Quantum vacuum effects induced by branes in AdS spacetime

Aram Saharian

Department of Physics, Yerevan State University

Gourgen Sahakyan Chair of Theoretical Physics

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Outline

- Anti-de Sitter spacetime as a background geometry in QFT

- Geometry, field and boundary conditions

- Vacuum forces and induced cosmological constant

- VEV of current density in models with compact dimensions

- Conclusions

Theoretical background

- The influence of gravity on quantum matter is considered within the framework of semiclassical theory in which the gravitational field is considered as a classical curved background

- Back-reaction of quantum effects is described by Einstein equations with the expectation value of the energy-momentum tensor for quantum fields in the right-hand side

- This hybrid but very useful scheme is an important intermediate step to the development of quantum gravity

- Among the most interesting effects in this field are the particle production and the vacuum polarization by strong gravitational fields

- Among the most important lessons: The notions of particle and vacuum are observer dependent

Maximally symmetric spaces

■ Exact results for gravity-induced quantum effects are obtained for highly symmetric background geometries

■ Three distinct maximally symmetric spacetimes which are solutions to the vacuum Einstein equations supplemented with a cosmological constant

◆ *Minkowski, M_{D+1},* ➡ Zero cosmological constant

◆ *De Sitter (dS), dS_{D+1},* ➡ Positive cosmological constant

$$\Lambda = \frac{(D-1)(D-2)}{2a^2}, \qquad R = \frac{D(D-1)}{a^2},$$

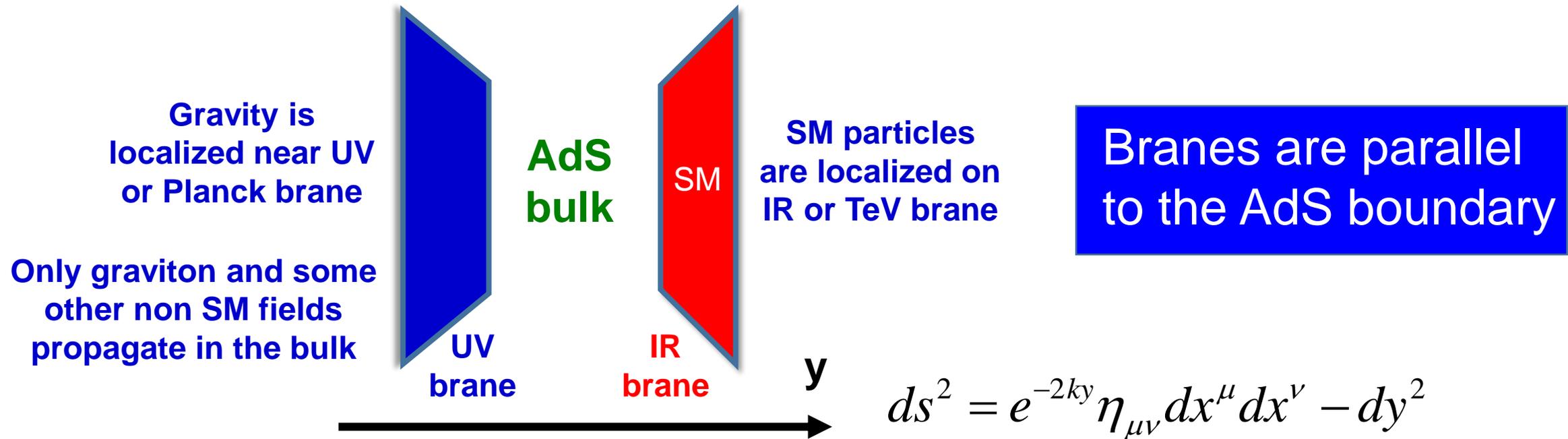◆ *Anti-de Sitter (AdS), AdS_{D+1},* ➡ Negative cosmological constant

$$\Lambda = -\frac{(D-1)(D-2)}{2a^2}, \qquad R = -\frac{D(D-1)}{a^2}$$

Importance of AdS in QFT on curved backgrounds

- Because of the high symmetry, numerous problems are exactly solvable on AdS bulk and this may shed light on the influence of a classical gravitational field on the quantum matter in more general geometries

- Questions of principal nature related to the quantization of fields propagating on curved backgrounds

- AdS spacetime generically arises as a ground state in extended supergravity and in string theories

- AdS/Conformal Field Theory correspondence: Relates string theories or supergravity in the bulk of AdS with a conformal field theory living on its boundary

- Braneworld models: Provide a solution to the hierarchy problem between the gravitational and electroweak scales

  - Braneworlds naturally appear in string/M-theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions

Randall-Sundrum-type braneworlds

- Original Randall-Sundrum model (RS1) offers a solution to the hierarchy problem by postulating 5D AdS spacetime bounded by two (3+1)-dimensional branes

Gravity is localized near UV or Planck brane

Only graviton and some other non SM fields propagate in the bulk

AdS bulk

SM

UV brane

IR brane

y

SM particles are localized on IR or TeV brane

Branes are parallel to the AdS boundary

$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

- Hierarchy problem between the gravitational and electroweak scales is solved for

k·distance between branes = 40

Boundary and periodicity conditions on quantum fields in AdS

- In field theoretical models on AdS bulk several types of boundary conditions arise imposed on field operators

- AdS spacetime has a timelike boundary at spatial infinity through which the data can propagate ➡ *Information may be lost to, or gained from, spatial infinity*

    ⬇

    To preserve the unitarity of the theory appropriate boundary conditions should be imposed on the AdS boundary

- In braneworld models boundary conditions are imposed on the fields at the location of the branes

- In models motivated by string theory compact spatial dimensions are present

    ⬇

    Compactification imposes periodicity conditions on the fields along compact dimensions

■ Both types of conditions modify the spectrum of vacuum fluctuations of quantum fields ➡ Vacuum expectation values of physical observables are changed ➡ Casimir effect

