

**IAA : Institut d'Astronomie et d'Astrophysique**  
*Université Libre de Bruxelles*



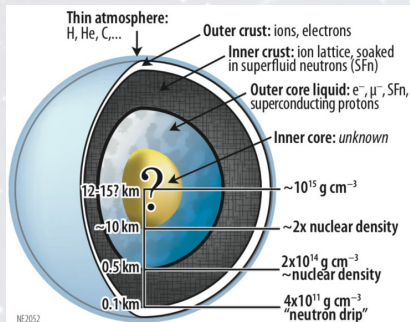
**ENTRAINMENT EFFECTS IN NEUTRON-PROTON  
SUPERFLUID MIXTURES WITHIN THE NUCLEAR  
ENERGY-DENSITY FUNCTIONAL THEORY**

**ALLARD Valentin**  
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**56th Karpacz Winter School**

# Neutron stars and nuclear superfluidity

**Nuclear superfluidity** was discovered well **before** the first discovery of NS.



- **Outer core** made of **superfluid** neutrons and **superconducting** protons.
- Nuclear superfluidity has found **strong evidence** : Pulsar **glitches** (and, NS cooling).
- Superfluid = not one but **two velocity fields**.

Constructing superfluid models of NS involves using **multifluid hydrodynamics**  $\implies$  There will be **additional microscopic inputs**.

# Entrainment effects (or *Andreev-Bashkin effects*)

The dynamics of **superfluid neutrons and protons** in the core **are not independent from each other** → Non-dissipative coupling.

## Entrainment effects

Due to the interactions between superfluids, the superfluid **flow** of **neutrons** of the mixture **entrains** the superfluid **flow** of the **protons** and vice versa.

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$$\rho_n = mn_n \mathbf{V}_{ex} + \rho_{nn} (\mathbf{V}_n - \mathbf{V}_{ex}) + \rho_{np} (\mathbf{V}_p - \mathbf{V}_{ex}) \neq mn_n \mathbf{V}_n$$

$$\rho_p = mn_p \mathbf{V}_{ex} + \rho_{pp} (\mathbf{V}_p - \mathbf{V}_{ex}) + \rho_{pn} (\mathbf{V}_n - \mathbf{V}_{ex}) \neq mn_p \mathbf{V}_p$$

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- $\rho_{qq'}$  = **Entrainment coefficients** = Intensity of the  $q - q'$  coupling ( $q, q' = n, p$ )
- **Entrainment effects** may leave their imprints in **the global dynamics** of the neutron star (e.g : frictional coupling between neutrons and electrons)

# Nuclear energy-density functional theory

## Preliminary assumptions

- NS temperature ( $\sim 10^8$  K)  $\ll$  Critical temperatures of neutrons and protons ( $\sim 10^9 - 10^{10}$  K).
- Neutrons and protons currents  $\ll$  Critical currents.

**The influence of the nuclear pairing on the entrainment matrix can be neglected** (equivalent to  $V_{ex} \simeq \mathbf{0}$  or  $T_{NS} \simeq 0$  K)

The total **energy** of a nucleon-matter system can be written as a **functional** of the **density matrix**  $n_q(\mathbf{r}\sigma, \mathbf{r}'\sigma', t)$  :

$$E(t) = E_{kin} + E_{Coul} + E_{nuc} = \int d\mathbf{r} \mathcal{E} \left[ n_q(\mathbf{r}, t), \nabla n_q(\mathbf{r}, t), \mathbf{j}_q(\mathbf{r}, t), \tau_q(\mathbf{r}, t), \dots \right]$$

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- Where :

$$n_q(\mathbf{r}, t) = \sum_{\sigma} n_q(\mathbf{r}\sigma, \mathbf{r}\sigma, t)$$

$$\mathbf{j}_q(\mathbf{r}, t) = \frac{1}{2i} (\nabla - \nabla') n_q(\mathbf{r}\sigma, \mathbf{r}'\sigma, t) |_{\mathbf{r}=\mathbf{r}'}$$

$$\tau_q(\mathbf{r}, t) = \sum_{\sigma} \nabla \cdot \nabla' n_q(\mathbf{r}\sigma, \mathbf{r}'\sigma, t) |_{\mathbf{r}=\mathbf{r}'}$$

Minimizing  $E(t)$  yields the **time-dependent Hartree-Fock (TDHF)** equations :

$$i\hbar\partial_t n_q(\mathbf{r}\sigma, \mathbf{r}'\sigma', t) = h_q(\mathbf{r}, t)n_q(\mathbf{r}\sigma, \mathbf{r}'\sigma', t) - h_q^*(\mathbf{r}', t)n_q(\mathbf{r}\sigma, \mathbf{r}'\sigma', t)$$

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Where  $h_q(\mathbf{r}, t)$  is the **single-particle hamiltonian** given by :

$$h_q(\mathbf{r}, t) = -\nabla \cdot \frac{\hbar^2}{2m_q^\oplus(\mathbf{r}, t)} \nabla + U_q(\mathbf{r}, t) + \frac{1}{2i} [\mathbf{I}_q(\mathbf{r}, t) \cdot \nabla + \nabla \cdot \mathbf{I}_q(\mathbf{r}, t)]$$

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Where the different fields (depending on the density matrix) are defined by :

$$\frac{\hbar^2}{2m_q^\oplus(\mathbf{r}, t)} = \frac{\delta E}{\delta \tau_q(\mathbf{r}, t)}, \quad U_q(\mathbf{r}, t) = \frac{\delta E}{\delta n_q(\mathbf{r}, t)}, \quad \mathbf{I}_q(\mathbf{r}, t) = \frac{\delta E}{\delta \mathbf{j}_q(\mathbf{r}, t)}$$

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**TDHF equations are self-consistent and highly non-linear !**

# Mass current and Skyrme functionals

A continuity equation can be derived from the TDHF equations

$$\partial_t (mn_q(\mathbf{r}, t)) + \nabla \cdot \boldsymbol{\rho}_q(\mathbf{r}, t) = \partial_t \rho_q(\mathbf{r}, t) + \nabla \cdot \boldsymbol{\rho}_q(\mathbf{r}, t) = 0$$

