# Relativistic SPHerical Dark Matter

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# Motivation for the Study of Dark Matter

- Visible matter: Only a small fraction of the universe
- Dark Matter: Massive, invisible component with no interaction with light except gravity.

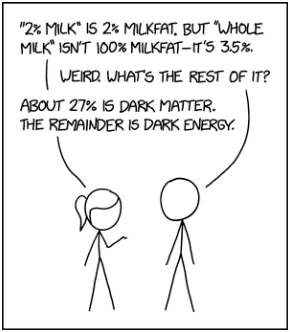



Figure: Matter Distribution in Universe (https://xkcd.com/2216/)

# Observational Evidence for Dark Matter

**Galaxy Rotation Curves:** Faster-than-expected star orbits imply more mass.

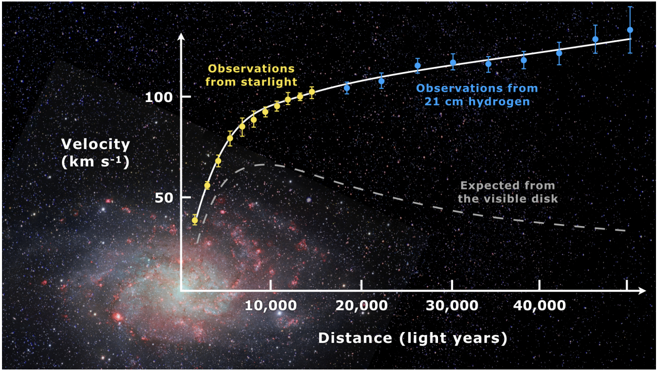



Figure: Rotation curve of spiral galaxy,a predicted one from distribution of the visible matter (gray line). $v(r) = \sqrt{GM(r)/r}$ (by Mario De Leo)

# Observational Evidence for Dark Matter

- **Galaxy Rotation Curves:** Faster-than-expected star orbits imply more mass.
- **Gravitational Lensing:** Bending of light around invisible mass points to dark matter.
- **Cosmic Microwave Background:** Fluctuations match models that include dark matter.

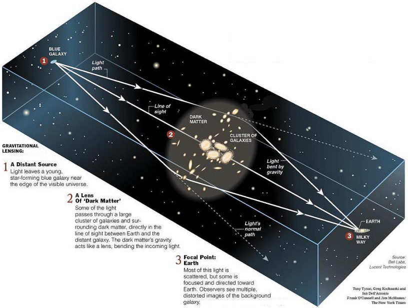



Figure: Frank O'Connell and Jim McManus/ The New York Times

Other evidence of dark matter include: **Galaxy Cluster Dynamics** and **Structure Formation**.

# Cosmological Model

The prevailing cosmological model that describes the large-scale structure and evolution of the universe.

- **Λ**: Represents dark energy, a mysterious force driving the universe's accelerated expansion.
- **CDM (Cold Dark Matter):** Accounts for the gravitational effects of unseen matter that doesn't interact with light i.e. Dark Matter.
  - ↪ Non-Baryonic
  - ↪ Collisionless

# Problems with Cold Dark Matter (CDM)

$\Lambda$CDM able to describe the universe at large scale, even if some issues remain open...

- On large scales, the $\Lambda$CDM model excellently describes cosmic phenomena like the CMB and structure formation.
- On small scales, such as in the Local Group, puzzles like the "core-cusp" problem arise, where dwarf galaxies($\sim$ 1000 to 10,000 lyrs) show less dark matter in their cores than predicted.

*There are problems with the $\Lambda$CDM model predictions on small scales core-cusp problem, the too big to fail and missing satellites.*

# Core-Cusp Problem

**Core-Cusp Problem:** CDM predicts a 'cuspy' dark matter distribution in the centers of galaxies (NFW profile Navarro, Frenk, White), where the density sharply increases, but observations show a flatter 'cored' distribution.

$$\rho_{NFW}(r) = \rho_s/((r/r_s)(1 + r/r_s)^2)$$

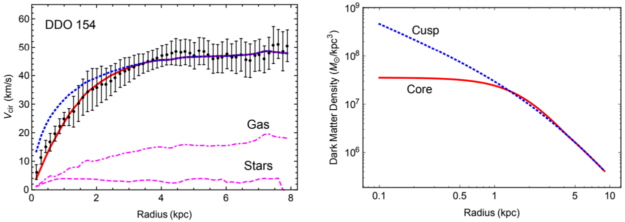



Figure: Left: Observed rotation curve of dwarf galaxy DDO 154 (black data points) compared to models with an NFW profile for $r_s \approx 3.4kpc$ and $\rho_s \approx 1.5 \times 10^7 M_\odot/kpc^3$ (Tulin & Yu (2017))

# Self-interacting Dark Matter (SIDM)

The challenges and discrepancies at small scales in the ΛCDM model, such as the cusp-core problem etc have prompted the exploration of alternative dark matter models.