AIP

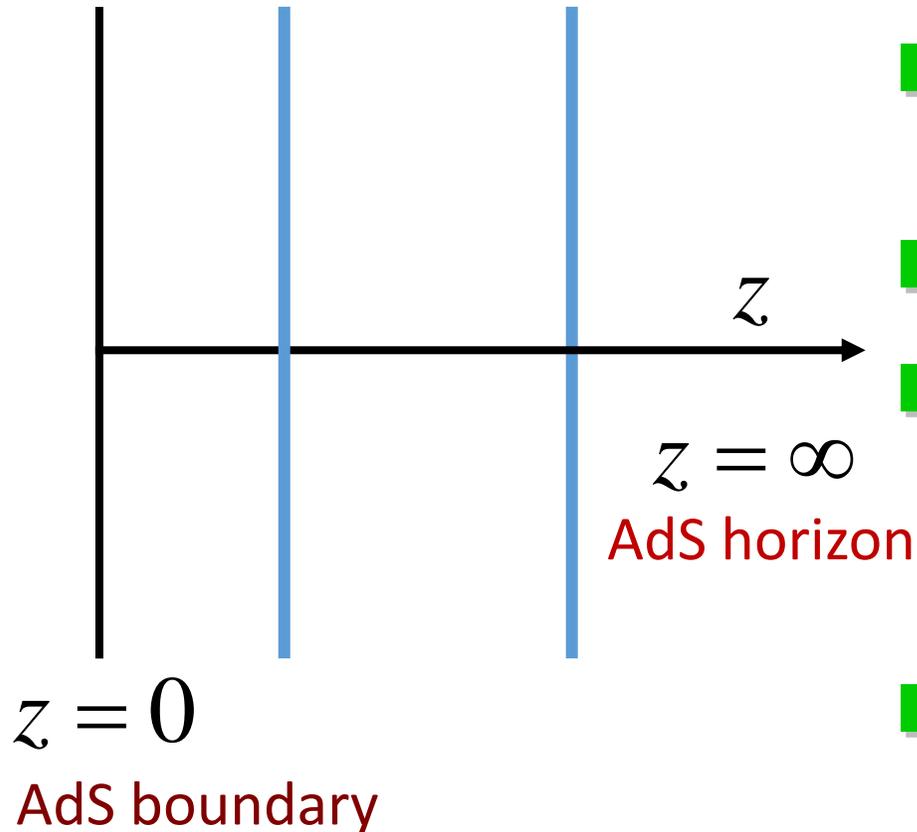Hendrik Brugt Gerhard Casimir, 1909 - 2000

$$F = -\frac{\pi^2 \hbar c}{240} \frac{A}{d^4}$$

Casimir plates

Vacuum fluctuations

Bulk and brane geometry

■ Line element for AdS bulk

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{-2y/a}\eta_{ik}dx^i dx^k - dy^2, \quad -\infty < y + \infty$$

$$ds^2 = (a/z)^2(\eta_{ik}dx^i dx^k - dz^2) \quad z = ae^{y/a}, \quad 0 \le z < \infty$$

z

$z = \infty$

AdS horizon

$z = 0$

AdS boundary

■ Geometry of branes: Two branes parallel to AdS boundary

■ Branes impose boundary conditions on bulk fields

■ Change of the spectrum for vacuum fluctuations give rise to vacuum forces acting on the branes

Casimir forces

■ The Casimir effect has been considered as a stabilization mechanism for the interbrane distance (radion field)

Fields

- **Scalar field** $(g^{\mu\nu}D_\mu D_\nu + m^2 + \xi R)\varphi(x) = 0$

  $$(1 + \beta_j n_j^\mu D_\mu)\varphi(x) = 0, \quad z = z_j \quad \Longleftarrow \quad \text{Robin boundary condition}$$

- **Fermion field** $(i\gamma^\mu D_\mu - m)\psi(x) = 0$

  $$(1 + i\gamma^\mu n_\mu^{(j)})\psi(x) = 0, \quad z = z_j \quad \Longleftarrow \quad \text{MIT bag boundary condition}$$

- **Vector field** $\nabla_\nu F^{\mu\nu} = \dfrac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}F^{\mu\nu}) = 0,$

  $$n^{\mu_1 *}F_{\mu_1\cdots\mu_{D-1}} = 0, \quad z = z_1, z_2, \quad \Longleftarrow \quad \text{Generalization of perfect conductor boundary condition for general number of spatial dimension}$$

  $$n^\mu F_{\mu\nu} = 0, \quad z = z_1, z_2$$

  Boundary condition used in bag models of hadrons for confinement of gluons

Vacuum expectation values

- Vacuum expectation values of the energy-momentum tensor

- Vacuum forces acting on the branes

- Cosmological constant induced on branes by vacuum fluctuations of bulk fields

- Vacuum force acting per unit surface of the brane

$$= \boxed{\text{Self-action force}} + \boxed{\text{Force acting on the brane due to the presence of the second brane}} \longleftarrow \boxed{\text{(Interaction force)}}$$
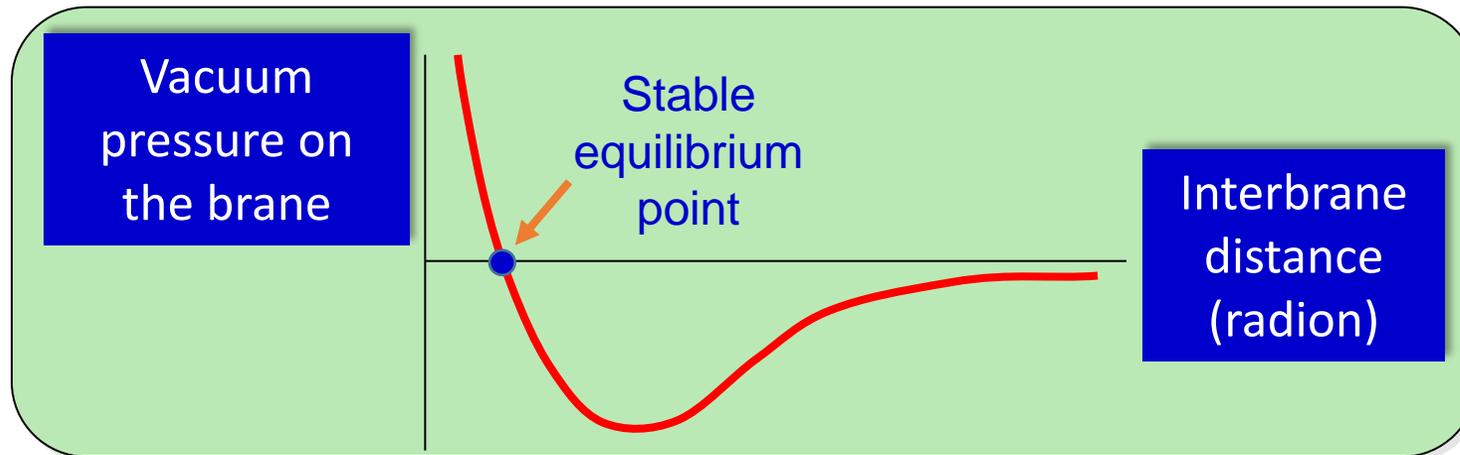
At large separations between the branes (scalar field)

$$\textit{Interaction force} \sim \begin{cases} e^{-2\nu(z_2 - z_1)/a} & \text{for the brane at } z = z \\ e^{-(2\nu + D)(z_2 - z_1)/a} & \text{for the brane at } z = z_1 \end{cases} \qquad \nu = \sqrt{D^2/4 - D(D+1)\xi + m^2 a^2}$$