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Giving the mass-current  $\boldsymbol{\rho}_q$  :

$$\boldsymbol{\rho}_q(\mathbf{r}, t) = \frac{m}{m_q^\oplus(\mathbf{r}, t)} \hbar \mathbf{j}_q(\mathbf{r}, t) + mn_q(\mathbf{r}, t) \frac{\mathbf{I}_q(\mathbf{r}, t)}{\hbar}$$

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We consider **generalized** Skyrme nuclear energy-density functionals:

$$E_{nuc} = \int d\mathbf{r} \mathcal{E}_{Skyrme}(\mathbf{r})$$

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We consider **generalized** Skyrme nuclear energy-density functionals:

$$E_{nuc} = \int d\mathbf{r} \mathcal{E}_{Skyrme}(\mathbf{r})$$

The nuclear energy terms contributing to the currents are given by :

$$\mathcal{E}_{nuc}^j = C_0^\tau(n) \left[ n_0(\mathbf{r}, t) \tau_0(\mathbf{r}, t) - \mathbf{j}_0(\mathbf{r}, t)^2 \right] + C_1^\tau(n) \left[ n_1(\mathbf{r}, t) \tau_1(\mathbf{r}, t) - \mathbf{j}_1(\mathbf{r}, t)^2 \right]$$

where  $0 = n + p =$  is the isoscalar index and  $1 = n - p =$  is the isovector index.

The superfluid velocity is :

$$\mathbf{V}_q = \frac{\hbar}{\rho_q(\mathbf{r}, t)} \mathbf{j}_q(\mathbf{r}, t) = \frac{\hbar}{mn_q(\mathbf{r}, t)} \mathbf{j}_q(\mathbf{r}, t)$$

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Writing  $m_q^\oplus(\mathbf{r}, t)$  and  $\mathbf{I}_q(\mathbf{r}, t)$  in terms of  $\mathbf{V}_q$  and using the mass-current formula gives the entrainment matrix :

## Entrainment matrix

$$\rho_{qq'} = mn_q \left[ \delta_{qq'} + \frac{2}{\hbar^2} \left( \frac{\delta E_{Nuc}^j}{\delta X_0(\mathbf{r}, t)} - \frac{\delta E_{Nuc}^j}{\delta X_1(\mathbf{r}, t)} \right) mn'_q (2\delta_{qq'} - 1) \right] \quad (q, q' = n, p)$$

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- Evaluating the derivatives  $\delta E_{Nuc}/\delta X_0$  and  $\delta E_{Nuc}/\delta X_1$  at the static ground state (i.e : where  $\mathbf{j}_q = \mathbf{0}$ ) gives the entrainment matrix obtained using the Fermi-Liquid theory.
- The **mass currents**  $\rho_q$  obtained using the TDHF equations **may depend nonlinearly on the superfluid velocities**.

# Parametrization of the entrainment matrix

The Galilean invariance implies the following identities :

$$\rho_n = mn_n = \rho_{nn} + \rho_{np} \quad \rho_p = mn_p = \rho_{pp} + \rho_{pn} \quad \rho_{np} = \rho_{pn}$$

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The **entrainment matrix** can thus be **parametrized** using **only one parameter** such as the **determinant of the entrainment matrix**  $\Upsilon(\rho) = m/m_v^\oplus$  (= inverse of the isovector effective mass) :

$$\Upsilon = \frac{\rho_{nn}\rho_{pp} - \rho_{np}^2}{\rho_n\rho_p} = \left( \frac{m}{m_n^\oplus} - \frac{n_n}{n_p} \frac{m}{m_p^\oplus} \right) \left( 1 - \frac{n_n}{n_p} \right)^{-1} = 1 + \frac{2}{\hbar^2} \left( \frac{\delta E_{Nuc}}{\delta X_0} - \frac{\delta E_{Nuc}}{\delta X_1} \right) \rho$$

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Such parametrization decouples the  $\rho$  and  $\eta$  dependencies :

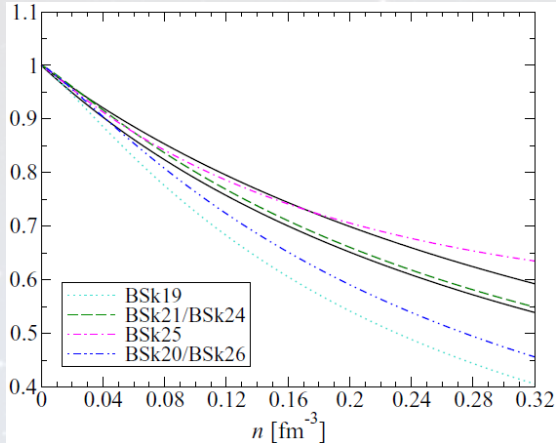
$$\eta = (n_n - n_p)/(n_n + n_p) \quad \rho = \rho_n + \rho_p$$

$$\rho_{nn} = \frac{1}{2}\rho(1 + \eta) \left[ 1 - \frac{1}{2}(1 - \eta)(1 - \Upsilon) \right]$$

$$\rho_{pp} = \frac{1}{2}\rho(1 - \eta) \left[ 1 - \frac{1}{2}(1 + \eta)(1 - \Upsilon) \right]$$

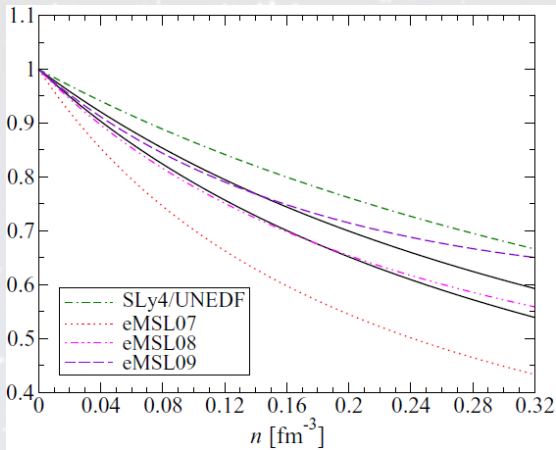
$$\rho_{np} = \rho_{pn} = \frac{1}{4}\rho(1 - \eta^2)(1 - \Upsilon)$$

The  $\Upsilon = \frac{m}{m_v^\oplus}$  parameter is **independent of the composition (i.e  $\eta$ ) for Skyrme-like functionals** and it can be estimated from functionals fitted to various nuclear data.



