\mu\,(2)} &= q^\mu_{MIS} = \kappa T\Delta^{\alpha\mu}\left[2\mathcal{S}_2^2\dot{q}_\alpha + \mathcal{S}_4^2\nabla_\alpha\Pi + \mathcal{S}_5^2\nabla^\beta\pi_{\langle\alpha\beta\rangle}\right] \\
&\quad +\kappa T\Delta^{\alpha\mu}\left[q_\alpha(\dot{\mathcal{S}}_2^2 + \mathcal{S}_2^2\nabla_\nu u^\nu) + \Pi(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2\dot{u}_\alpha)\right. \\
&\quad \left.+\pi_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_5^2 - \mathcal{S}_5^2\dot{u}^\beta)\right] \\
\pi^{\langle\mu\nu\rangle\,(2)} &= \pi^{\langle\mu\nu\rangle}_{MIS} = 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[2\mathcal{S}_3^2\dot{\pi}_{\langle\alpha\beta\rangle} + \mathcal{S}_5^2\nabla_{\langle\alpha}q_{\beta\rangle}\right] \\
&\quad +2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^2 + \mathcal{S}_3^2\nabla_\lambda u^\lambda)\right. \\
&\quad \left.+q_{\langle\alpha}(\nabla_{\beta\rangle}\mathcal{S}_5^2 - \mathcal{S}_5^2\dot{u}_{\beta\rangle})\right]
\end{aligned}
$$

Dissipative fluxes

- The terms in red are neglected in the original MIS formulation. Terms of the general form $\Pi\partial_\nu u^\mu$, $\Pi\partial_\lambda n$, $\Pi\partial_\lambda\varepsilon$, $q_\alpha\partial_\nu u^\mu$, $q_\alpha\partial_\lambda n$, $q_\alpha\partial_\lambda\varepsilon$, $\pi_{\langle\alpha\beta\rangle}\partial_\nu u^\mu$, $\pi_{\langle\alpha\beta\rangle}\partial_\lambda n$, $\pi_{\langle\alpha\beta\rangle}\partial_\lambda\varepsilon$ have been considered non-linear and thus ignored. These terms have been shown to be important in heavy ion collisions. They will be even more important at low energies and high densities.
- Derivations of the equations from kinetic theory reveals terms that are not explicit from phenomenological considerations (e.g., vorticity terms)

Dissipative fluxes: Third order: Bulk equation

$$
\begin{aligned}
\Pi^{(3)} = & -\zeta\Big[3\mathcal{S}_1^3\dot{\Pi} + 2\mathcal{S}_2^3\dot{q}_\lambda q^\lambda + 2\mathcal{S}_3^3\dot{\pi}_{\langle\alpha\beta\rangle}\pi^{\langle\alpha\beta\rangle} \\
& +\mathcal{S}_6^3(\Pi\nabla_\alpha q^\alpha + q^\alpha\nabla_\alpha\Pi) + \mathcal{S}_9^3(\pi^{\langle\alpha\beta\rangle}\nabla_\alpha q_\beta + q_\beta\nabla_\alpha\pi^{\langle\alpha\beta\rangle})\Big] \\
& -\zeta\Big[\Pi^2(\dot{\mathcal{S}}_1^3 + \mathcal{S}_1^3\nabla_\alpha u^\alpha) - q^\alpha q_\alpha(\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3\nabla_\alpha u^\alpha) \\
& +\pi^{2\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\alpha u^\alpha) \\
& +\Pi q^\alpha(\nabla_\alpha\mathcal{S}_6^3 - \mathcal{S}_6^3 a_\alpha) + \pi^{\langle\alpha\beta\rangle}q_\beta(\nabla_\alpha\mathcal{S}_9^3 - \mathcal{S}_9^3 a_\alpha)\Big]
\end{aligned}
$$