- For Dirichlet scalar the interaction forces are attractive

Vacuum forces

■ In general case, in dependence of the coefficients in the boundary conditions the vacuum interaction forces between the branes can be either attractive or repulsive

■ There is a region in the region in the space of the Robin parameter in which the interaction forces are repulsive at small separations and attractive at large distances

Stabilization of the interbrane distance (radion) by the vacuum forces

Induced cosmological constant

- Vacuum expectation value of the surface EMT on the brane at $z = z_j$

$$\langle 0 | T_M^{(s)N} | 0 \rangle = \operatorname{diag}(\varepsilon_j^{(s)}, \ldots, -p_j^{(s)}, \ldots), \quad \varepsilon_j^{(s)} = -p_j^{(s)}$$

- Corresponds to the generation of the cosmological constant on the branes by quantum effects

- Induced cosmological constant is a function of the interbrane distance, AdS curvature radius, and of the coefficients in the boundary conditions

- In dependence of these parameters the induced cosmological constant can be either positive or negative

■ *D*-dimensional Newton's constant G_{Dj} measured by an observer on the brane at $z = z_j$ is related to the fundamental *(D+1)*-dimensional Newton's constant G_{D+1} by the formula

$$G_{Dj} = \frac{(D-2)G_{D+1}\,/\,a}{e^{(D-2)(z_2-z_1)/a} - 1}\, e^{(D-2)(z_2-z_j)/a}$$

■ For large interbrane distances the gravitational interactions on the brane $z = z_2$ are exponentially suppressed

■ This feature is used in the Randall-Sundrum model to address the hierarchy problem

■ Same mechanism also allows to obtain a naturally small cosmological constant on the brane generated by vacuum fluctuations

$$\Lambda_{Dj} = 8\pi G_{Dj}\,\varepsilon_j^{(s)} \sim 8\pi G_{Dj} M_{Dj}^{D}\, e^{-(D+2\nu)(z_2-z_1)/a}$$

Cosmological constant on the brane $z = z_j$

Effective Planck mass on the brane $z = z_j$

Geometry

■ (D+1)-dimensional AdS spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{-2y/a}\eta_{ik} dx^i dx^k - dy^2 \qquad \eta_{ik} = \mathrm{diag}(1, -1, \ldots, -1)$$

■ New coordinate $z = ae^{y/a}, \ 0 \leqslant z < \infty$

$$ds^2 = (a/z)^2(\eta_{ik} dx^i dx^k - dz^2)$$

■ AdS boundary ➡ z=0, horizon ➡ z=∞

■ Branes perpendicular to AdS boundary

Proper distance between the branes $(a/z)(a_2 - a_1)$
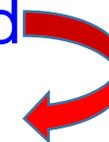
Field, boundary conditions and VEVs

■ Scalar field with general curvature coupling parameter

$$(g^{ik}\nabla_i\nabla_k + m^2 + \xi R)\varphi(x) = 0, \ \ R = -D(D+1)/\alpha^2$$

■ Boundary conditions $(1 + \beta_j n_j^i \nabla_i)\varphi(x) = 0, \ n_j^i$ ⬅ Normal to the brane at $x^1 = a_j$

■ Boundary condition imposed on the field modify the spectrum of vacuum fluctuations and the VEVs of physical observables are changed

Casimir effect

■ We are interested in the VEVs of the field squared and of the energy-momentum tensor

■ Scheme of calculation:

Two-point function ⟹ Mean field squared ⟹ VEV of the energy-momentum tensor
⟹ Forces acting on the branes

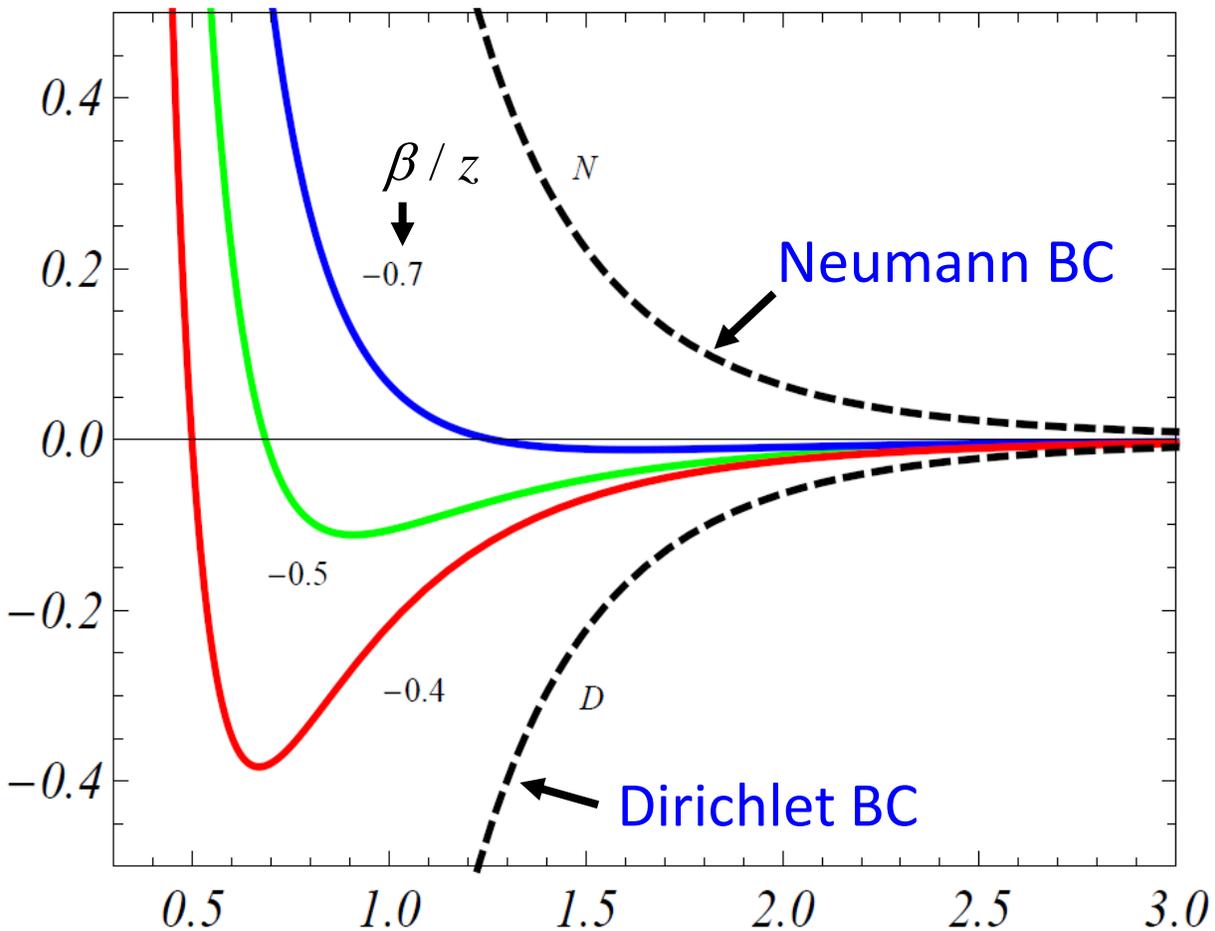- Geometry with a single brane at $x^1 = 0$