- Solid lines = microscopic calculations of effective masses (LNS and  $\text{Sk}\chi\text{m}^*$ ).
- **BSk** fitted to **nuclear masses** and different **neutron-matter equations of state** (BSk19-BSk21) or **symmetry energies** (BSk22-BSk25).

BSk21, BSk24 BSk25 consistent with microscopic predictions. Moreover, **unified EoS using the BSk family have been recently calculated.**



- Solid lines = microscopic calculations of effective masses (LNS and  $\text{Sk}\chi\text{m}^*$ ).
- **eMSL fitted to a few nuclear masses and different effective masses.**

The eMSL family gives similar predictions as the BSk ones although **no unified EoS using eMSL have been constructed.**

# Conclusions and prospects

## Conclusions

- We have **analytical formula for the entrainment matrix** of neutron and proton mixture at  $T=0K$ .

<https://doi.org/10.1103/PhysRevC.100.065801>

- Applicable to a large class of Energy density functionals and some of which (like the BSk family) have unified EoS  $\rightarrow$  We have **consistent calculations of the EoS and entrainment !**

## Prospects

- Entrainment at **finite temperature**  $\rightarrow$  **Three fluid model** : neutron superfluid, proton superconductor and normal fluid (thermal excitations).
- The problem **can be reduced to a two-fluid problem** with a **temperature dependent entrainment matrix**.
- We won't use the TDHF equations but we will use the **time-dependent Hartree-Fock Bogoliubov equations instead**.

Thank you for your attention

# Backup slides

# Skyrme functionals (1)

The Skyrme interaction is given by :

$$\begin{aligned}v_{ij}^{(Skyrme)} = & t_0 (1 + x_0 P_\sigma) \delta(\mathbf{r}_{ij}) + \frac{1}{2\hbar^2} t_1 (1 + x_1 P_\sigma) \left[ \mathbf{p}_{ij}^2 \delta(\mathbf{r}_{ij}) + \delta(\mathbf{r}_{ij}) \mathbf{p}_{ij}^2 \right] \\ & + \frac{t_2}{\hbar^2} (1 + x_2 P_\sigma) \mathbf{p}_{ij} \cdot \delta(\mathbf{r}_{ij}) \mathbf{p}_{ij} + \frac{1}{6} t_3 (1 + x_3 P_\sigma) n(\mathbf{r})^\alpha \delta(\mathbf{r}_{ij}) \\ & + \frac{1}{2\hbar^2} t_4 (1 + x_4 P_\sigma) \left[ \mathbf{p}_{ij}^2 n(\mathbf{r})^\beta \delta(\mathbf{r}_{ij}) + \delta(\mathbf{r}_{ij}) n(\mathbf{r})^\beta \mathbf{p}_{ij}^2 \right] \\ & + \frac{t_5}{\hbar^2} (1 + x_5 P_\sigma) \mathbf{p}_{ij} \cdot n(\mathbf{r})^\gamma \delta(\mathbf{r}_{ij}) \mathbf{p}_{ij} + \frac{i}{\hbar^2} W_0 (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) \cdot \mathbf{p}_{ij} \times \delta(\mathbf{r}_{ij}) \mathbf{p}_{ij} \\ & + \frac{i}{\hbar^2} W_1 (\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) \cdot \mathbf{p}_{ij} \times (n_{qi} + n_{qj})^\nu \delta(\mathbf{r}_{ij}) \mathbf{p}_{ij}\end{aligned}$$

Where  $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ ,  $\mathbf{r} \equiv (\mathbf{r}_i + \mathbf{r}_j)/2$ . The momentum operator is given by  $\mathbf{p}_{ij} = -i\hbar(\nabla_i - \nabla_j)/2$ . We have :  $P_\sigma \equiv (1 + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)/2$  where  $\boldsymbol{\sigma}_i$  is a vector made by using Pauli matrices. The Skyrme interaction gives the Skyrme energy :

$$\mathcal{E}_{Skyrme} = \langle HF | v_{ij}^{(Skyrme)} | HF \rangle$$

# Skyrme functionals (2)

**Phenomenological functional**  $\Leftrightarrow$  **parameters adjusted using experiments and observations.**

$$\mathcal{E}_{Skyrme} = \sum_{t=0,1} \left[ C_t^n n_t^2 + C_t^\tau n_t \tau_t + C_t^j J_t^2 + C_t^s \mathbf{s}_t^2 + C_t^{sT} \mathbf{s}_t \cdot \mathbf{T}_t + C_t^j J_t^2 \right]$$

- Isoscalar ( $t = 0 \Leftrightarrow n+p$ ) and isovector ( $t = 1 \Leftrightarrow n-p$ )

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## Special case : Neutron star matter

- **Homogeneous spin unpolarized inner core**, made of  $n, p, e, \mu$ .
- **Charge neutrality** :  $n_p = n_e + n_\mu$
- **$\beta$  Equilibrium** :  $n \leftrightarrow p + e + \nu_e$  and  $\mu \leftrightarrow e + \nu_\mu + \bar{\nu}_e$

