

# Diffusion as the main source of dissipation in superconducting neutron stars

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# Introduction

- Oscillations of neutron stars (NSs) is an important phenomenon, since they can imprint in various observational manifestations of neutron stars, such as, e.g., gravitational signal from binary NS inspirals. Whether an oscillation mode can actually be excited depends on the efficiency of dissipative mechanisms in the stellar matter.
- So far the role of dissipation agents was attributed mostly to shear and bulk viscosities.
- Here we will analyze the role of particle diffusion in dissipation of various oscillation modes and will demonstrate that diffusion is the dominating dissipative mechanism in superconducting (SC) NSs.
- The mechanism of dissipation due to diffusion is very similar to that discussed in the context of magnetic field dissipation (e.g., [Goldreich & Reisenegger 1992](#)), with the only difference – the source of perturbation is stellar oscillations rather than the magnetic field.

# Introduction

In what follows we will consider  
normal core matter and SC core matter

Considering sound waves, p-, g-, and f- modes we will confront the  
dissipation due to diffusion to the dissipation due to shear viscosity

# Normal matter

neutrons are nonsuperfluid, protons are nonsuperconducting

each particle species is characterized with

$$e_\alpha, n_\alpha, \mu_\alpha, \mathbf{v}_\alpha$$

The velocities of different particle species are allowed to differ from each other

$$\mathbf{v}_\alpha \neq \mathbf{v}_\beta$$

However, since the hydrodynamic regime is relevant,  
due to effective particle collisions the relative velocities are small  
in comparison to the global velocity of the fluid

$$\mathbf{w}_{\alpha\beta} \equiv \mathbf{v}_\alpha - \mathbf{v}_\beta \ll \mathbf{v}_\alpha \sim \mathbf{v}_\beta$$

Hydrodynamic limit

# Normal matter

In the hydrodynamic regime, integrating Boltzmann equations, one can show that the friction force between different particle species

$$\mathbf{F}_{\alpha\beta} \propto \mathbf{w}_{\alpha\beta} \quad \mathbf{F}_{\alpha\beta} = J_{\alpha\beta} \mathbf{w}_{\alpha\beta} \quad (\text{e.g., Yakovlev, Shalybkov 1991})$$

$J_{\alpha\beta}$  is the momentum transfer rate  
to be calculated from microscopic theory

Dissipated energy  $\dot{E}_{\alpha\beta} = -\mathbf{F}_{\alpha\beta} \mathbf{w}_{\alpha\beta}$

Summing up over particle species and integrating

$$\dot{E}_{\text{diff}} = -\frac{1}{2} \int \sum_{\alpha\beta} J_{\alpha\beta} w_{\alpha\beta}^2 dV \quad \text{Braginski (1965)}$$

To calculate the energy dissipation rate we need to know  $w_{\alpha\beta}$

# Normal *npe* matter

Due to effective electromagnetic interaction the matter is quasineutral

$$n_e = n_p, \quad \mathbf{v}_e = \mathbf{v}_p \quad \text{with extremely high accuracy.}$$

$$\mathbf{v}_n \neq \mathbf{v}_e = \mathbf{v}_p \quad \mathbf{w}_{ne} = \mathbf{w}_{np}$$

To find  $\mathbf{w}_{n\alpha}$  we use linearized Euler-like equations for each particle species

$$\frac{n_{\alpha 0} \mu_{\alpha 0}}{c^2} \frac{\partial \mathbf{v}_\alpha}{\partial t} = e_\alpha n_{\alpha 0} \mathbf{E} - n_{\alpha 0} \nabla \mu_\alpha - \sum_\beta J_{\alpha\beta} \mathbf{w}_{\alpha\beta}$$

electric field
(e.g. Goldreich & Reisenegger 1992)
Yakovlev & Shalybkov 1991)
← all dissipative terms except for particle diffusion are ignored