Boundary condition on the brane

$$(1 + \beta \partial_1)\, \varphi(x) = 0, \quad x^1 = 0$$

Brane-induced vacuum energy density as a function of the proper distance measured in units of the AdS curvature radius

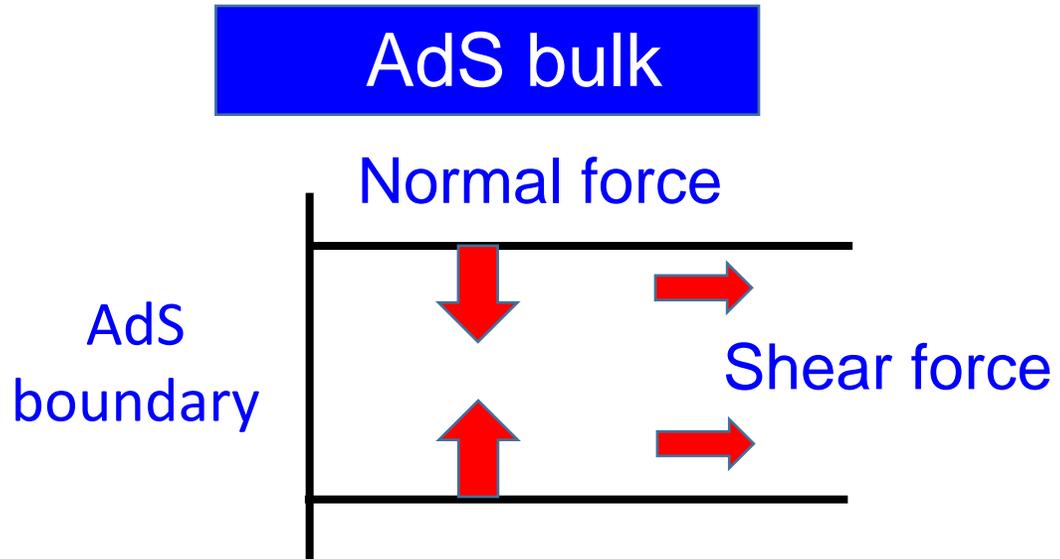$$10^4 \alpha^{D+1} <T^0_0>_b$$

$$D = 4, \ m\alpha = 0.5$$

Conformally coupled field

Minimally coupled field

$$x^1 / z \longrightarrow$$

The Casimir forces

AdS bulk

Minkowski bulk

Normal force

Normal force

AdS boundary

Shear force

Directions of the forces depend on the coefficients in the boundary conditions on the branes

Models with compact dimensions

■ (D+1)-dimensional AdS spacetime

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{-2y/a}\eta_{ik}dx^i dx^k - dy^2 \qquad \eta_{ik} = \mathrm{diag}(1,-1,\ldots,-1)$$

■ New coordinate $\quad z = ae^{y/a}, \ 0 \leqslant z < \infty$
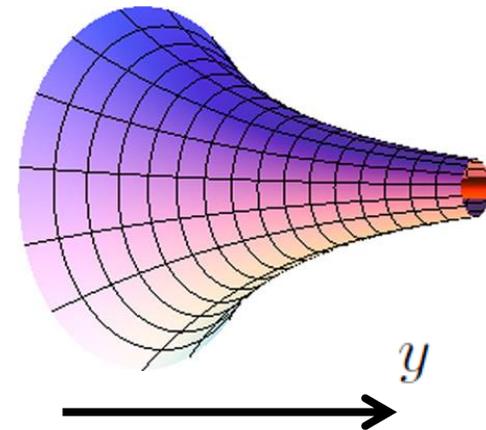
$$ds^2 = (a/z)^2(\eta_{ik}dx^i dx^k - dz^2)$$

■ Topology $\quad R^p \times (S^1)^q, \ q+p = D-1$

q-dimensional torus

■ Cartesian coordinates along uncompactified and compactified dimensions

$$\mathbf{x}_q = (x^{p+1},\ldots,x^{D-1}) \qquad \mathbf{x}_p = (x^1,\ldots,x^p)$$

Compactification $\quad R^1 \implies S^1$

$L_l \implies$ length of the l-th compact dimension
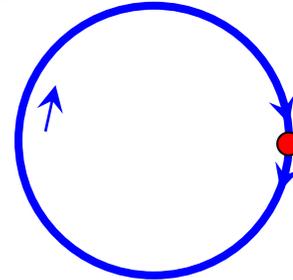
y

Fields

■ Charged scalar field with general curvature coupling

$$\left(g^{\mu\nu} D_\mu D_\nu + m^2 + \xi R\right) \varphi(x) = 0, \, D_\mu = \nabla_\mu + ie A_\mu$$

External classical gauge field

■ In models with nontrivial topology one need also to specify the periodicity conditions obeyed by the field operator along compact dimensions

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l} \varphi(t, \mathbf{x}_p, \mathbf{x}_q)$$

■ Special cases:
- Untwisted fields $\alpha_l = 0$
- Twisted fields $\alpha_l = \pi$