# Skyrme functionals (3)

The C coefficients are given by :

$$C_0^n \equiv \frac{3}{8}t_0 + \frac{3}{48}t_3n_b^\alpha$$

$$C_1^n \equiv -\frac{t_0}{4}\left(\frac{1}{2} + x_0\right) - \frac{t_3}{24}\left(\frac{1}{2} + x_3\right)n_b^\alpha$$

$$C_0^s \equiv -\frac{t_0}{4}\left(\frac{1}{2} - x_0\right) - \frac{t_3}{24}\left(\frac{1}{2} - x_3\right)n_b^\alpha$$

$$C_1^s \equiv -\frac{t_0}{8} - \frac{t_3}{48}n_b^\alpha$$

$$C_0^\tau \equiv \frac{3}{16}t_1 + \frac{t_2}{4}\left(\frac{5}{4} + x_2\right) + \frac{3}{16}t_4n_b^\beta + \frac{t_5}{4}n_b^\gamma\left(\frac{5}{4} + x_5\right)$$

$$C_1^\tau \equiv -\frac{t_1}{8}\left(\frac{1}{2} + x_1\right) + \frac{t_2}{8}\left(\frac{1}{2} + x_2\right) - \frac{t_4}{8}n_b^\beta\left(\frac{1}{2} + x_4\right) + \frac{t_5}{8}n_b^\gamma\left(\frac{1}{2} + x_5\right)$$

$$C_T^j = -C_T^\tau$$

$$n_b = n_n + n_p$$

# Skyrme functionals (4)

The energy per nucleon  $e$  in the infinite nuclear matter can be expressed as a function of the asymmetry parameter  $\eta = (n_n - n_p)/n$  and  $\epsilon = (n - n_0)/n_0$  (where  $n_0$  is the saturation density  $n_0 \simeq 0,16 \text{ fm}^{-3}$ ) :

$$e(n, \eta) = e(n, \eta = 0) + e_{Sym}^{(1)}(n)\eta^2 + \mathcal{O}(\eta^4)$$

$$e_{SNM}(n) = e(n, \eta = 0) = a_v + \frac{1}{18}K_v\epsilon^2 - \frac{1}{162}K'\epsilon^3 + \dots$$

$$e_{Sym}^{(1)}(n) = J + \frac{1}{3}L\epsilon + \frac{1}{18}K_{Sym}\epsilon^2 + \dots$$

$$e_{Sym}^{(2)}(n) = e(n, \eta = 1) - e(n, \eta = 0)$$

$$J = e_{Sym}^{(1)}(n_0) \quad L = 3n_0 \left. \frac{de_{Sym}^{(1)}}{dn} \right|_{n=n_0}$$

# Skyrme functionals (5)

Functional	J (MeV)	L (MeV)	$K_v$ (MeV)	$M_{Max}$ ( $M_\odot$ )	$m_v^\oplus/m$
<b>BSk19</b>	30	31,9	237,3	1,86	,61
<b>BSk20</b>	30	37,4	241,4	2,15	,65
<b>BSk21</b>	30	46,6	245,8	2.28	,71
<b>BSk24</b>	30	46.4	245,5	2.28	,71
<b>BSk25</b>	29	36.9	236,0	2.22	,75
<b>BSk26</b>	30	37,5	240,8	2.15	,65

# BSk19, BSk20 and BSk21 functionals

- **Experimental data** : Fit over 2149 experimental masses (AME 2003) with an rms deviation of 0,58 MeV such that  $J = 30$  MeV and  $K_{\nu} = 240 \pm 10$  MeV and  $m_s^{\oplus}/m = 0,80$  at saturation density.
- **N-body calculations** : fit to realistic  $^1S_0$  pairing gaps in neutron and symmetric matter + fit to realistic neutron matter EoS with different degrees of stiffness : FP EoS (BSk19), APR EoS (BSk20) and LS2 EoS (BSk21).
- **Implications on NS** : NeuM core and no direct URCA process for  $M_{NS,max} = 1,86 M_{\odot}$  (BSk19), unlikely direct URCA process for  $M_{NS,max} = 2,15 M_{\odot}$  (BSk20), possible direct URCA process for  $M_{NS,max} = 2,28 M_{\odot}$  (BSk21).

## Retrospective constraints

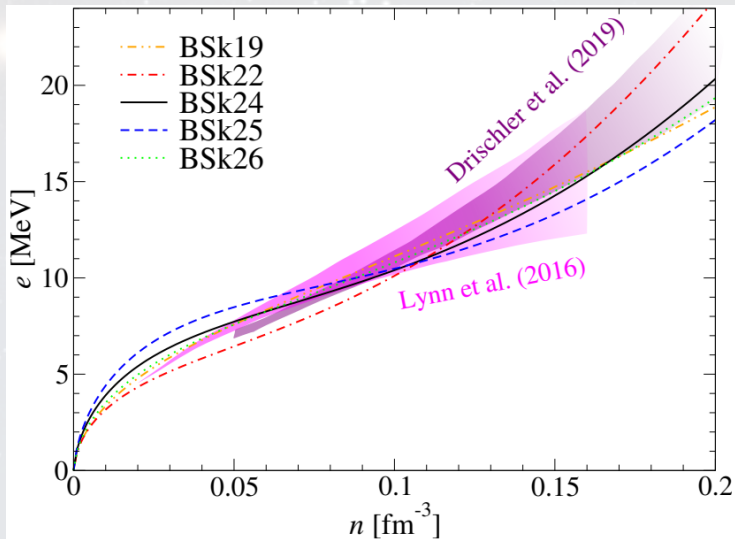
- $M_{PSRJ1614-2230} = 2,01 \pm 0,04 M_{\odot}$  and  $M_{PSRJ0348-0432} = 1,97 \pm 0,04 M_{\odot} \implies$  BSk19 is disfavored by astrophysical arguments !
- Measurements of IVGDR favor  $m_v^{\oplus}/m \simeq 0,7-1 \implies$  BSk21 seems to be the most reliable functional.