After some algebra we express  $\mathbf{w}_{n\alpha}$  through  $\delta\mu \equiv \mu_n - \mu_p - \mu_e$  and find

$$\dot{E}_{\text{diff}} = - \int (J_{ne} w_{ne}^2 + J_{np} w_{np}^2) dV = - \int \frac{1}{J_{np} + J_{en}} \left[ \frac{n_{n0} n_{e0}}{n_{b0}} \nabla \delta\mu \right]^2 dV$$

$$J_{np}/J_{en} \sim 10^5$$

$$\dot{E}_{\text{diff}} \approx - \int \frac{1}{J_{np}} \left[ \frac{n_{n0} n_{e0}}{n_{b0}} \nabla \delta\mu \right]^2 dV$$

due to friction between neutrons and protons

# Superconducting (SC) *npe* matter

neutrons are nonsuperfluid, protons are strongly superconducting  $T \ll T_{cp}$

$$n_e = n_p, \quad \mathbf{v}_e = \mathbf{v}_p$$

$$\mathbf{v}_e \neq \mathbf{v}_n \quad \mathbf{w}_{ne} \equiv \mathbf{v}_n - \mathbf{v}_e \ll \mathbf{v}_n \sim \mathbf{v}_e$$

$$\dot{E}_{\text{diff}} = - \int J_{ne} w_{ne}^2 dV$$

Equations to calculate  $w_{ne}$

$$\frac{n_n \mu_n}{c^2} \frac{\partial \mathbf{v}_n}{\partial t} = -n_n \nabla \mu_n - J_{ne} \mathbf{w}_{ne}$$

$$\frac{n_e \mu_e}{c^2} \frac{\partial \mathbf{v}_e}{\partial t} = -en_e \mathbf{E} - n_e \nabla \mu_e + J_{ne} \mathbf{w}_{ne}$$

$$\frac{\mu_{p0}}{c^2} \frac{\partial \mathbf{v}_{sp}}{\partial t} = e\mathbf{E} - \nabla \mu_p$$

$$\dot{E}_{\text{diff}} = - \int \frac{1}{J_{en}} \left[ \frac{n_{n0} n_{e0}}{n_{b0}} \nabla \delta \mu \right]^2 dV$$

due to friction between neutrons and electrons

# Normal matter vs Superconducting matter (SC)

normal  $\dot{E}_{\text{diff}} \approx - \int \frac{1}{J_{\text{np}}} \left[ \frac{n_{\text{n}0} n_{\text{e}0}}{n_{\text{b}0}} \nabla \delta \mu \right]^2 dV$  due to friction between neutrons and protons

SC  $\dot{E}_{\text{diff}} = - \int \frac{1}{J_{\text{en}}} \left[ \frac{n_{\text{n}0} n_{\text{e}0}}{n_{\text{b}0}} \nabla \delta \mu \right]^2 dV$  due to friction between neutrons and electrons

$$J_{\text{np}} \propto J_{\text{ne}} \propto T^2$$

at low temperatures diffusion is more efficient  
thus we compare diffusion with shear viscosity

$\delta \mu$  is specified by oscillation type. Once  $\delta \mu$  is given,  $\dot{E}_{\text{diff}}$  can be calculated.

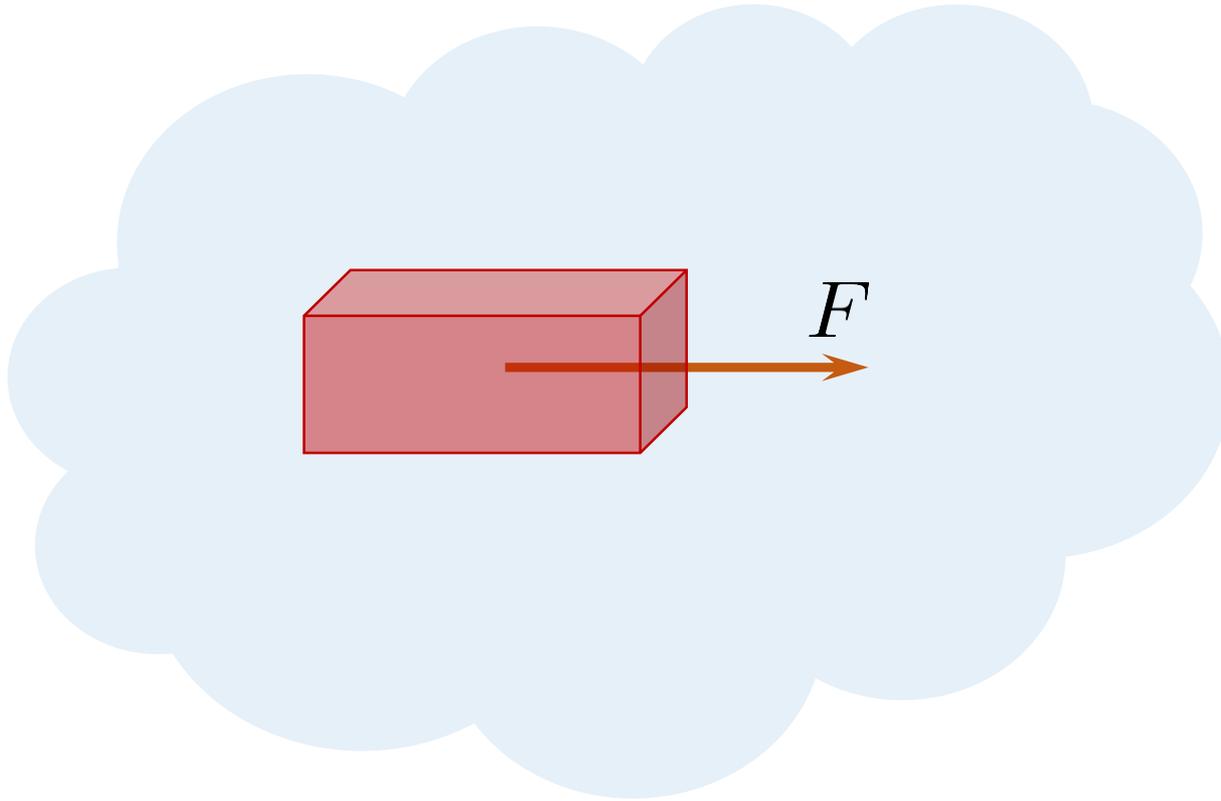
Small dissipation  $\Rightarrow \delta \mu$  from equations of non-dissipative hydrodynamics.