**What is SIDM?**

Self-Interacting Dark Matter (SIDM): A proposed form of dark matter where particles interact with each other through forces other than gravity, where collisions and scatterings between DM particles are allowed.

$$R_{\text{scat}} = \frac{\sigma v_{\text{rel}} \rho_{\text{dm}}}{m}$$

*For dwarf galaxy we want at least one scattering per particle over 10 Gyr*

# Self-interacting Dark Matter (SIDM)

SIDM yields a cored density distribution as favoured by observations.

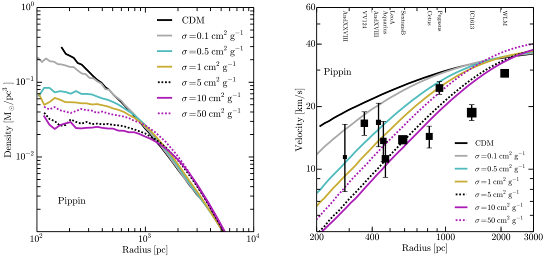



Figure: Elbert et al.

Given this, it is very well possible that the nature of DM is far richer and more complex than what the CDM model assumes.

# N-body simulations


Figure: Randall Munroe, xkcd

# N-body simulations

- *N*-body simulations are computational methods used to study the dynamics of systems with many interacting particles, such as galaxies, galaxy clusters, or the universe's large-scale structure.
- They simulate the gravitational interactions between a large number of particles ($\sim 10^9$) , representing dark matter and baryonic matter, to model the formation and evolution of cosmic structures in a "cosmological box"

# Simulations

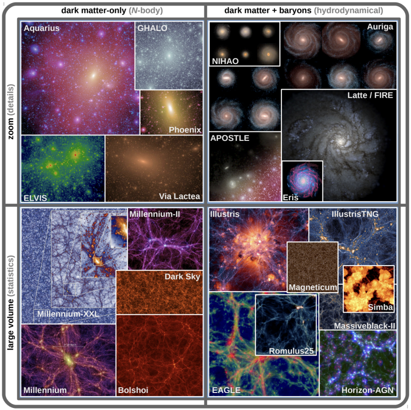



Figure: Visual representations of some selected recent structure and galaxy formation simulations. (vogelsberger2019)

# Complexity of N-body Simulations

- Simulating both DM and BM is highly resource-intensive, often requiring millions of CPU hours on supercomputers.
- Due to these demands, most simulations prioritize the simpler CDM model, limiting the exploration of more complex dark matter models.
- While SIDM offers potential solutions to small-scale structure problems, its inclusion in simulations further increases complexity and computational costs.

# Smooth Particle Hydrodynamic

SPH is a computational method used to simulate fluids and gases:

- ↪ each particle carrying properties like mass, velocity, and energy.
- ↪ particle represents a portion of the fluid or gas, interacting with nearby particles to simulate the fluid's behavior.

# Euler Hydrodynamics Equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla p}{\rho} = \mathbf{g}$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + \frac{p}{\rho} \nabla \cdot \mathbf{v} = 0$$

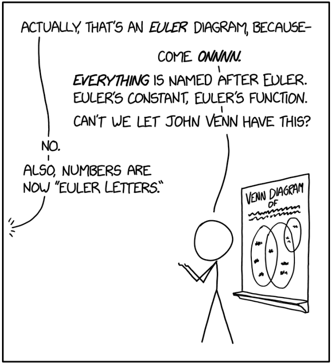
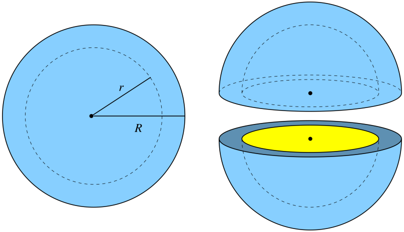



Figure: https://xkcd.com/2721/

+ heat conduction via Fourier's law and Equation State for monatomic ideal gas

# `SPHerical` Method

- **`SPHerical` assumes spherical symmetry:**
  - ↪ Problem reduces from 3D to 1D
- **Concentric Shells:**
  - ↪ Instead of simulating individual particles in a 3D space, the particles can be grouped into concentric spherical shells.
  - ↪ Each shell represents a layer of particles at a specific distance from the center of the sphere.