# BSk24, BSk25 and BSk26 functionals

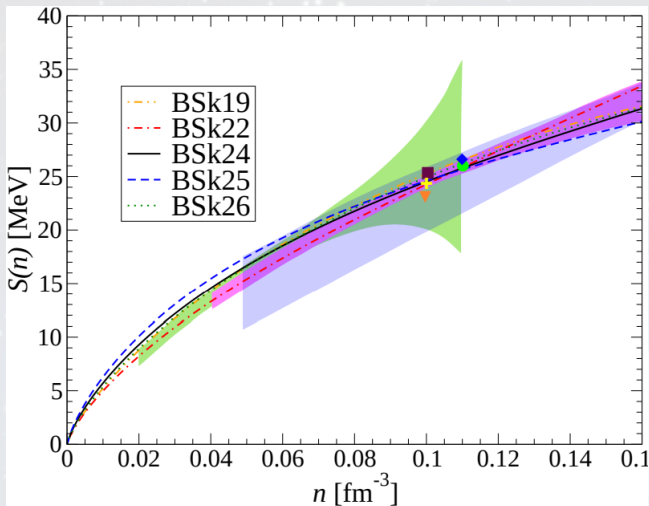
- **Experimental data** : Fit over 2353 experimental masses (AME 2012) with an rms deviation of 0,549 MeV such that  $K_\nu = 240 \pm 10$  MeV and  $m_s^\oplus/m = 0,80$  at saturation density.
- **N-body calculations** : fit to realistic  $^1S_0$  pairing gaps in neutron and symmetric matter + fit to realistic neutron matter LS2 EoS (BSk24 and BSk25)/ APR EoS (BSk26) with different symmetry energy :  $J = 30$  MeV (BSk24 and BSk26),  $J = 29$  MeV (BSk25).
- **Implication on NS masses** :  $M_{NS,max} = 2,28 M_\odot$  (BSk24),  $M_{NS,max} = 2,22 M_\odot$  (BSk25),  $M_{NS,max} = 2,15 M_\odot$  (BSk26).

## Restrospective constraints

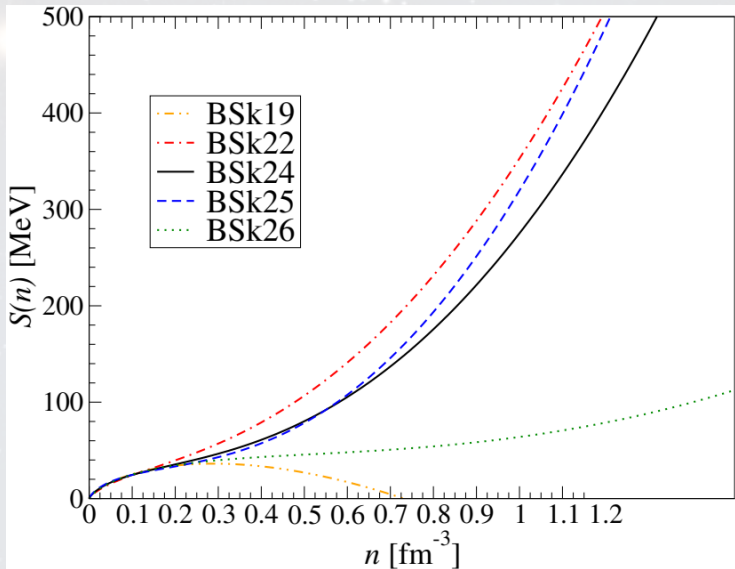
Measurements of IVGDR favoir  $m_v^\oplus/m \sim 0,7-1 \rightarrow$  BSk24 (remake of BSk21) and BSk25 seem to be the most reliable functionals.



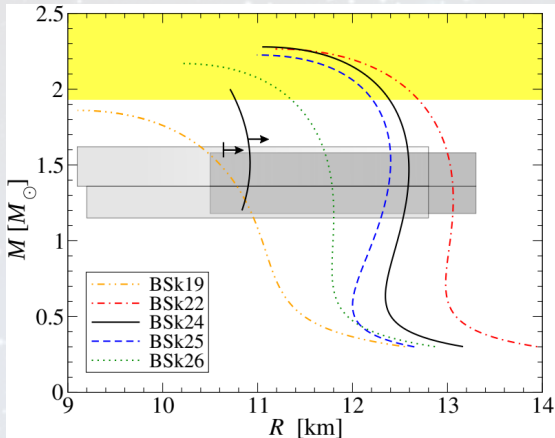
- EoS of pure neutron matter.
- Shaded areas = Constraints from chiral effective field theory.
- BSk20 (resp. BSk21) is indistinguishable from BSk26 (resp. BSk24).



- Symmetry energy at subsaturation densities.
- Constraints : heavy-ion collisions (blue), isobaric-analog and neutron skins (purple), electric dipole polarisability of  $^{208}\text{Pb}$  (green).
- BSk20 (resp. BSk21) is indistinguishable from BSk26 (resp. BSk24).



- Symmetry energy at supersaturation densities.
- BSk20 (resp. BSk21) is indistinguishable from BSk26 (resp. BSk24).



- Gravitational masses vs radius (non-rotating NS).
- Constraints inferred from the analysis of GW170817 : masses and radii of the two NS, with 90% CL (grey boxes) expected values for the maximal mass using various studies (yellow).
- BSk20 (resp. BSk21) is indistinguishable from BSk26 (resp. BSk24).

- **Experimental data** : Fit over the binding energy of 12 spherical even-even (e-e) nuclei + charge rms radii of 10 spherical e-e nuclei + ISGMR energies of 4 spherical e-e nuclei such that the neutron  $3p_{1/2} - 3p_{3/2}$  energy level splitting in  $^{208}\text{Pb}$  is in the range  $0,8 - 1,0$  MeV + fixed values for  $m_s^\oplus$  and  $m_v^\oplus$  at saturation density.
- **Constraints on nuclear matter** : fit to constant pairing gap +  $\rho_{cr} > 1,2$   $\text{fm}^{-3}$  + pressure of symmetric nuclear matter consistent with constraints in  $2\rho_0 < \rho < 4,6\rho_0$  + EoS of pure neutron matter consistent with predictions of the latest chiral effective field theory (Tews and al. 2013) +  $J = 26,65$  MeV and  $L = 47,3$  MeV at  $\rho_c = 0,11$   $\text{fm}^{-3}$ .
- **Constraints on effective masses** (at saturation density) :  $m_s^\oplus/m = 0,7$  and  $m_v^\oplus/m = 0,6$  (for eMSL07),  $m_s^\oplus/m = 0,8$  and  $m_v^\oplus/m = 0,7$  (for eMSL08),  $m_s^\oplus/m = 0,9$  and  $m_v^\oplus/m = 0,75$  (for eMSL09).
- **Implications on NS masses** :  $M_{NS,max} = 2,17 M_\odot$  (eMSL07),  $M_{NS,max} = 2,19 M_\odot$  (eMSL08),  $M_{NS,max} = 2,21 M_\odot$  (eMSL09)