$\delta \mu$  does not depend on proton state.

$$\frac{\dot{E}_{\text{diff SC}}}{\dot{E}_{\text{diff N}}} = \frac{J_{\text{np}}}{J_{\text{ne}}} \sim 10^5$$

friction smaller – dissipation higher

# Friction smaller – Dissipation higher



$$\mathbf{F}_{\text{fr}} = -\gamma \mathbf{V}$$

stronger friction –  
higher  $\gamma$

in equilibrium

$$\mathbf{F} = -\mathbf{F}_{\text{fr}}$$

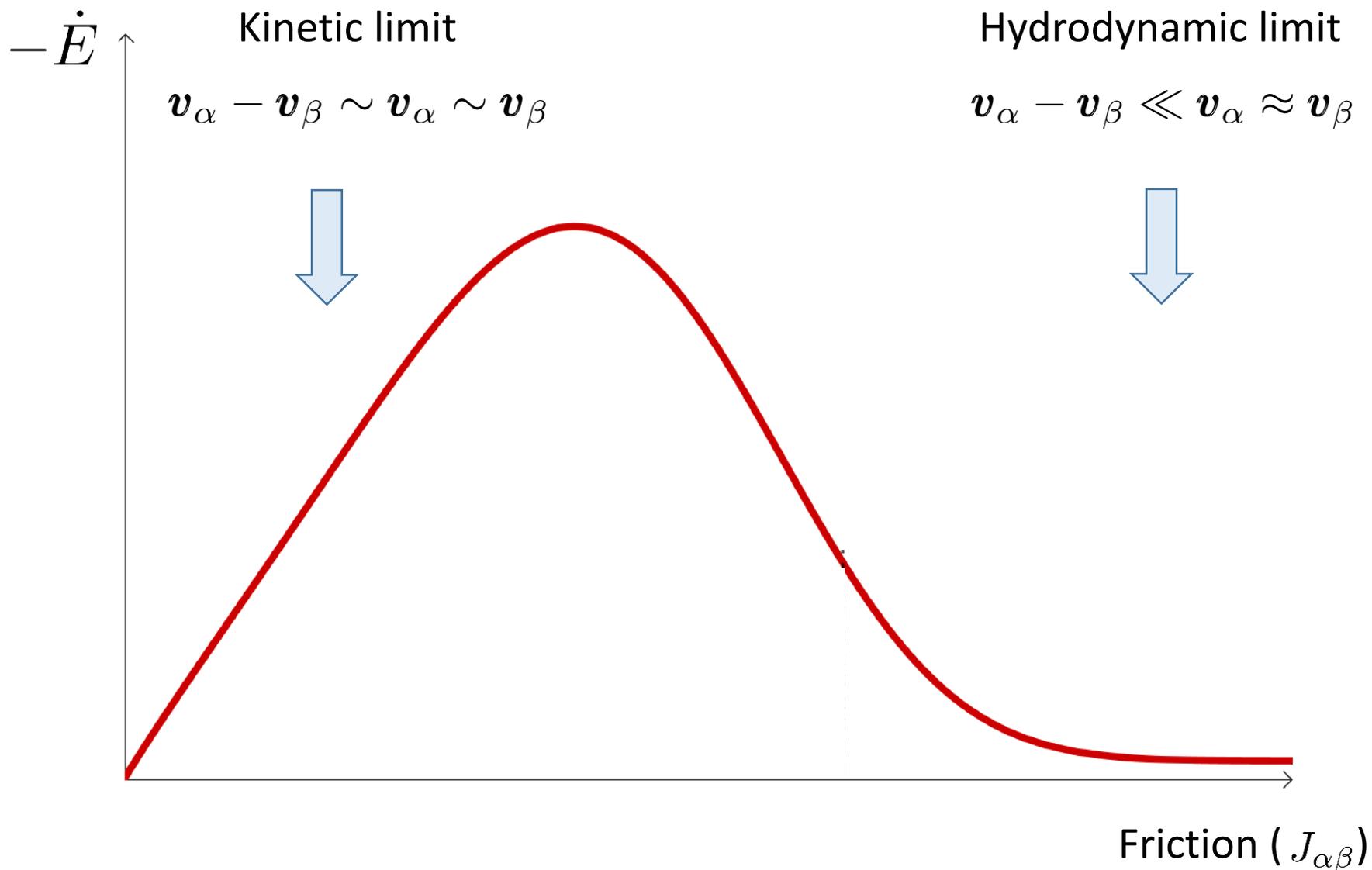
$$\mathbf{V} = \frac{\mathbf{F}}{\gamma}$$