Dissipative fluxes: Third order: Heat equation

$$
\begin{aligned}
q^{\mu\,(3)} \quad = \quad & \kappa T \Delta^{\alpha\mu}\Big[-\mathcal{S}_2^3(2\Pi\dot{q}_\alpha + q_\alpha\dot{\Pi}) + \mathcal{S}_4^3(2\dot{q}^\beta\pi_{\langle\alpha\beta\rangle} + q^\beta\dot{\pi}_{\langle\alpha\beta\rangle}) \\
& +2\mathcal{S}_6^3\Pi\nabla_\alpha\Pi - 2\mathcal{S}_7^3 q^\beta\nabla_\alpha q_\beta + \mathcal{S}_9^3(\Pi\nabla^\beta\pi_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle}\nabla^\beta\Pi) \\
& +2\mathcal{S}_{10}^3\pi_{\langle\beta\nu\rangle}\nabla_\alpha\pi^{\langle\beta\nu\rangle}\Big] \\
& +\kappa T\Delta^{\alpha\mu}\Big[\Pi q_\alpha(\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3\nabla_\nu u^\nu) + q^\beta\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3\nabla_\nu u^\nu) \\
& +(\Pi^2\nabla_\alpha\mathcal{S}_6^3 - q^\lambda q_\lambda\nabla_\alpha\mathcal{S}_7^3 + \pi^{2\langle\lambda\lambda\rangle}\nabla_\alpha\mathcal{S}_8^3) \\
& +\Pi\pi_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_9^3 - \mathcal{S}_9^3 a^\beta) + \pi^2_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a^\beta) \\
& +\mathcal{S}_7^3 q_\alpha q^\lambda a_\lambda\Big]
\end{aligned}
$$

Dissipative fluxes: Third order: Shear equation

$$
\begin{aligned}
\pi^{\langle\mu\nu\rangle\,(3)} \quad = \quad & 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\mathcal{S}_3^3(2\Pi\dot{\pi}_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle}\dot{\Pi}) + 2\mathcal{S}_4^3\dot{q}_{\langle\alpha}q_{\beta\rangle} \\
& + 3\mathcal{S}_5^3\dot{\pi}_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle} + \mathcal{S}_8^3\pi_{\langle\alpha\beta\rangle}\nabla_\lambda q^\lambda + \mathcal{S}_9^3(\Pi\nabla_{\langle\alpha}q_{\beta\rangle} + q_{\langle\alpha}\nabla_{\beta\rangle}\Pi) \\
& + \mathcal{S}_{10}^3(q_{(\beta}\nabla^\lambda\pi_{\langle\alpha)\lambda\rangle} + \pi_{\langle\lambda(\alpha\rangle}\nabla^\lambda q_{\beta)})\Big] \\
& + 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\Pi\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\lambda u^\lambda) \\
& + q_{\langle\alpha}q_{\beta\rangle}(\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3\nabla_\lambda u^\lambda) + \pi_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle}(\dot{\mathcal{S}}_5^3 + \mathcal{S}_5^3\nabla_\lambda u^\lambda) \\
& + \pi_{\langle\alpha\beta\rangle}q^\lambda(\nabla_\lambda\mathcal{S}_8^3 - \mathcal{S}_8^3 a_\lambda) + \Pi q_{\langle\beta}(\nabla_{\alpha\rangle}\mathcal{S}_9^3 - \mathcal{S}_9^3 a_{\alpha\rangle}) \\
& + \pi_{\langle\alpha\lambda}q^\lambda(\nabla_{\beta\rangle}\mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a_{\beta\rangle})\Big]
\end{aligned}
$$

We derive the third order entropy 4-current as well the non-classical coefficients by going beyond Israel-Stewart entropy 4-current expression in kinetic theory. The kinetic expression for entropy, can be written as

$$S^{\mu} = -\int dwp^{\mu}\psi[f(x,p)] \ ,$$

where

$$\psi[f(x,p)] = f(x,p)\left\{\ln[A_0^{-1}f(x,p)] - 1\right\} \ ,$$

and $f(x,p)$ is the out of the equilibrium distribution function. Expanding $\psi(f)$ around $\psi(f^{eq})$ up to third order we get,

$$\begin{aligned}\psi(f) &= \psi(f^{eq}) + \psi'(f^{eq})(f - f^{eq}) + \frac{1}{2}\psi''(f^{eq})(f - f^{eq})^2 \\ &\quad + \frac{1}{6}\psi'''(f^{eq})(f - f^{eq})^3 + .. ,\end{aligned}$$