# BSk VS eMSL

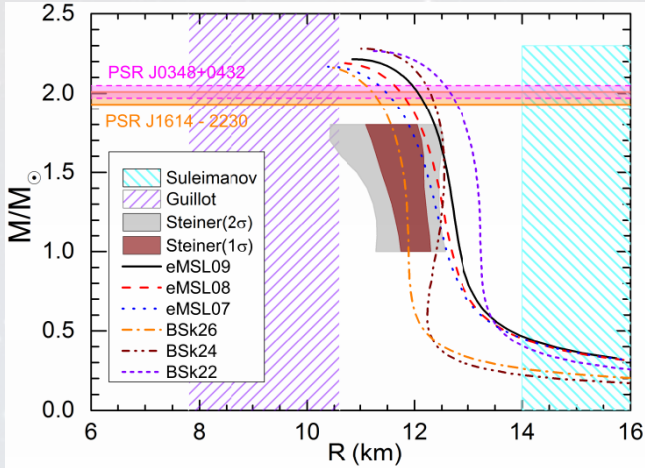
## BSk

- Fit to 2000 masses.
- Fit to microscopic calculations of NuM EoS, effective masses, pairing gaps, ...
- Constraints on incompressibility  $K_\nu$  and stability (eg.  $c_s < c$ , etc.).
- Existence of unified EoS constructed using this family.
- Constructed in order to describe nuclei as well as nuclear matter in NS.
- $M_{NS,max} \sim 2 M_\odot$ .

## eMSL

- Fit to a dozen of masses.
- Constraints on the effective masses.
- No unified EoS constructed using this family.
- Constructed in order to describe and explain the existence of massive NS.
- $M_{NS,max} \sim 2 M_\odot$ .

The BSk and eMSL family give quite similar predictions (in terms of entrapment).



- Gravitational masses vs radius (non-rotating NS).
- Constraints : masses of the two heaviest NS (horizontal bands), 3 x-ray bursters and transient low-mass x-ray binaries (Steiner constraints), atmosphere models (Suleimanov) and quiescent low mass X-ray binaries (Guillot).
- BSk20 (resp. BSk21) is indistinguishable from BSk26 (resp. BSk24).

# LNS functional and $\text{Sk}\chi\text{m}^*$ functionals

## LNS

- **Experimental data** : fit over 9 closed-(sub)shell nuclei (giving the  $t_1, x_1, t_2, x_2$ )
- **N-body calculations** : fit over nucleon effective masses in symmetric and non-symmetric nuclear matter obtained using the Brueckner Hartree-Fock method (giving the  $t_0, x_0, t_3, x_3$  parameters) where  $\eta = 0, 4 + K_\nu = 200 - 240 \text{ MeV} + \text{Symmetry energy at saturation density} = 34 \text{ MeV}$ .

## $\text{Sk}\chi\text{m}^*$

- **Experimental data** : fit the binding energies of double closed-shell nuclei
- **N-body calculations** : fitting the EoS and the nucleon effective masses in asymmetric nuclear matter described by chiral effective field theory.

# UNEDF and SLy4 functionals

## UNEDF0 and UNEDF1

- **Experimental data** : fitting the binding energies for 44 well deformed even-even nuclei + fitting the binding energies and charge radii for 28 spherical nuclei (+ fitting the excitation energies of four fission isomers + fitting 3 additional masses of deformed nuclei, for UNEDF1 case).
- **N-body calculations** : fit to density-dependent gap with fixed isovector effective mass  $m/m_v^\oplus = 1,249$  and zero tensor coupling constants (and no CM correction, for UNEDF1 case).

## SLy4

- **Experimental data** : fitting the binding energies and rms radii of 5 doubly magic nuclei + fitting the neutron  $3p_{1/2} - 3p_{3/2}$  single particle energy splitting in  $^{208}\text{Pb}$  + constraints on the surface properties.
- **N-body calculations** : fit to contact density-dependent pairing gap +  $E/A \simeq -16 \text{ MeV}$  at  $\rho_0 \simeq 0,16 \text{ fm}^{-3}$  +  $K_v \simeq 230 \text{ MeV}$  + reproduction of Wiringa and al. EoS for NeUM.

# Entrainment matrix at low temperature under linear response theory

- At low T : **the entrainment matrix does not depend on the pairing.**  
HFB reduces to HF:

$$\boxed{\hat{h}_q \phi_{\mathbf{k}}^q = e_{\mathbf{k}}^q \phi_{\mathbf{k}}^q} \quad \mathbf{v}_{\mathbf{k}}^q = \frac{1}{\hbar} \nabla_{\mathbf{k}} e_{\mathbf{k}}^q \quad n_q = \frac{2}{\Omega} \sum_{\mathbf{k}} \tilde{n}_{\mathbf{k}}^q \quad \mathbf{j}_q = \frac{2}{\Omega} \sum_{\mathbf{k}} \tilde{n}_{\mathbf{k}}^q \mathbf{k}$$

- Static ground state :  $\tilde{n}_{\mathbf{k}}^q = \theta(\epsilon_F^q - e_{\mathbf{k}}^q) \Rightarrow \mathbf{j}_q = \mathbf{0}$
- Dynamical ground state :  $\tilde{n}_{\mathbf{k}}^q = \theta(\epsilon_F^q + \boldsymbol{\pi}_q \cdot \mathbf{v}_{\mathbf{k}}^q - e_{\mathbf{k}}^q) \Rightarrow \mathbf{j}_q \neq \mathbf{0}$