energy dissipated in to heat per unit time

$$\dot{E} = -\mathbf{F}\mathbf{V} = -\frac{F^2}{\gamma}$$

smaller friction ( $\gamma$ ) – higher relative velocity – stronger dissipation

# Friction smaller – Dissipation higher



# Diffusion vs Shear Viscosity

$$\tau_{\text{diff}} = - \frac{2E}{\langle \dot{E}_{\text{diff}} \rangle}$$

$$\tau_{\eta} = - \frac{2E}{\langle \dot{E}_{\eta} \rangle}$$

normal

$$\dot{E}_{\text{diff}} = - \int \frac{1}{J_{\text{np}}} \left[ \frac{n_{\text{n}0} n_{\text{e}0}}{n_{\text{b}0}} \nabla \delta \mu \right]^2 dV$$

$$\dot{E}_{\eta} = - \int \frac{\eta}{2} \left( \frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \text{div } \mathbf{v} \right)^2 dV$$

$\eta$  – shear viscosity coefficient in normal matter

SC

$$\dot{E}_{\text{diff}} = - \int \frac{1}{J_{\text{en}}} \left[ \frac{n_{\text{n}0} n_{\text{e}0}}{n_{\text{b}0}} \nabla \delta \mu \right]^2 dV$$

$$\dot{E}_{\eta} = - \int \frac{\eta_{\text{L}}}{2} \left( \frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \text{div } \mathbf{v} \right)^2 dV$$