# Smoothing Kernel

Mathematical function, $W(\mathbf{r}, \mathbf{r}', \mathbf{h})$ used in SPH to interpolate and smooth physical quantities like density, pressure, and velocity over a set of particles.

- $W$ is Maximum at $\mathbf{r} = \mathbf{r}'$
- Only depends on $r$ and $W(\mathbf{r}, \mathbf{r}', h) \geq 0$ for all $r, h$
- Continuous up to second derivatives everywhere
- Approximates a Dirac delta function in the limit $\lim_{h \to 0} W(\mathbf{r}, \mathbf{r}', h) = \delta^3(\mathbf{r})$

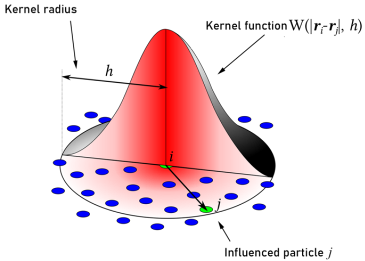



Figure: Truong et al.

# Re-driving Euler Hydrodynamics Equations

Fluid is broken down into discrete units called "particles", which travel along with the overall flow. We define an arbitrary quantity of fluid as follows

$$A(r) = \sum_j \frac{m_j}{\rho_j} A_j W(r, r_j, h)$$

For example, "smoothed" density, $\rho$ is

$$\rho(r) = \sum_j m_j W(r, r_j, h)$$

# Re-driving Euler Hydrodynamics Equations

This transforms Euler equations: SPH equations of motion for spherically-symmetric, self-gravitating, conducting fluid with variable smoothing length

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \frac{1}{\Omega_i} \sum_j m_j \left( v_i \frac{\partial W(r_i, r_j, h_i)}{\partial r_i} + v_j \frac{\partial W(r_i, r_j, h_i)}{\partial r_j} \right)$$

$$\frac{\mathrm{d}e_i}{\mathrm{d}t} = \frac{P_i}{\Omega_i \rho_i^2} \sum_j m_j \left( v_i \frac{\partial W(r_i, r_j, h)}{\partial r_i} + v_j \frac{\partial W(r_i, r_j, h_i)}{\partial r_j} \right)$$

$$\frac{dv_i}{dt} = -\frac{GM_i}{r_i^2} - \sum_j m_j \left( \frac{P_i}{\Omega_i \rho_i^2} \frac{\partial W(r_i, r_j, h_i)}{\partial r_i} + \frac{P_j}{\Omega_j \rho_j^2} \frac{\partial W(r_j, r_i, h_j)}{\partial r_i} \right)$$

where $\Omega_j = 1 - \frac{1}{3} \frac{h_j}{\rho_j} \sum_k m_k \frac{\partial W(r_j, r_k, h_j)}{\partial h_j}$

# Results Of SPHerical

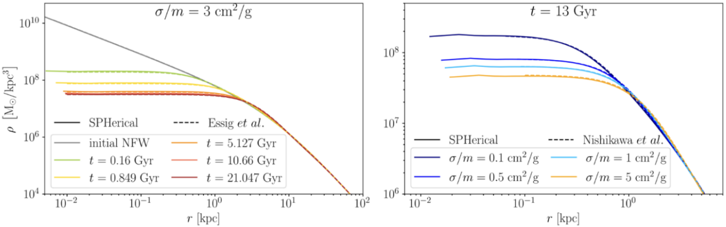



Figure: Comparisons vs previous results from Essig et al (2019); Nishikawa et al (2019) & Gad-Nasr et al (2023)

# Conclusion

- General Relitivistic corrections
  - ↪ SMBHs are formed often involve the direct collapse of dense regions in the early universe. These processes occur under extreme gravitational conditions where Newtonian gravity is inadequate because the gravitational forces are immensely strong and the density is very high.
- Revise collapse condition in `SPHerical`