Entropy from kinetic theory

$$S^{\mu(1)} = \frac{q^\mu}{T} \ ,$$

$$S^{\mu(2)} = \frac{1}{2}\beta u^\mu \left[\mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^\alpha q_\alpha + \mathcal{S}_3^2 \pi^{\nu\alpha}\pi_{\nu\alpha}\right] + \beta\left[\mathcal{S}_4^2 q^\mu \Pi + \mathcal{S}_5^2 q_\alpha \pi^{\mu\alpha}\right] \ ,$$

$$\begin{aligned} S^{\mu(3)} \quad = \quad & \frac{1}{6}\beta u^\mu \left\{\mathcal{S}_1^3 \Pi^3 + \mathcal{S}_2^3 \Pi q^\alpha q_\alpha + \mathcal{S}_3^3 \Pi \pi^{\nu\alpha}\pi_{\nu\alpha} + \mathcal{S}_4^3 q_\nu q_\alpha \pi^{\nu\alpha} + \mathcal{S}_5^3 \pi_{\nu\alpha}\pi^\nu_\beta \pi^{\alpha\beta}\right\} \\ & -\frac{1}{6}\beta q^\mu \left\{\mathcal{S}_6^3 \Pi^2 + \mathcal{S}_7^3 q^\alpha q_\alpha - \mathcal{S}_8^3 \pi^{\nu\alpha}\pi_{\nu\alpha}\right\} - \beta \mathcal{S}_9^3 \Pi q_\alpha \pi^{\mu\alpha} \\ & +\frac{1}{2}\beta \mathcal{S}_{10}^3 q_\alpha \pi^{\nu\alpha}\pi^\mu_\nu \ , \end{aligned}$$

Up to third order in dissipative fluxes the the entropy 4-current can be written as,

$$
\begin{aligned}
S^\mu &= S_1^0 u^\mu + S_1^1 \Pi u^\mu + S_2^1 q^\mu \\
&+ \left(S_1^2 \Pi^2 - S_2^2 q^\alpha q_\alpha - S_3^2 \pi^{2\langle\alpha\alpha\rangle} \right) \beta u^\mu + \beta \left(S_4^2 \Pi q^\mu \right. \\
&+ \left. S_5^2 \pi^{\langle\mu\alpha\rangle} q_\alpha \right) + \left(S_1^3 \Pi^3 - S_2^3 \Pi q_\alpha q^\alpha + S_3^3 \Pi \pi^{2\langle\alpha\alpha\rangle} \right. \\
&+ \left. S_4^3 q_\alpha q_\beta \pi^{\langle\alpha\beta\rangle} - S_5^3 \pi^{3\langle\alpha\alpha\rangle} \right) \beta u^\mu + \left(S_6^3 \Pi^2 - S_7^3 q_\alpha q^\alpha \right. \\
&+ \left. S_8^3 \pi^{2\langle\alpha\alpha\rangle} \right) \beta q^\mu + \beta \left(S_9^3 \Pi \pi^{\langle\mu\alpha\rangle} q_\alpha + S_{10}^3 \pi^{2\langle\mu\alpha\rangle} q_\alpha \right)
\end{aligned}
$$

Third order coefficients

As function of m/T in the large temperature limit

$$\left.\begin{array}{rcl} \mathcal{S}_1^2 &=& \infty \\ \mathcal{S}_2^2 &=& \frac{5}{4p} \\ \mathcal{S}_3^2 &=& \frac{3}{4p} \\ \mathcal{S}_4^2 &=& \infty \\ \mathcal{S}_5^2 &=& \frac{1}{4p} \end{array}\right\} \Longrightarrow \text{Second order coefficients known}$$

