

# Elastic properties of crystals in neutron stars crust

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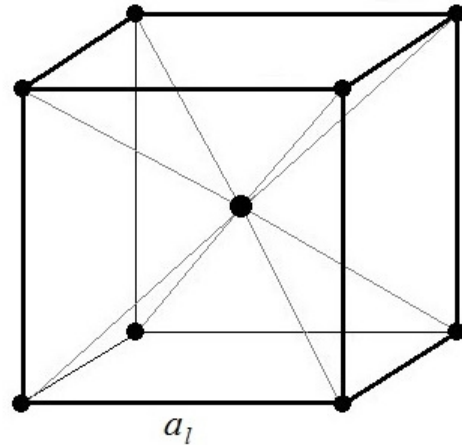
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**Elastic properties of binary crystals in neutron stars and white dwarfs**

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$$\Gamma = \frac{Z^2 e^2}{aT} > 175$$



### body centred cubic (bcc) lattice

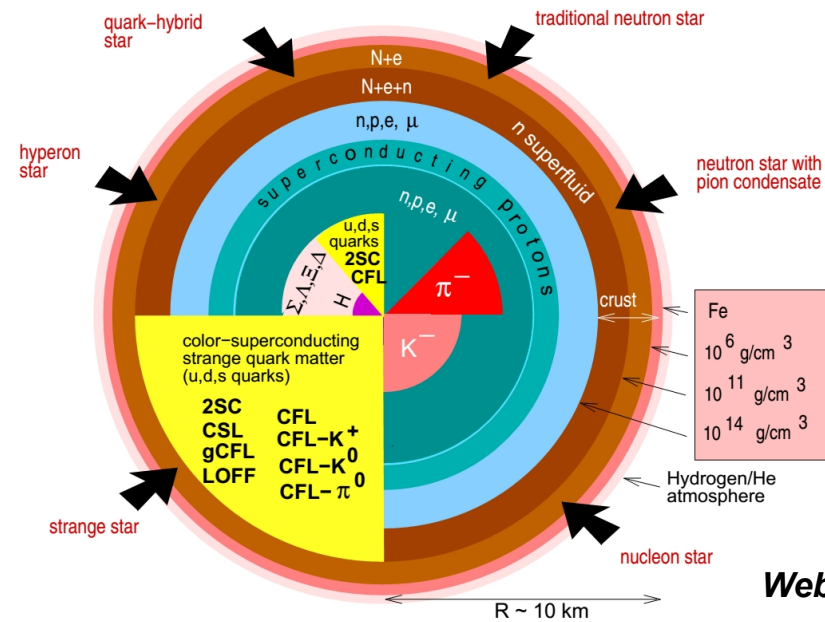
It is generally accepted that the neutron star crust consists of a degenerate electron gas and fully ionized atoms arranged into a crystal lattice. This system of electrons and ions can be described by the **Coulomb crystal** model, which suggests that ions can be considered as point particles, and electrons as a uniform or weakly polarized neutralizing background.

Elastic (resistance to deformation) properties play an important role in neutron stars and white dwarfs. They are crucial for modeling stellar oscillations and different processes in magnetars and in degenerate stars, which enter compact binary systems.

$$\mathcal{Y}'' + \left[ \left( \frac{4}{r} + \Phi' - \Lambda' \right) + \frac{\mu'}{\mu} \right] \mathcal{Y}' + \left[ \frac{\tilde{H}}{\mu} \omega^2 e^{-2\Phi} - \frac{(\ell+2)(\ell-1)}{r^2} \right] e^{2\Lambda} \mathcal{Y} = 0$$

effective shear modulus

TORSIONAL OSCILLATIONS



$$U = \frac{1}{2} \sum'_{lp'l'p'} \frac{Z^2 e^2}{|\mathbf{r}_{lp} - \mathbf{r}_{l'p'}|} - n_e \sum_{lp} \int d\mathbf{r} \frac{Ze}{|\mathbf{r}_{lp} - \mathbf{r}|} + \frac{n_e^2}{2} \int \int \frac{d\mathbf{r} d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{r}_{lp} = \mathbf{R}_l + \boldsymbol{\chi}_p + \mathbf{u}_{lp}$$

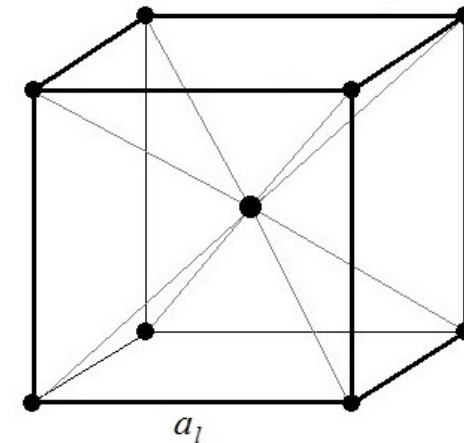
body centred cubic (bcc) lattice

$$U \approx U_M + \frac{1}{2} \sum_{i,j=1}^N u_i^\mu u_j^\nu \frac{\partial^2 U}{\partial u_i^\mu \partial u_j^\nu} \Big|_{u_i^\mu, u_j^\nu=0}.$$