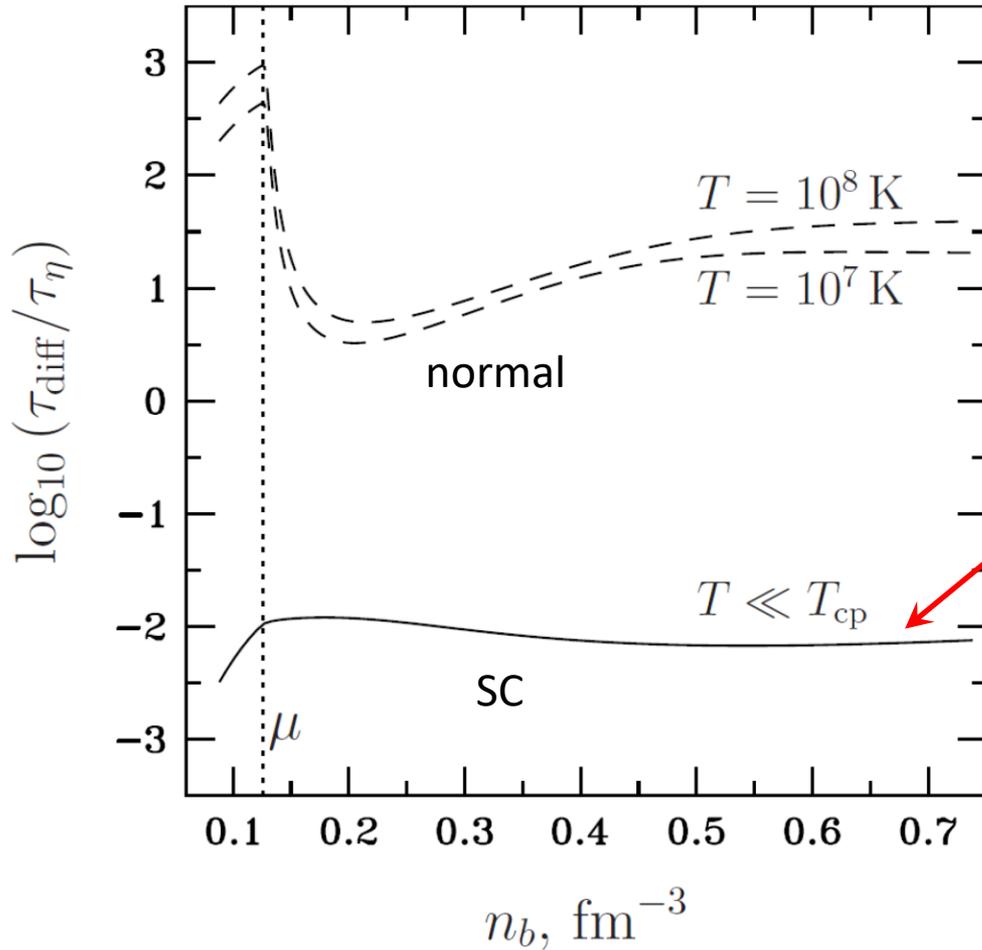
$\eta_{\text{L}}$  – shear viscosity coefficient in SC

$$v \sim v_{\text{e}} \sim v_{\text{p}} \sim v_{\text{n}}$$

$J_{\text{np}}, J_{\text{ne}}$  from [Dommes, Gusakov, Shternin PRD 2020](#)

$\eta, \eta_{\text{L}}$  from [Shternin PRD 2018](#)