Third order coefficients

$$\begin{array}{rcl} \mathcal{S}_1^3 &=& \infty, \quad \mathcal{S}_2^3 = \infty, \quad \mathcal{S}_3^3 = \infty \\ \mathcal{S}_4^3 &=& \frac{6}{p^2}, \quad \mathcal{S}_5^3 = \frac{3}{4p^2} = \frac{\mathcal{S}_3^2}{p}, \quad \mathcal{S}_6^3 = \infty \\ \mathcal{S}_7^3 &=& \frac{2}{p^2} = 2\frac{\mathcal{S}_2^2}{p}, \quad \mathcal{S}_8^3 = \frac{27}{32p^2}, \quad \mathcal{S}_9^3 = \infty \\ \mathcal{S}_{10}^3 &=& \frac{9}{32p^2} \end{array}$$

From the second law of thermodynamics the third order shear viscous equation can be written as,

$$u_\lambda \partial^\lambda \pi^{\langle\mu\nu\rangle} = - \frac{\pi^{\langle\mu\nu\rangle}}{\tau_\pi} + \frac{2\eta\,\partial^\mu u^\nu}{\tau_\pi} - \frac{2\eta T}{\tau_\pi}\pi^{\langle\mu\nu\rangle}\partial^\lambda\left(\frac{S_3^2}{2T}u_\lambda\right)$$
$$- \frac{3\eta S_5^3}{\tau_\pi} u_\lambda(\partial^\lambda \pi^{\langle\mu\delta\rangle})\pi^{\nu\rangle}_{\langle\delta}$$
$$- \frac{2\eta T}{\tau_\pi}\pi^{\langle\mu\delta\rangle}\pi^{\nu\rangle}_{\langle\delta}\partial^\lambda\left(\frac{S_5^3}{2T}u_\lambda\right).$$

where $\tau_\pi = 2\eta S_3^2$ is relaxation time for the shear pressure.
The term $3\eta S_5^3 .\pi$ can be referred as a relaxation time that comes from third order equation. This term also provides a correction factor to second order relaxation time τ_π as follows

$$\tau_\pi^{(3)} u_\lambda \partial^\lambda \pi^{\langle\mu\delta\rangle}$$
$$= -\pi^{\langle\mu\nu\rangle} + 2\eta\,\partial^\mu u^\nu - 2\eta T\pi^{\langle\mu\nu\rangle}\partial^\lambda\left(\frac{S_3^2}{2T}u_\lambda\right)$$
$$-2\eta T\pi^{\langle\mu\delta\rangle}\pi^{\nu\rangle}_{\langle\delta}\partial^\lambda\left(\frac{S_5^3}{2T}u_\lambda\right).$$

Where $\tau_\pi^{(3)} = \tau_\pi\left(1 + \frac{3\eta S_5^3}{\tau_\pi}\pi^{\nu\rangle}_{\langle\delta}\right)$ can be termed as the modified relaxation time for the third order shear viscous pressure

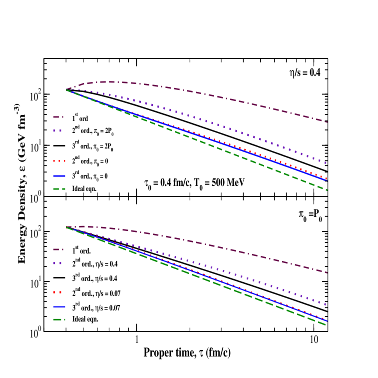
After simplification and keeping all the terms, the final equations for shear pressure for 3rd order viscous and massless fluids is found to be

$$\dot{\pi} \quad = \quad -\frac{\pi}{\tau_\pi} - \frac{1}{2}\frac{\pi}{\tau} + \frac{3}{10}\frac{\varepsilon}{\tau} + \frac{5}{8}\frac{\pi}{\varepsilon}\dot{\varepsilon} - \frac{3}{2}\frac{\pi^2}{\varepsilon\tau} + \frac{27}{8}\frac{\pi^2}{\varepsilon^2}\dot{\varepsilon} - \frac{12}{5}\frac{\pi}{\varepsilon}\dot{\pi}$$