QUESTIONS
FOUND IN GOOGLE AUTOCOMPLETE
WHY DO I SAY UH
WHY IS SEA SALT BETTER
WHY ARE THERE TREES IN THE MIDDLE OF FIELDS
WHY IS THERE NOT A POKEMON MMO
WHY IS THERE LAUGHING IN TV SHOWS
WHY ARE THERE DOORS ON THE FREEWAY
WHY ARE THERE SO MANY SVCHOST.EXE RUNNING
WHY AREN'T THERE ANY COUNTRIES IN ANTARCTICA
WHY ARE THERE SCARY SOUNDS IN MINECRAFT
WHY IS THERE KICKING IN MY STOMACH
WHY ARE THERE TWO SLASHES AFTER HTTP
WHY ARE THERE CELEBRITIES
WHY DO SNAKES EXIST
WHY DO OYSTERS HAVE PEARLS
WHY ARE DUCKS CALLED DUCKS
WHY DO THEY CALL IT THE CLAP
WHY ARE KYLE AND CARTMAN FRIENDS
WHY IS THERE AN ARROW ON AANG'S HEAD
WHY ARE TEXT MESSAGES BLUE
WHY ARE THERE MUSTACHES ON CLOTHES
WHY ARE THERE MUSTACHES ON CARS
WHY ARE THERE MUSTACHES EVERYWHERE
WHY ARE THERE SO MANY BIRDS IN OHIO
WHY IS THERE SO MUCH RAIN IN OHIO
WHY IS OHIO WEATHER SO WEIRD
AREN'T THERE DINOSAUR GHOSTS
WHY DO IGUANAS DIE
WHY AREN'T ECONOMISTS RICH
WHY DO AMERICANS CALL IT SOCCER
WHY ARE MY EARS RINGING
WHY ARE THERE SO MANY AVENGERS
WHY ARE THE AVENGERS FIGHTING THE X MEN
WHY IS WOLVERINE NOT IN THE AVENGERS
WHY IS THERE LAVA
WHY ARE THERE SWARMS OF GNATS
WHY IS THERE PHLEGM
IS YKK ON ALL ZIPPERS
WHY IS THERE A RED LINE THROUGH HTTPS ON FACEBOOK
WHY IS HTTPS IMPORTANT
WHY AREN'T MY ARMS GROWING
WHY DO I FEEL DIZZY
WHY ARE THERE WEEKS
WHY ARE THERE SO MANY CROWS IN ROCHESTER, MN
WHY IS PSYCHIC WEAK TO BUG
WHY DO CHILDREN GET CANCER
WHY IS POSEIDON ANGRY WITH ODYSSEUS
WHY IS THERE ICE IN SPACE
WHY ARE THERE ANTS IN MY LAPTOP
WHY IS THERE NO KING IN ENGLAND
WHY ARE DOGS AFRAID OF FIREWORKS
WHY IS EARTH TILTED
WHY IS SPACE BLACK
WHY IS OUTER SPACE SO COLD
WHY ARE THERE PYRAMIDS ON THE MOON
WHY IS NASA SHUTTING DOWN
WHY ARE THERE GHOSTS
WHY IS THERE AN OWL IN MY BACKYARD
WHY IS THERE AN OWL OUTSIDE MY WINDOW
WHY IS THERE AN OWL ON THE DOLLAR BILL
WHY DO OWLS ATTACK PEOPLE
WHY ARE AK 47s SO EXPENSIVE
WHY ARE THERE HELICOPTERS CIRCLING MY HOUSE
WHY ARE THERE GODS
WHY ARE THERE TWO SPOCKS
WHY ARE THERE MALE AND FEMALE BIKES
WHY ARE THERE FEMALE MR MIMES
WHY ARE THERE BRIDESMAIDS
WHY DO DYING PEOPLE REACH UP
WHY AREN'T THERE VARICOSE ARTERIES
WHY ARE OLD KLINGONS DIFFERENT
WHY ARE THERE SQUIRRELS
WHY IS THERE HELL IF GOD FORGIVES
WHY ARE THERE TINY SPIDERS IN MY HOUSE
WHY DO SPIDERS COME INSIDE
WHY ARE THERE HUGE SPIDERS IN MY HOUSE
WHY ARE THERE LOTS OF SPIDERS IN MY HOUSE
WHY ARE THERE SPIDERS IN MY ROOM
WHY ARE THERE SO MANY SPIDERS IN MY ROOM
WHY DO SPIDER BITES ITCH
WHY IS DYING SO SCARY
WHY IS THERE NO GPS IN LAPTOPS
WHY DO KNEES CLICK
WHY AREN'T THERE E GRADES
WHY IS ISOLATION BAD
WHY DO BOYS LIKE ME
WHY DON'T BOYS LIKE ME
WHY IS THERE ALWAYS A JAVA UPDATE
WHY ARE THERE RED DOTS ON MY THIGHS
WHY IS LYING GOOD
WHY IS GPS FREE
WHY IS SEX SO IMPORTANT
WHY IS PROGRAMMING SO HARD
WHY IS THERE A 0 OHM RESISTOR
WHY DO AMERICANS HATE SOCCER
WHY DO RHYMES SOUND GOOD
WHY DO TREES DIE
WHY IS THERE NO SOUND ON CNN
WHY AREN'T POKEMON REAL
WHY AREN'T BULLETS SHARP
WHY DO DREAMS SEEM SO REAL
WHY IS MT VESUVIUS THERE
WHY DO THEY SAY T MINUS
WHY ARE THERE OBELISKS
WHY ARE WRESTLERS ALWAYS WET
WHY ARE OCEANS BECOMING MORE ACIDIC
WHY IS ARWEN DYING
WHY AREN'T MY QUAIL LAYING EGGS
WHY AREN'T MY QUAIL EGGS HATCHING
WHY AREN'T THERE ANY FOREIGN MILITARY BASES IN AMERICA
WHY IS LIFE SO BORING
WHY ARE MY BOOBS ITCHY
WHY ARE CIGARETTES LEGAL
WHY ARE THERE DUCKS IN MY POOL
WHY IS JESUS WHITE
WHY IS THERE LIQUID IN MY EAR
WHY DO Q TIPS FEEL GOOD
WHY DO GOOD PEOPLE DIE
WHY AREN'T THERE GUNS IN HARRY POTTER
WHY ARE ULTRASOUNDS IMPORTANT
WHY ARE ULTRASOUND MACHINES EXPENSIVE
WHY IS STEALING WRONG