# Sound waves



BSk24 EOS

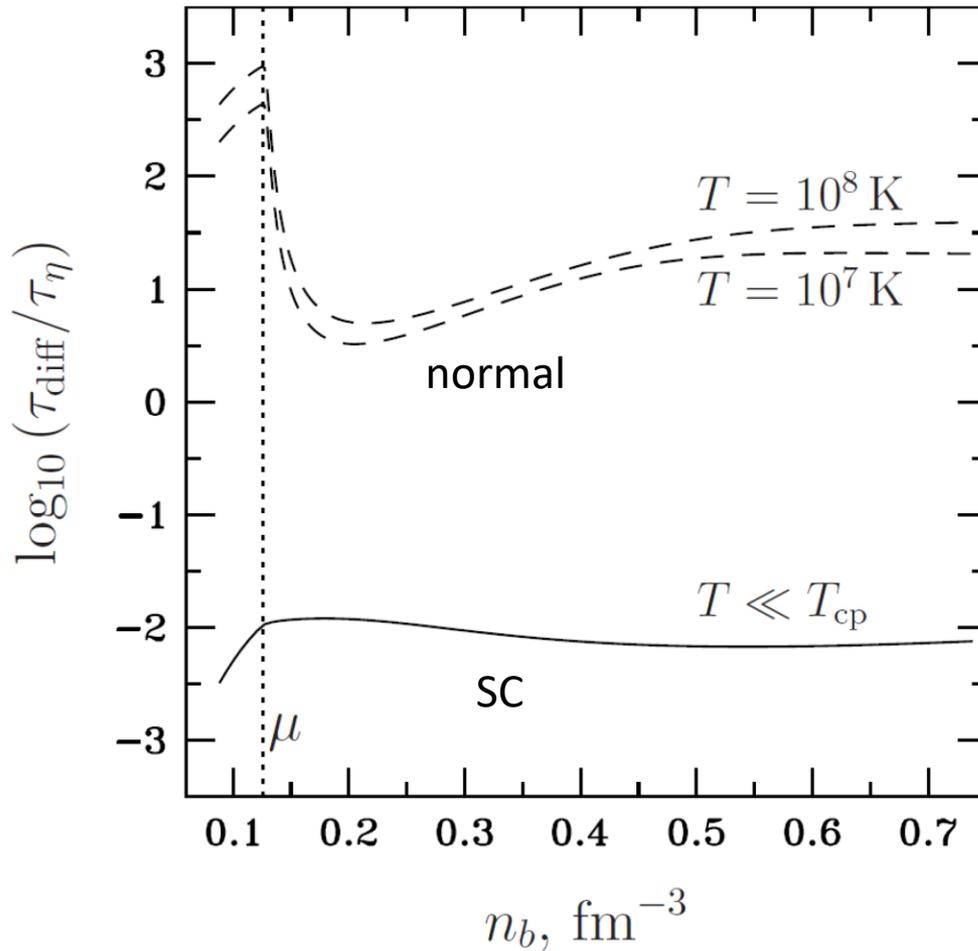
*Goriely, Chamel, Pearson, PRC, 2013*

practically temperature independent

the curves do not depend on wave frequency since

$$\tau_{\eta}, \tau_{\text{diff}} \propto \sigma^{-2}$$

# Sound waves



- In normal npe matter protons and electrons are locked by electromagnetic forces and move together; neutrons effectively scatter off protons, the relative velocity of neutrons is low, the diffusion is inefficient.
- Appearance of muons allows for relative velocity between charge particles and increases diffusion efficiency in normal matter.
- In SC neutrons do not scatter off protons and their relative motion is driven by the scattering off leptons. Effective dissipation due to diffusion!

The diffusion is by a factor of **100**  
more efficient than shear viscosity in SC

# Global oscillation modes in SC NS

We calculate eigenfrequencies and eigenfunctions of f-mode, g- and p-modes in General Relativity using Cowling approximation.

To calculate the dissipation of global oscillation modes we adopt the fully Relativistic hydrodynamics, allowing for the particle diffusion, developed in [Dommes, Gusakov, Shternin, PRD, 2020](#); [Dommes, Gusakov, submitted to PRD](#)

$$\partial_\mu S^\mu = -D_{ik} d_{(i)}^\mu d_{(k)\mu} \quad \dot{E}_{\text{diff}} = \int \partial_\mu S^\mu dV$$

$$d_{(j)\mu} \equiv {}^\perp\nabla_\mu \left( \frac{\mu_j}{T} \right) - \frac{e_j E_\mu}{T}$$

$$\mathcal{D}_{11} = \frac{n_1^2 T (J_{12} \mu_3^2 n_3^2 + J_{13} \mu_2^2 n_2^2 + J_{23} (\mu_2 n_2 + \mu_3 n_3)^2)}{c (J_{12} (J_{13} + J_{23}) + J_{13} J_{23}) (\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3)^2}$$

$$\mathcal{D}_{12} = -\frac{n_1 n_2 T (-J_{12} \mu_3^2 n_3^2 + \mu_1 \mu_2 n_1 n_2 (J_{13} + J_{23}) + \mu_3 n_3 (J_{13} \mu_2 n_2 + J_{23} \mu_1 n_1))}{c (J_{12} (J_{13} + J_{23}) + J_{13} J_{23}) (\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3)^2}$$

$$\mathcal{D}_{13} = -\frac{n_1 n_3 T (\mu_3 n_3 (\mu_1 n_1 (J_{12} + J_{23}) + J_{12} \mu_2 n_2) + \mu_2 n_2 (J_{23} \mu_1 n_1 - J_{13} \mu_2 n_2))}{c (J_{12} (J_{13} + J_{23}) + J_{13} J_{23}) (\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3)^2}$$

$$\mathcal{D}_{22} = \frac{n_2^2 T (J_{12} \mu_3^2 n_3^2 + J_{13} (\mu_1 n_1 + \mu_3 n_3)^2 + J_{23} \mu_1^2 n_1^2)}{c (J_{12} (J_{13} + J_{23}) + J_{13} J_{23}) (\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3)^2}$$