Also in (1+1)D Bjorken flow, the energy and number density equations calculated from energy-momentum conservation to yield

$$\dot{\varepsilon} = -\frac{\varepsilon + P}{\tau} + \frac{\pi}{\tau}\,.$$

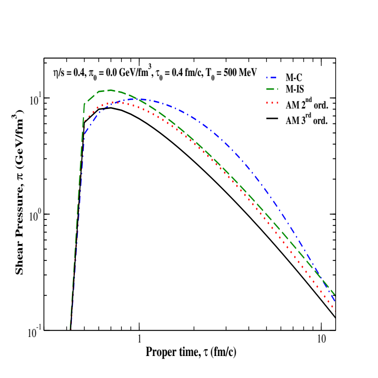
M. Younus & A. Muronga 2019...energy density evolution order by order



parameters are tuned so as to compare with similar studies

M. Younus & A. Muronga 2019

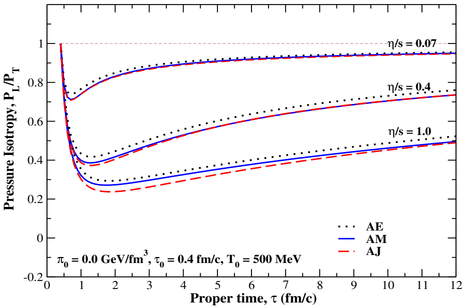
A. Younus & A. Muronga 2019 ...shear pressure term by term or order by order



Strong dependence on which terms you include

M. Younus &A. Muronga 2019.

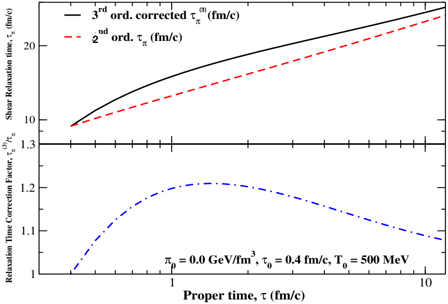
M. Younus & A. Muronga 2019 ...Pressure isotropy



M. Younus & A.Muronga, 2019

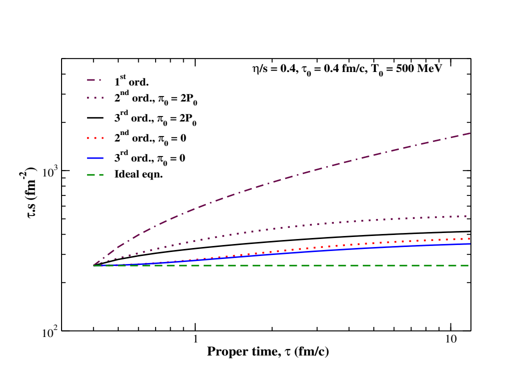
The models are very much close, they differ by EoS-calculated coefficients.

Relaxation time beyond second order



Strong dependence on the order. Still needs proper interpretation

M. Younus & A. Muronga 2019 ...Entropy generation



M. Younus & A.Muronga, 2019

This is important since it is linked to final multiplicity - refer to Jean-Yves Ollitrault talk

Conclusions

- For a one component fluid there are three main mechanisms for entropy production. One is related to the dynamic preassure, one due to heat flux and one due to shear stress.
- For the entropy production to be non-negative the coefficients related to bulk viscosity, shear viscosity and heat flux must satisfy some inequality relations.
- Third order relativistic fluid dynamics reveals the couplings and relaxation times that are not present up to second order
- The many identities one encounter in deriving the equations implies that perhaps it is possible to constrain the lower order known functions such as the equation of state and the transport, relaxation and coupling coefficients
- The equations presented here are the same whether derived via divergence theory or kinetic theory
- The relaxation and coupling coefficients are not *new* parameters - but may be determined from the equation of state

To Jean-Paul, Miklos, and Larry

"I have walked that long road to freedom. I have tried not to falter; I have made missteps along the way. But I have discovered the secret that after climbing a great hill, one only finds that there are many more hills to climb. I have taken a moment here to rest, to steal a view of the glorious vista that surrounds me, to look back on the distance I have come. But I can only rest for a moment, for with freedom come responsibilities, and I dare not linger, for my long walk is not ended"

Long Walk to Freedom
–Nelson Mandela

Larry McLerran