■ We assume that the gauge field is constant: $A_\mu = \text{const}$

■ Though the corresponding field strength vanishes, the nontrivial topology gives rise to Aharonov-Bohm-like effects
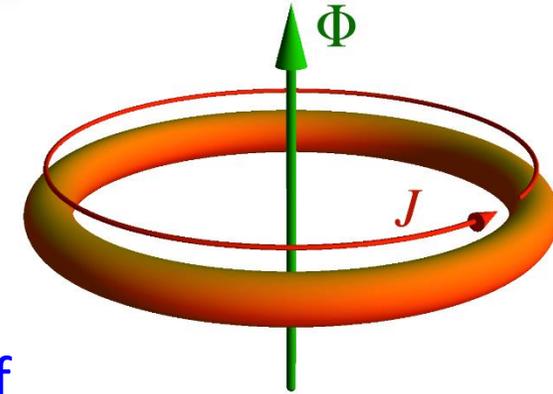
Current density

- We are interested in the effects of non-trivial topology and gravity on the vacuum expectation value (VEV) of the current

$$j_\mu(x) = ie[\varphi^+(x)D_\mu\varphi(x) - (D_\mu\varphi^+(x))\varphi(x)]$$

- This VEV is among the most important quantities that characterize the properties of the quantum vacuum

- Although the corresponding operator is local, due to the global nature of the vacuum, the VEV carries important information about the global properties of the background space-time

- Current acts as the source in the Maxwell equations and therefore plays an important role in modeling a self-consistent dynamics involving the electromagnetic field
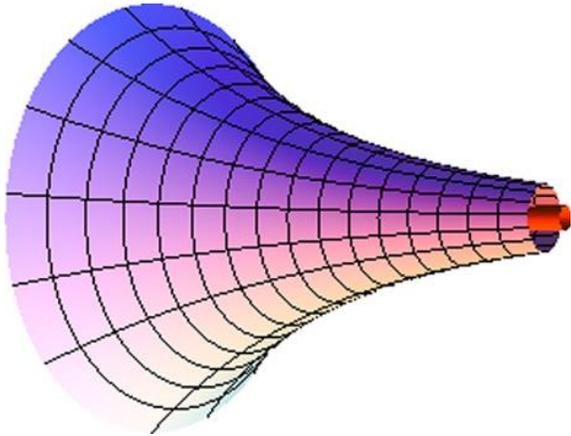
Analog from condensed matter physics: Persistent currents

Φ

J

- Persistent currents in metallic rings are predicted in *M. Büttiker, Y. Imry, R. Landauer, Phys. Lett. A **96**, 7 (1983)*

- Existence of persistent currents in normal metal rings is a signature of phase coherence in mesoscopic systems and an example of the Aharonov-Bohm effect

- Temperature must be sufficiently low to reduce the probability of inelastic scattering and the circumference of the ring short enough that the phase coherence of the electronic wave functions is preserved around the loop

- Measurements of persistent currents in nanoscale gold and aluminum rings: A.C. Bleszynski-Jayich et. al., *Science* **326** (2009); H. Bluhm et. al., *Phys. Rev. Lett.* **102** (2009)

Induced currents in models with compact dimensions

■ In the problem under consideration the presence of a constant gauge field is equivalent to the magnetic flux enclosed by the compact dimension

■ Flux of the field strength which threads the *l*-th compact dimension

$$A_l L_l = \oint dx^l A_l$$

■ Gauge transformation

$$\varphi = \varphi' e^{-ie\chi}, \ A_\mu = A'_\mu + \partial_\mu \chi \quad \chi = A_\mu x^\mu$$

■ Problem with a constant gauge field is reduced to the problem in the absence of the gauge field with the shifted phases in the periodicity conditions:

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\widetilde{\alpha}_l} \varphi'(t, \mathbf{x}_p, \mathbf{x}_q), \ \ \widetilde{\alpha}_l = \alpha_l + e A_l L_l$$

Vacuum current density

■ Charge density vanishes

■ Components of the current density along uncompact dimensions vanish

■ Current density along *l*-th compact dimension

$$\langle j^l \rangle = \frac{4ea^{-1-D}L_l}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1}) q_{\nu-1/2}^{(D+1)/2} (1 + g_{\mathbf{n}_q}^2 /(2z^2))$$

$$\mathbf{n}_{q-1} = (n_{p+1}, \dots, n_{l-1}, n_{l+1}, \dots, n_{D-1}), \quad -\infty < n_i < +\infty, \ i \neq l$$