# General Relativity and `SPHerical`

# General Relativity Overview

The Einstein Field Equation is

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where

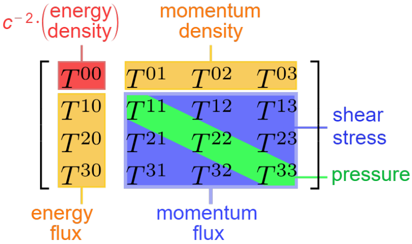
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$T_{\mu\nu}$ is the energy-momentum tensor
Conservation of momentum and energy requires

$$\nabla_{\mu} T^{\mu\nu} = 0$$

Together with the geodesic equation, EFE determines the path of the particles and radiation through the geometry of the space-time caused by the source terms.

$$\frac{d^2 x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{d\tau} \frac{dx^{\gamma}}{d\tau} = 0$$




Describes the density and flux of energy and momentum in spacetime.

The component $T_{\mu\nu}$ describes the flux of $\mu$ th component of momentum vectors across the surface of constant $x^{\nu}$. The

stress-energy tensor and metric tensor or both symmetric The

stress-energy tensor for an ideal fluid is

$$
\begin{aligned}
T^{\mu\nu} &= (\rho + \rho u + P)U^{\mu}U^{\nu} + Pg^{\mu\nu} \\
&= \rho\omega(U^0)^2 v^{\mu}v^{\nu} + Pg^{\mu\nu}
\end{aligned}
$$

# General Relativistic Corrections

Using the GR variational principle, Monaghan and Price (2001) derived the SPH equations in general relativistic formalism.
The conserved density, momentum and energy are

$$\begin{aligned} \rho^* &= \sqrt{-g}\rho U^0 \\ p_i &= U^0 w g_{i\mu} v^\mu \\ e &= U^0 \left( w g_{i\mu} v^\mu v^i - (1+u) g_{\mu\nu} v^\mu v^\nu \right) \end{aligned}$$

where $g = \det(g_{\mu\nu})$ (covariant metric) and enthalpy is defined $w = 1 + u + \frac{P}{\rho}$ where $u$, $P$ and $\rho$ are specific internal energy, pressure and density in the rest frame of the fluid. The four-velocity components are :

$$U^\mu = \frac{dx^\mu}{d\tau}$$

and the coordinate velocities $v^\mu$ are given by

$$v^\mu \equiv \frac{dx^\mu}{dt} = \frac{U^\mu}{U^0},$$

where the normalization condition $U_\mu U^\mu = -1$ gives

$$U^0 \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{-g_{\mu\nu} v^\mu v^\nu}},$$

where relativistic units $c = G = 1$ and the metric signature $(-,+,+,+)$ is assumed.