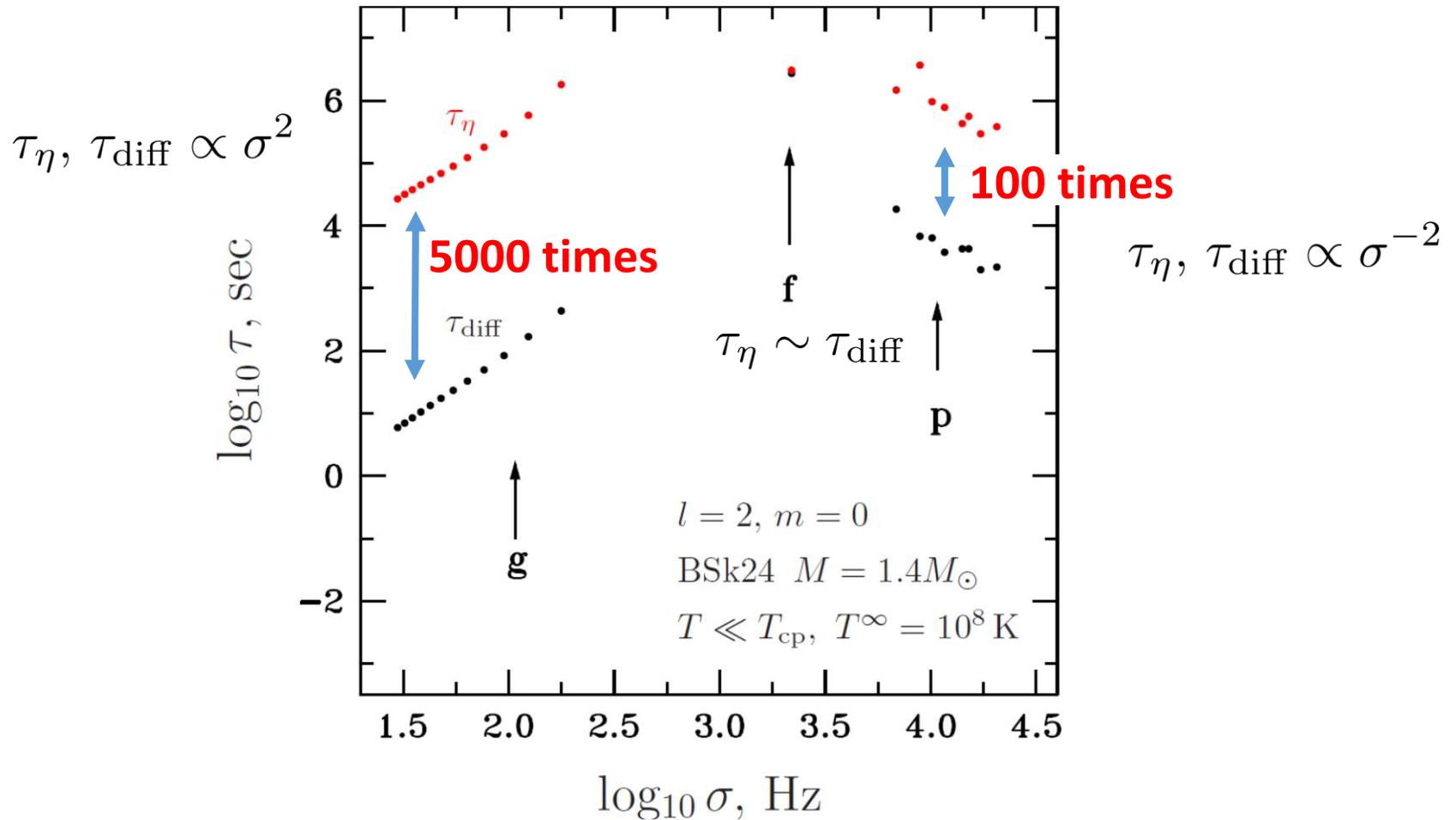
$$\mathcal{D}_{23} = -\frac{n_2 n_3 T (\mu_3 n_3 (\mu_2 n_2 (J_{12} + J_{13}) + J_{12} \mu_1 n_1) + \mu_1 n_1 (J_{13} \mu_2 n_2 - J_{23} \mu_1 n_1))}{c (J_{12} (J_{13} + J_{23}) + J_{13} J_{23}) (\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3)^2}$$

$$\mathcal{D}_{33} = \frac{n_3^2 T (J_{12} (\mu_1 n_1 + \mu_2 n_2)^2 + J_{13} \mu_2^2 n_2^2 + J_{23} \mu_1^2 n_1^2)}{c (J_{12} (J_{13} + J_{23}) + J_{13} J_{23}) (\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3)^2}$$