$$\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1} = \sum_{i=1,\neq l}^{D-1} \tilde{\alpha}_i n_i, \ g_{\mathbf{n}_q} = \left(\sum_{i=p+1}^{D-1} n_i^2 L_i^2 \right)^{1/2} \qquad \tilde{\alpha}_l = \alpha_l + eA_l L_l$$
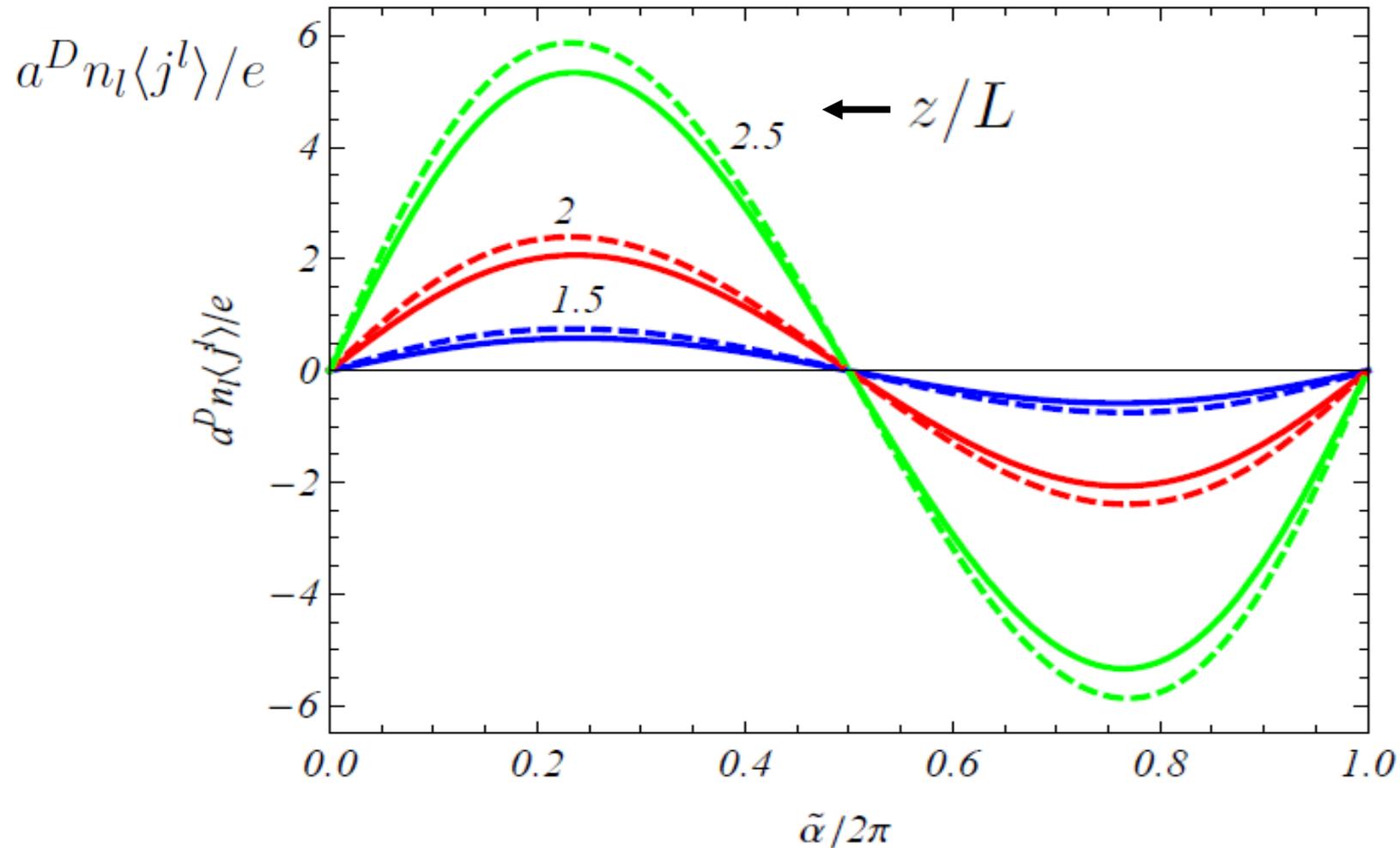
$$q_\alpha^\mu(x) = \frac{e^{-i\pi\mu} Q_\alpha^\mu(x)}{(x^2 - 1)^{\mu/2}} \quad \leftarrow \quad \text{Associated Legendre function of the second kind}$$

- Current density along the *l*-th compact dimension is an odd periodic function of the phase $\tilde{\alpha}_l$ and an even periodic function of the phases $\tilde{\alpha}_i,\ i \neq l$, with the period 2π

- In particular, the current density is a periodic function of the magnetic fluxes with the period equal to the flux quantum $2\pi/|e|$

- In the absence of the gauge field, the current density along the *l*-th compact dimension vanishes for untwisted and twisted fields along that direction

- Charge flux through (*D -1*)-dimensional hypersurface $x^l = \mathrm{const}$

$$n_l \langle j^l \rangle,\ n_l = a/z \impliedby \text{Normal to the hypersurface}$$

- Charge flux depends on the coordinate lengths of the compact dimensions in the form of

$$L_i/z \ = \text{proper length of the compact dimension, measured}$$
in units of the AdS curvature radius