$$\frac{\mathrm{d}\rho_a^*}{\mathrm{d}t} = \frac{1}{\Omega_a} \sum_b m_b \left(v_a^i - v_b^i\right) \frac{\partial W_{ab}\left(h_a\right)}{\partial x^i}$$

$$\frac{\mathrm{d}p_i^a}{\mathrm{d}t} = -\sum_b m_b \left[ \frac{\sqrt{-g_a} P_a}{\Omega_a \rho_a^{*2}} \frac{\partial W_{ab}\left(h_a\right)}{\partial x^i} + \frac{\sqrt{-g_b} P_b}{\Omega_b \rho_b^{*2}} \frac{\partial W_{ab}\left(h_b\right)}{\partial x^i} \right] + f_i^a$$

$$\frac{\mathrm{d}e_a}{\mathrm{d}t} = -\sum_b m_b \left[ \frac{\sqrt{-g_a} P_a v_b^i}{\Omega_a \rho_a^{*2}} \frac{\partial W_{ab}\left(h_a\right)}{\partial x^i} + \frac{\sqrt{-g_b} P_b v_a^i}{\Omega_b \rho_b^{*2}} \frac{\partial W_{ab}\left(h_b\right)}{\partial x^i} \right] + \Lambda_a$$

where we define

$$\Omega_a = 1 + \frac{1}{d} \frac{h_a}{\rho_a^*} \sum_b m_b \frac{\partial W_{ab}(h_a)}{\partial h_a}$$

where the source term containing the derivative of the metric is defined accordingly

$$f_i = \frac{\sqrt{-g}}{2\rho^*}\left(T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^i}\right)$$
$$\Lambda = -\frac{\sqrt{-g}}{2\rho^*}\left(T^{\mu\nu}\frac{\partial g_{\nu\mu}}{\partial t}\right)$$

In flat space-time these partial derivatives are zero and there is no additional term.

# Schwarzschild Metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

- Simplest solution to EFE under time invariance and spherical symmetry
- Singularity at origin and $r_S$
- Infalling time is infinite

# Tolman Oppenheimer Volkoff Equations

$$\frac{dm}{dr} = 4\pi r^2 \epsilon$$

$$\frac{dP}{dr} = -(\epsilon + P)\frac{m + 4\pi r^3 P}{r(r-2m)}$$

$$\frac{d\Phi}{dr} = -\frac{1}{\epsilon + P}\frac{dP}{dr}$$

Can be integrated from $r = 0$ outwards, until P vanishes.

Boundary condition is

$$\Phi(r = R) = \frac{1}{2} ln(1 - \frac{2M}{R})$$

The assumed equation of state for SIDM is chosen to be polytropic $P = (\gamma_{ad} - 1)\rho u$ where $\gamma_{ad}$ is the adiabatic index which is equal to $\frac{5}{3}$ in the newtonian code. The internal energy $u$ is related to the gas temperature $T$ through the ideal gas law

$$P = \frac{\rho k_B T}{m}$$

where $m$ is the mass of the dark matter particle.
Therefore the specific enthalpy in the comoving coordinate would be given by

$$\omega = 1 + u + \frac{P}{\rho} = 1 + u + \frac{2}{3}u = 1 + \frac{5}{3}u$$

The coefficients of the partial derivatives in $f_i$ and $\Lambda$ becomes

$$\frac{\sqrt{-g}\,T^{\mu\nu}}{2\rho^*} = \frac{1}{2}\left[\omega U^0 v^\mu v^\nu + \frac{P g^{\mu\nu}}{\rho U^0}\right]$$