# Global oscillation modes in SC NS

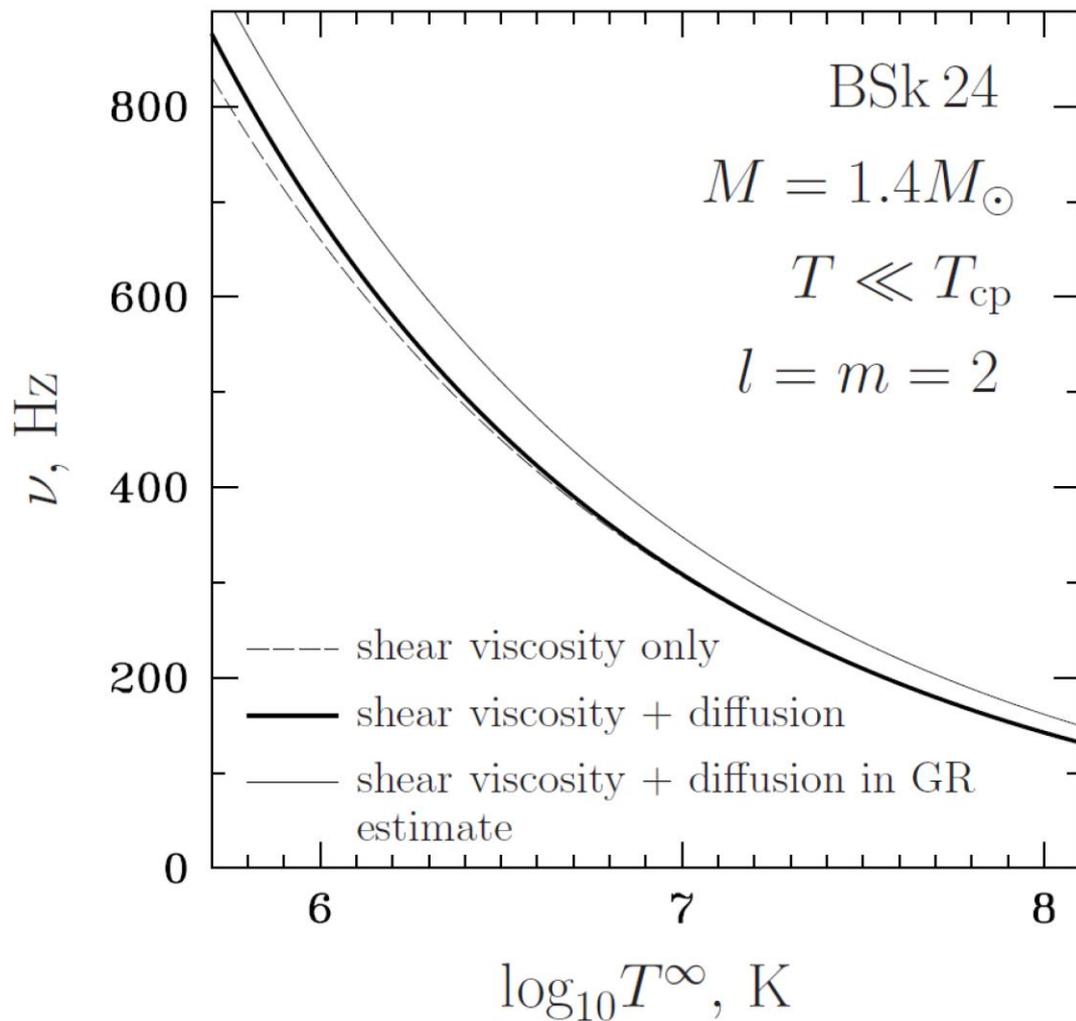
damping times versus oscillation frequency of a mode in a superconducting neutron star

Red points – shear viscosity, black points – diffusion



# Global oscillation modes in SC NS

## Instability window for r-mode



# Conclusion

Particle diffusion is the main dissipative mechanism  
in SC neutron stars

f-modes

$$\tau_{\eta} \sim \tau_{\text{diff}}$$

p-modes

$$\tau_{\eta} \sim 100 \tau_{\text{diff}}$$

g-modes

$$\tau_{\eta} \sim 5000 \tau_{\text{diff}}$$

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All the phenomenon involving excitation of oscillation modes  
should be revised

May affect the modelling of gravitational signal from the late binary NS  
inspirals:

- (i) Suppress the resonance tidal excitation of g-modes
- (ii) Suppress the p-g instability (the tidal bulge excites a pair of a low-frequency g-mode and a high-frequency p-mode)