$$L_{(p)i} = aL_i/z \impliedby \text{ proper length of the compact dimension}$$

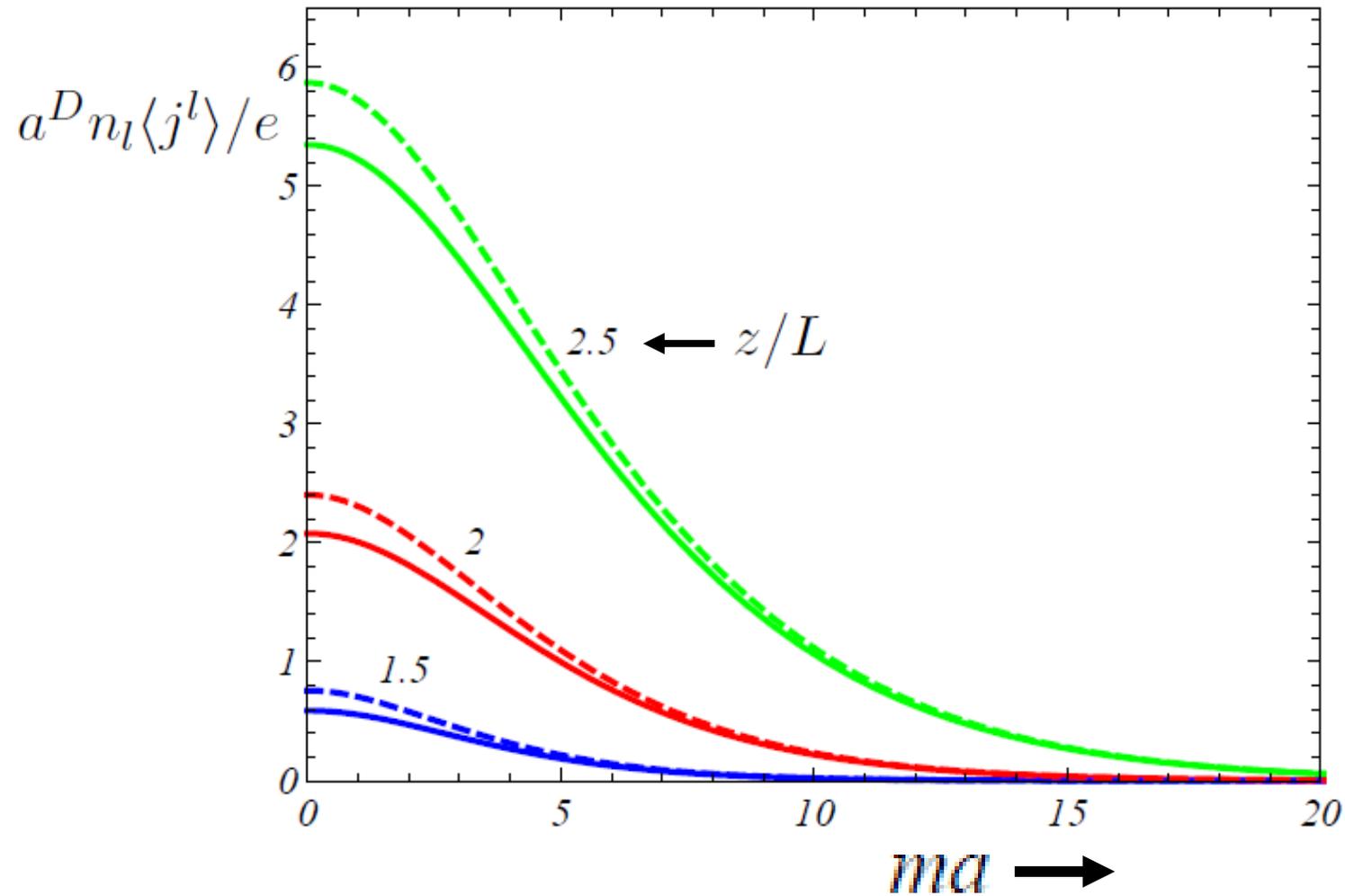# Numerical example

- D=4 minimally (full curves) and conformally (dashed curves) coupled fields
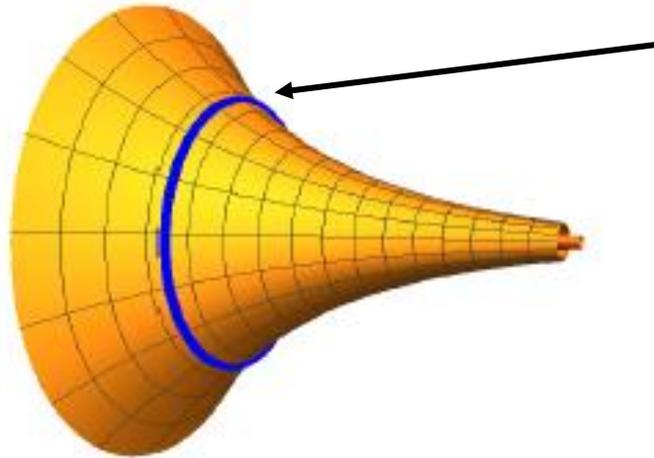
- Single compact dimension

$$\tilde{\alpha} = \pi/2$$

Brane at $\quad y = y_0$

■ Boundary condition on the brane

$$(1 + \beta n^{\mu} D_{\mu})\varphi(x) = 0, \quad y = y_0$$
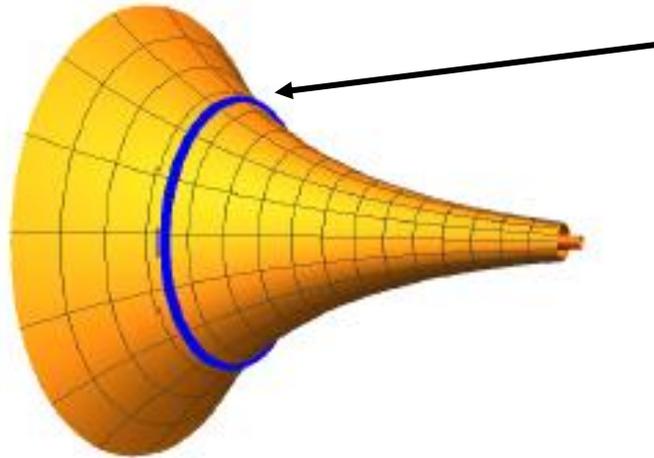
Constant   Normal to the brane

■ Robin boundary condition

■ Special cases: Dirichlet $(\beta = 0)$ and Neumann $(\beta = \infty)$

■ There is a region in the space of the parameter β in which the vacuum becomes unstable

■ Critical value for the Robin coefficient depends on the lengths of the compact dimensions, on the phases in periodicity conditions and on the mass of the field

Brane at $y = y_0$

🟩 Boundary condition on the brane

$$(1 + \beta n^\mu D_\mu)\varphi(x) = 0, \quad y = y_0$$

Constant   Normal to the brane

🟩 Robin boundary condition

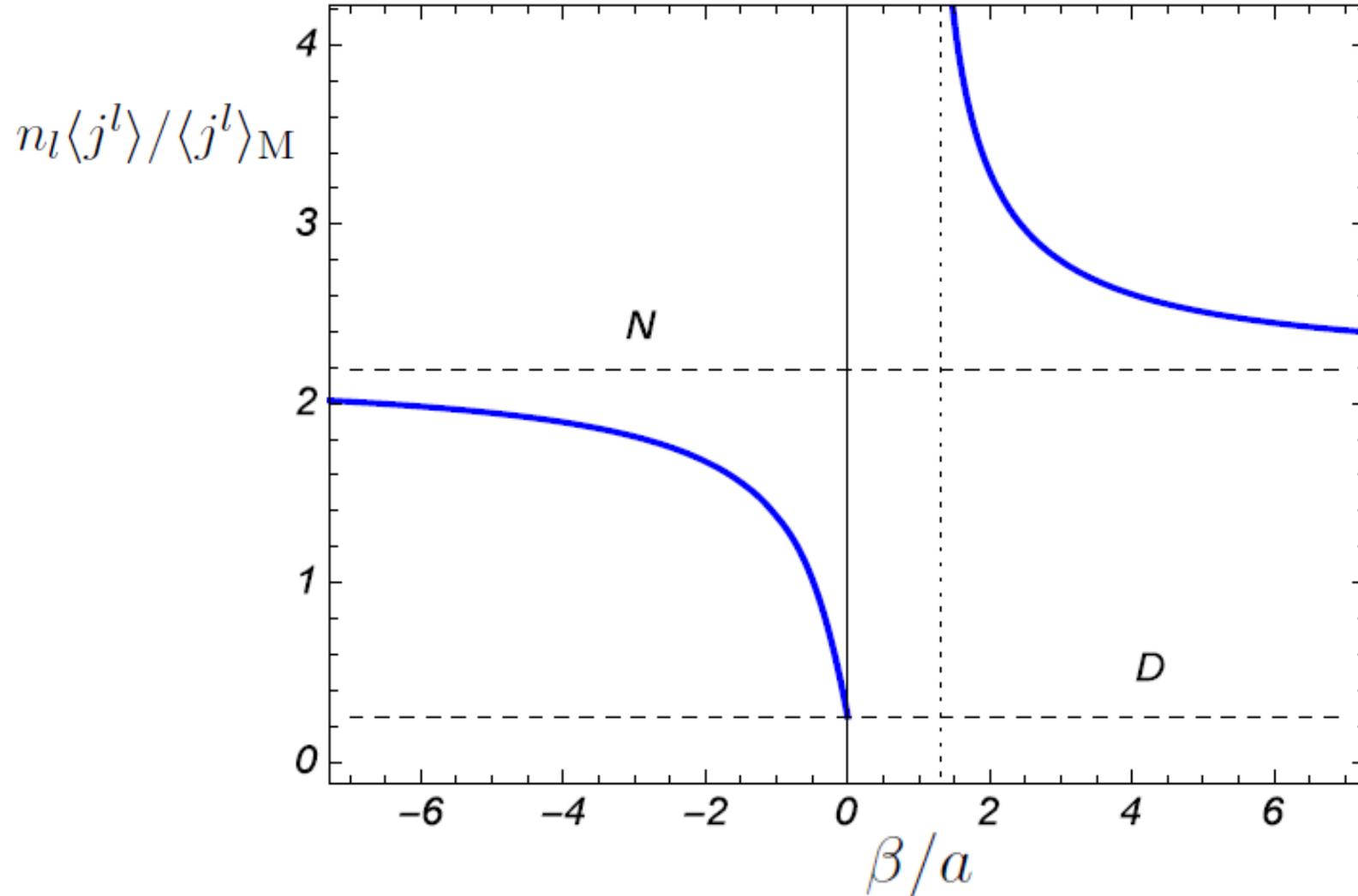🟩 Special cases: Dirichlet $(\beta = 0)$ and Neumann $(\beta = \infty)$