# Computational Procedure

For the case of spherically symmetric shells with only possible motion in radial direction, we note that the only non-zero $v^\mu$ components are $v^0 = \frac{U^0}{U^0} = 1$ and $v^1 \equiv v^r$. Another quantity in the equations is $U^0$ which for our simulation would be given by

$$U^0 = \frac{1}{\sqrt{-g_{tt} - g_{rr}(v^r)^2}}$$

Also,

$$\frac{P}{\rho^*} = \frac{P}{\sqrt{-g}\rho U^0} = \frac{2}{3}\frac{u}{\sqrt{-g}U^0}$$

from which the sph equations turn into:

$$\frac{\mathrm{d}\rho_a^*}{\mathrm{d}t} = \frac{1}{\Omega_a}\sum_b m_b\left(v_a^i - v_b^i\right)\frac{\partial W_{ab}(h_a)}{\partial x^i}$$

$$\frac{\mathrm{d}p_i^a}{\mathrm{d}t} = -\frac{2}{3}\sum_b m_b\left[\frac{u_a}{\Omega_a\rho_a^* U_a^0}\frac{\partial W_{ab}(h_a)}{\partial x^i} + \frac{u_b}{\Omega_b\rho_b^* U_b^0}\frac{\partial W_{ab}(h_b)}{\partial x^i}\right] + f_i^a$$

$$\frac{\mathrm{d}e_a}{\mathrm{d}t} = -\frac{2}{3}\sum_b m_b\left[\frac{u_a v_b^i}{\Omega_a\rho_a^* U_a^0}\frac{\partial W_{ab}(h_a)}{\partial x^i} + \frac{u_b v_a^i}{\Omega_b\rho_b^* U_b^0}\frac{\partial W_{ab}(h_b)}{\partial x^i}\right] + \Lambda_a,$$

where

$$f_i^a = \frac{1}{2}[(1+\frac{5}{3}u_a)U_a^0 v^\mu v^\nu + \frac{2}{3}\frac{u_a g_a^{\mu\nu}}{U_a^0}]\ \frac{\partial g_{\mu\nu}}{\partial x^i}$$

$$\Lambda_a = \frac{1}{2}[(1+\frac{5}{3}u_a)U_a^0 v^\mu v^\nu + \frac{2}{3}\frac{u_a g_a^{\mu\nu}}{U_a^0}]\ \frac{\partial g_{\mu\nu}}{\partial t}$$

The basic variables are $u$, $\rho^*$ and $v_i$. Knowing these allows for evolving the variables.

Unlike the Newtonian case, the basic variables $v_i$ and $u$ are not evolved directly in GRSPH, rather their dependant functions $p_r$ and $e$. Therefore, after each time step evolution one needs to use two of the equations to solve for the two variables $v_i$ and $u$.

$$p_r = U^0\,(1 + \frac{5}{3}u) g_{rr} v^r$$
$$e = U^0\,[(1 + \frac{5}{3}u) g_{rr} (v^r)^2 - (1+u)(g_{tt} + g_{rr}(v^r)^2)]$$

The left hand side of these equations are the time evolved canonical momentum and energy which are known. Solving the $p_r$ equation for $v^r$ and substituting in $e$ equation gives

$$e = \frac{p_r^2}{U^0\,(1+\frac{5}{3}u)\,g_{rr}} - U^0\,(1+u)(g_{tt} + \frac{p_r^2}{g_{rr}\,(1+\frac{5}{3}u)^2\,(U^0)^2})$$

which can be solved numerically for $u$ and obtain $v^r$ from it.
Knowing all the new time-step variables, the next time-step evolution is possible.

# Collapse of Dark Matter

Gravothermal arises due to the negative heat capacity of the self-gravitating, self-interacting dark matter fluid.

We set further dynamical conditions on core collapse based on the adiabatic index.

The system becomes unstable if the pressure-averaged adiabatic index $< \gamma >$ is less than the critical adiabatic index $\gamma_{cr}$, which is the well-known $\frac{4}{3}$ with some velocity dependant GR correction terms