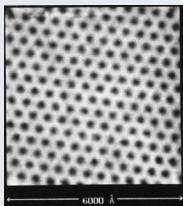
Slight perturbation of the Fermi surface by a vector  $\delta \mathbf{k} = \mathbf{Q}_q$ .

$$\boldsymbol{\pi}_q = m \mathbf{V}_q = \hbar \mathbf{Q}_q \quad \mathbf{j}_q = n_q \mathbf{Q}_q$$

# Superfluidity-Superconductivity

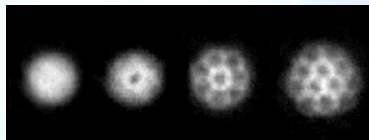
## Superconductivity

- Zero resistivity
- Magnetic flux expulsion (Meissner Effect)
- EM flux quantization (Fluxon)



## Superfluidity

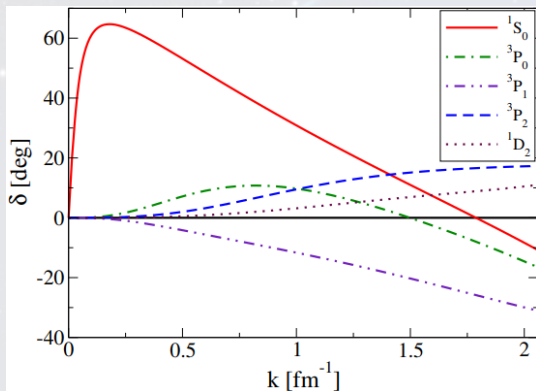
- Zero viscosity
- Angular momentum decrease (Hess-Fairbank Effect)
- Momentum quantization (Vortex)



- At first sight : no huge difference between superconductivity and superfluidity : Superconductor  $\simeq$  Charged superfluid.

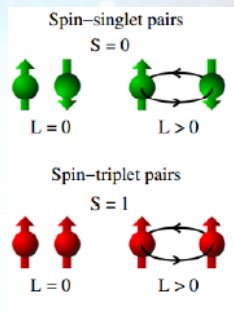
# Pairing inside NS

At low temperatures, the nuclear interaction induces nucleon pairs which can condense into a superfluid and superconducting phase.



(Picture from V.Dexheimer)

- Pairing channels :  $2S+1 L_J$ .



(Source : D. Page, J. M. Lattimer and al., arXiv:1406.6109)

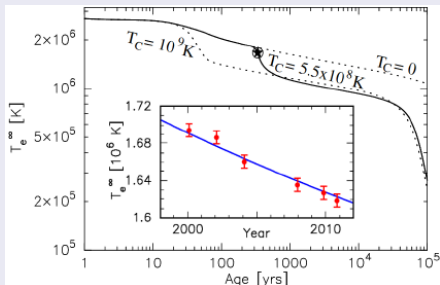
## Most attractive pairing channels

Neutrons  $^1S_0$  (crust and core), neutrons  $^2P_3$  (core) et protons  $^1S_0$  (core).

# Superfluidity in neutron stars

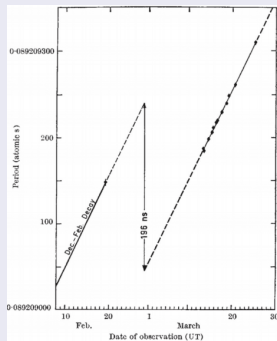
Nuclear superfluidity has **strong support** with observations of NS.

## Rapid cooling of Cassiopeia A



- NS cool mainly due to  $\nu$ ,  $\bar{\nu}$  emission.
- Superfluidity increases the number of emitted  $\nu$ ,  $\bar{\nu}$ .

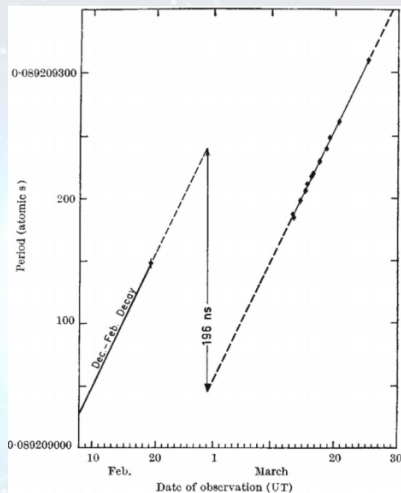
## Pulsar glitches



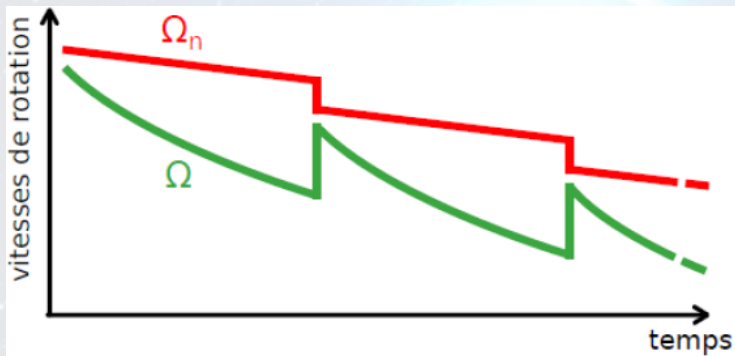
- Sudden spin up of pulsar explained by superfluid dynamics (vortices and fluxons).

# Pulsar frequency glitches

- Pulsar spin down BUT sudden spin up following by an exponential relaxation : **Pulsar Glitch**
- Coupling between the charged particles (in the outer core) and the crust (due to magnetic fields) :  $I$  et  $\omega$ .
- Neutron superfluid weakly coupled to the charged superfluid :  $I_n$  et  $\omega_n$ .