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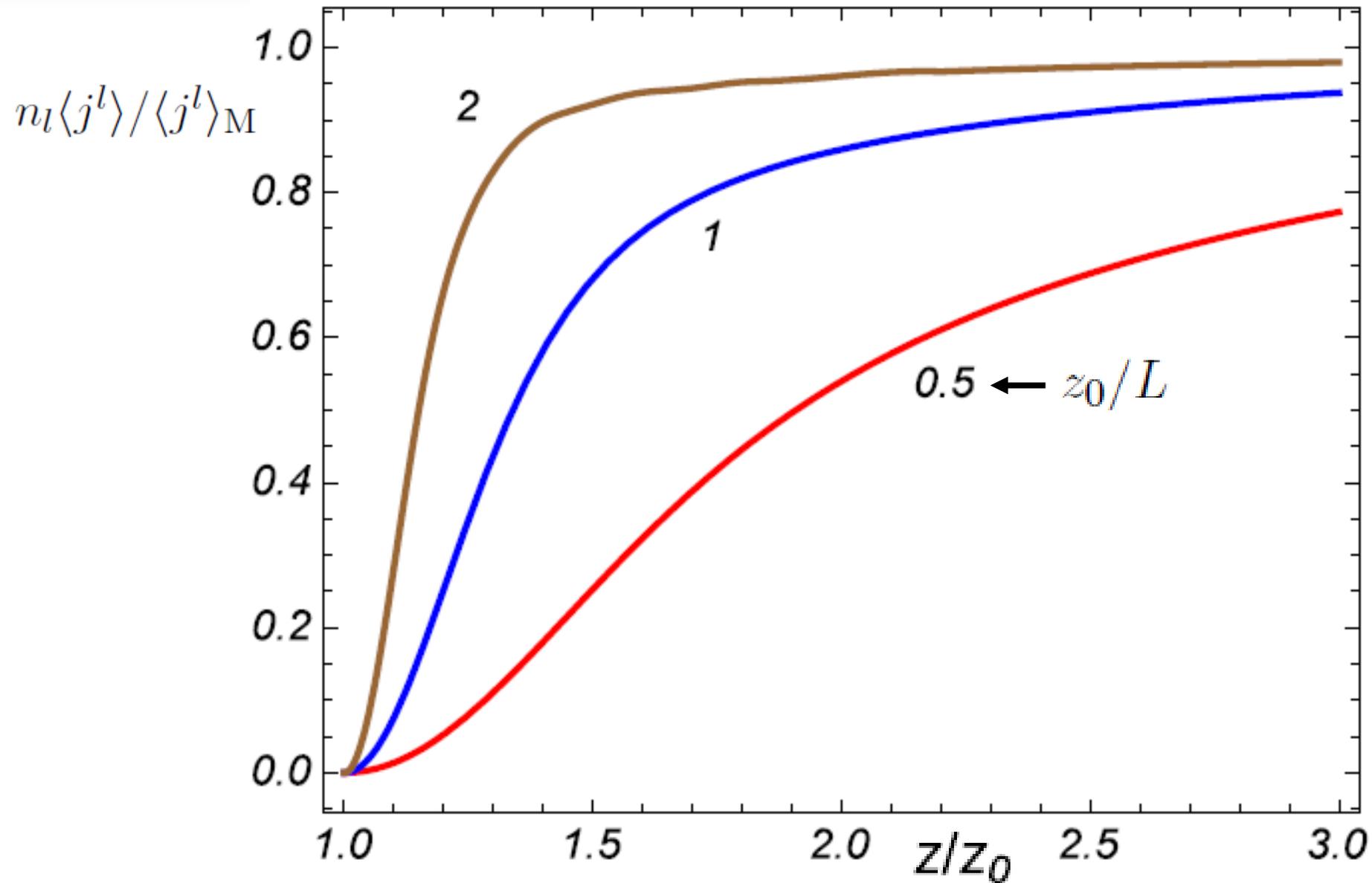
D = 4 AdS space with a single compact dimension

$$\tilde{\alpha} = \pi/2, \ z/z_0 = 1.2, \ z_0/L = 1$$

Conclusions

- Depending on the field and boundary conditions, the vacuum energy density can be either positive or negative

- The vacuum forces can be either attractive or repulsive

- The effect of gravity is essential at distances from the brane larger than the curvature radius

- In models with branes perpendicular to the AdS boundary off-diagonal component of the vacuum energy-momentum tensor is induced that gives arise to shear force acting on branes

- VEVs vanish on the AdS boundary and diverge on the horizon

- Signs for both the normal and shear Casimir forces depend on the Robin coefficients

Conclusions

- Charge density and the components along the uncompactified dimensions vanish

- Current density along compactified dimensions is a periodic function of the magnetic flux with the period of the flux quantum

- Current density vanishes on the AdS boundary

- Near the horizon the effects induced by the background curvature are small

- In Kaluza-Klein-type models the current with the components along compact dimensions is a source of cosmological magnetic fields

Thank you