Figure: M70 Gravothermal Collapse

$$\langle T \rangle = -\frac{1}{2} \sum_{k=1}^{N} \langle \mathbf{F}_k \cdot \mathbf{r}_k \rangle_\tau = \frac{n}{2} \langle V_{\text{tot}} \rangle$$

$$\langle E_{tot} \rangle = (\frac{n}{2} + 1) \langle V_{tot} \rangle$$

The critical adiabatic index is (Chandrasekhar 1964)

$$\gamma_{\mathrm{cr}} \equiv \frac{4}{3} + \frac{1}{36} \frac{\int_0^R e^{3\Phi+\Lambda} \left[16p + \left(e^{2\Lambda} - 1\right)(\rho + p)\right] \left(e^{2\Lambda} - 1\right) r^2 \, \mathrm{d}r}{\int_0^R e^{3\Phi+\Lambda} p r^2 \, \mathrm{d}r}$$

$$+ \frac{4\pi}{9} \frac{\int_0^R e^{3(\Phi+\Lambda)} \left[8p + \left(e^{2\Lambda} + 1\right)(\rho + p)\right] p r^4 \, \mathrm{d}r}{\int_0^R e^{3\Phi+\Lambda} p r^2 \, \mathrm{d}r} + \frac{16\pi^2}{9} \frac{\int_0^R e^{3\Phi+5\Lambda} (\rho + p) p^2 r^6 \, \mathrm{d}r}{\int_0^R e^{3\Phi+\Lambda} p r^2 \, \mathrm{d}r}$$

and the pressure averaged adiabatic index is

$$\langle\gamma\rangle \equiv \frac{\int_0^R e^{3\Phi+\Lambda} \gamma p r^2 \, \mathrm{d}r}{\int_0^R e^{3\Phi+\Lambda} p r^2 \, \mathrm{d}r}$$

The condition of stability is

$$\langle\gamma\rangle > \gamma_{\mathrm{cr}}$$

for generic spherically symmetric line element

$$ds^2 = -e^{2\Phi(r)} c^2 dt^2 + e^{2\Lambda(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

For a monoatomic ideal gas, $<\gamma>$ can be any number between $\frac{5}{3}$ and $\frac{4}{3}$. The instability hardly occurs in Newtonian relativity at ultra-relativistic regime where $<\gamma>= \frac{4}{3}$

However in GR the value of $\gamma_{cr}$ increases and instability can occur before particles become ultra-relativistic.

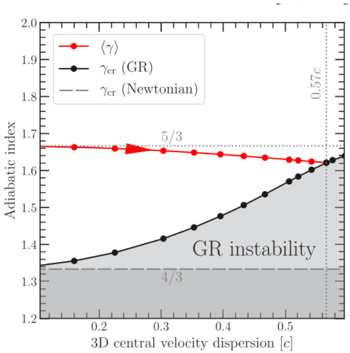



Figure: $\gamma$ vs 3D velocity dispersion (Feng, Yu, Zhong 2021)

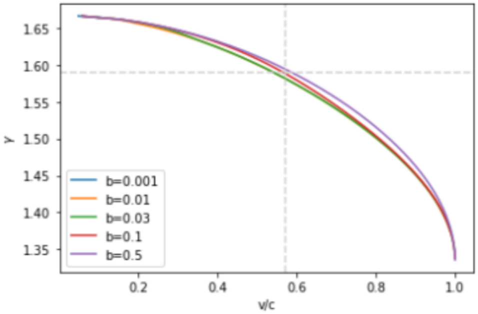



Figure: Gamma (pressure averaged adiabatic index) vs 3D velocity dispersion comparison [Maheen Hemani]

The boundary temperature doesn't vary the results as much and we see a similar trend at all different temperature (b) values.
This means using the velocity condition is good enough to initiate collapse.

# SPHINCS

- Rosswog and Diener (2021) present a new methodology for simulating self-gravitating general-relativistic fluids
- Fluid is modelled by means of Lagrangian particles in the framework of a GRSPH formulation, while the spacetime is evolved on a mesh according to the BSSN formulation
- Particles need from mesh: the metric $g_{\mu\nu}$ and the "metric acceleration terms" for the momentum and energy equations at the particle locations
- Mesh needs from particles: the energy-momentum tensor $T_{\mu\nu}$ for the source terms in BSSN

# BSSN

EFE are highly nonlinear partial differential equations, extremely challenging to solve, especially in dynamic and strong-field situations.

Traditional methods can lead to numerical instabilities, making it difficult to obtain accurate and stable simulations.

The BSSN formalism was developed to overcome these challenges by introducing a set of variables and equations that are more numerically stable and better suited for long-term evolution in computational simulations.

# Outlook

- 3+1 formalism
- Both mesh and particle method combined
- Need for sophisticated interpolation algorithms and BSSN formalism.
- Spherical averaging of the SPH equation in SPHINCS
- Apply the adiabatic condition for collapse (most likely through a condition on energy density)