- Velocity decrease of the charged fluid (EM emission)  $\Rightarrow$  Decrease of the vortex density IF NO pinning on the nuclei inside the inner crust  $\Rightarrow$  Magnus force
- Magnus force becoming greater than the pinning force  $\Rightarrow$  Unpinning and vortex ejection ( $\omega_n \searrow$ )
- The pulsar spins up (due to the angular momentum conservation) : **Glitch**



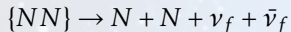
# NS cooling

- "Classical" cooling :



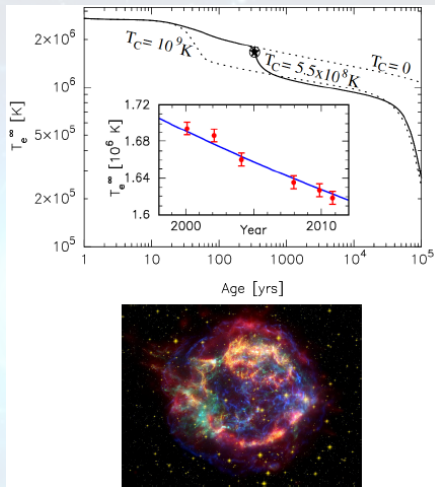
Such process are suppressed in presence of superfluidity.

- Cooling with superfluidity :



Where  $\{NN\}$  = Cooper pair.

- Observed in Cassiopeia A.



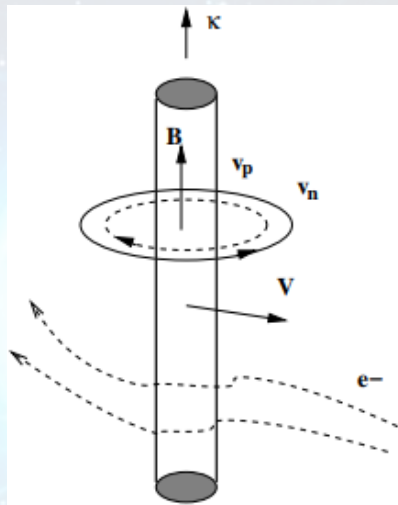
(Source : D. Page et M. Prakash, arXiv:1011.6142)

# Influence of entrainment effects in pulsars

- Induced **circulation of protons around neutron vortices**.
- **Induced magnetic field  $B$  and magnetic flux  $\phi_*$**  :

$$\phi_* = \frac{hc}{2e} \frac{\rho_{np}}{\rho_{pp}} \propto \rho_{np}$$

- **Electrons and muons scattering off the magnetic field**  $\rightarrow$  Effective viscosity in the neutron superfluid.
- **Frictional coupling between the crust and the core**  $\Rightarrow$  Influences the pulsar dynamics.



(Picture from K.Glampedakis)

# Nuclei, $e$ and $\mu$ : normal or superconducting phase ?

## Nuclei inside the outer crust

- Outer crust made of iron nuclei.
- For a density  $\rho \sim 8,2 \text{ g cm}^{-3}$  :  $T_c^{(Fe)} \simeq 2\text{K} \ll T_{NS} \Rightarrow$  **Fe is at normal state** (Source : Shimizu and al., Nature 412, 316 (2001)) !

## Electrons (and muons)

- Atoms completely ionized inside NS (where  $\rho \geq 10^4 \text{ g cm}^{-3}$ ) : electrons = Fermi gas.
- Critical temperature of a NR electron gas in a ionic background :  
 $T_c^{(e)} \propto e^{-\rho^{1/3}} \Rightarrow T_c^{(e)}$  decreases with density  $\rho$
- For  $\rho \geq 10^6 \text{ g cm}^{-3}$ , the electrons become relativistic :  $T_c^{(e)} \propto e^{-8/\pi\alpha} \sim 0$
- **Electrons (and muons) are in normal phase !**

# Bardeen-Pines interaction

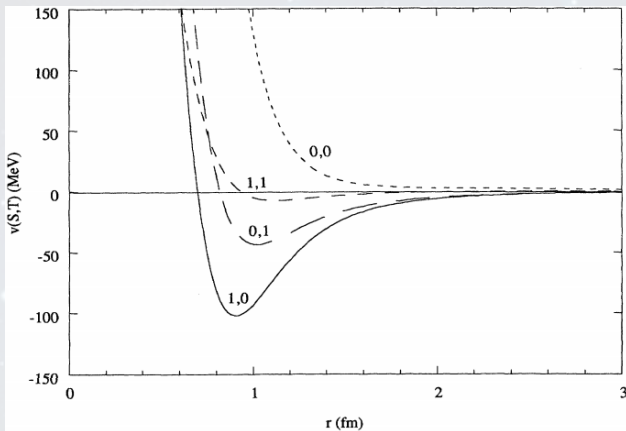
- Bare Coulomb interaction :  $V_c(\mathbf{k}) = \frac{4\pi e^2}{\Omega \mathbf{k}^2}$
- BUT interactions between conducting electrons with other electrons and with ions  $\Rightarrow$  Renormalization of the interaction (Jellium model) :

$$V_{eff}(\mathbf{k}, \omega) = \underbrace{\frac{4\pi e^2}{\mathbf{k}^2 + \mathbf{k}_{TF}^2}}_{\text{Screening}} + \underbrace{\frac{4\pi e^2}{\mathbf{k}^2 + \mathbf{k}_{TF}^2} \frac{\omega_{\mathbf{k}}^2}{\omega^2 - \omega_{\mathbf{k}}^2}}_{\text{Ionic lattice Polarization}}$$

Where :  $\omega_{\mathbf{k}} \simeq c_s \mathbf{k} \sim \omega_D$  et  $\mathbf{k}_{TF} = \sqrt{4\pi e^2 \frac{\partial n}{\partial \mu}}$

- The renormalized interaction can be attractive (for :  $\omega < \omega_{\mathbf{k}}$ )

# Nuclear interaction



- We have  $V_{nuc-nuc}(\mathbf{r}, \mathbf{S}, \mathbf{L}, \mathbf{T}) = V_{nuc-nuc}^{EM} + V_{nuc-nuc}^{\pi} + V_{nuc-nuc}^{Repulsion}$
- The long-range term  $V_{nuc-nuc}^{\pi}$  is attractive and is therefore responsible of the pairing phenomenon. Its mediator is the pion.