$$\mathbf{R}_l = l_1 \mathbf{a}_1 + l_2 \mathbf{a}_2 + l_3 \mathbf{a}_3$$

$a_1$  is the lattice constant

$a \equiv (4\pi n/3)^{-1/3}$  is the ion sphere radius



The potential energy of any one-component static lattice can be written as

$$U_M = \frac{1}{2} \sum_{p,p'=1}^{N_{\text{cell}}} \sum'_{l,l'=1}^N \frac{Z^2 e^2}{|\mathbf{R}_l + \boldsymbol{\chi}_p - \mathbf{R}_{l'} - \boldsymbol{\chi}_{p'}|} - \sum_{p=1}^{N_{\text{cell}}} \sum_{l=1}^N \int_V \frac{Z n_e d\mathbf{r}}{|\mathbf{R}_l + \boldsymbol{\chi}_p - \mathbf{r}|} + \frac{n_e^2}{2} \int_V \int_V \frac{d\mathbf{r} d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

$$U_M = \frac{1}{2} \sum_{p,p'=1}^{N_{\text{cell}}} \sum_{l,l'=1}^N \frac{Z^2 e^2}{|\mathbf{R}_l + \boldsymbol{\chi}_p - \mathbf{R}_{l'} - \boldsymbol{\chi}_{p'}|} - \sum_{p=1}^{N_{\text{cell}}} \sum_{l=1}^N \int_V \frac{Z n_e d\mathbf{r}}{|\mathbf{R}_l + \boldsymbol{\chi}_p - \mathbf{r}|} + \frac{n_e^2}{2} \int_V \int_V \frac{d\mathbf{r} d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

It can be rewritten as

$$U_M = N \frac{Z^2 e^2}{a} \zeta,$$

$$\zeta = \frac{a}{2N_{\text{cell}}} \sum_{lpp'} (1 - \delta_{pp'} \delta_{\mathbf{R}_l 0}) \frac{\text{erfc}(AY_{lpp'})}{Y_{lpp'}} - \frac{Aa}{\sqrt{\pi}} - \frac{3}{8A^2 a^2}$$

$$+ \frac{3}{2N_{\text{cell}}^2 a^2} \sum_{mpp'} (1 - \delta_{\mathbf{G}_m 0}) \frac{1}{G_m^2} \exp \left[ -\frac{G_m^2}{4A^2} + i\mathbf{G}_m(\boldsymbol{\chi}_p - \boldsymbol{\chi}_{p'}) \right].$$

where  $\mathbf{Y}_l = \mathbf{R}_l + \boldsymbol{\chi}_p - \boldsymbol{\chi}_{p'}$ ,  $\text{erfc}(x)$  is a complementary error function,  $Aa_1 \simeq 2$ .

$$\zeta = \zeta' \left( \frac{3}{4\pi n a_1^3} \right)^{1/3}$$

$$\zeta_{\text{bcc}} = -0.895929255682$$

$a_1$  is the lattice constant

$a \equiv (4\pi n/3)^{-1/3}$  is the ion sphere radius

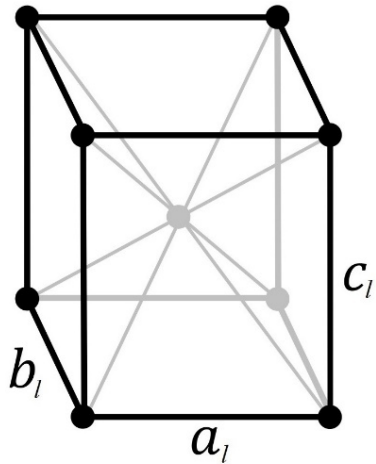
In this form the electrostatic energy of any lattice can be calculated (including deformed one)

$$c_1 \equiv b_1/a_1, \quad c_2 \equiv c_1/a_1$$

For example, a **stretched bcc lattice**.

It is characterized by two deformation parameters:

Consider  $c_1 - 1$  and  $c_2 - 1$  be small, than we can expand the electrostatic energy as



$$\zeta'(c_1, c_2) \approx \zeta'_{\text{bcc}} - p'_{\text{bcc}} [(c_1 - 1) + (c_2 - 1)] + 0.25s_{\text{bcc}}^{xxxx} [(c_1 - 1)^2 + (c_2 - 1)^2] + 0.5s_{\text{bcc}}^{xyxy} (c_1 - 1)(c_2 - 1)$$

$\zeta'_{\text{bcc}} = -1.81961673$  is the Madelung constant in  $a_1$  units,

$p'_{\text{bcc}} = \zeta'_{\text{bcc}}/3$  is the electrostatic pressure,

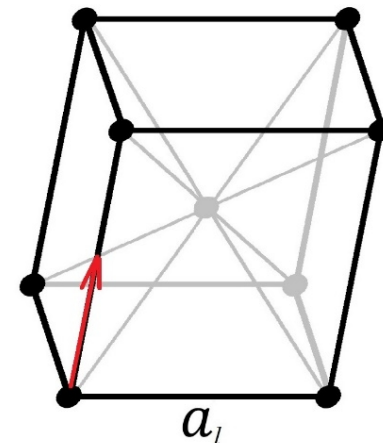
$s_{\text{bcc}}^{xxxx} = -1.4848079$  and  $s_{\text{bcc}}^{xyxy} = -0.47067387$  are elastic moduli.

Similarly, for the **bcc lattice with a shift**

$$a_1(0, 0, 1) \rightarrow a_1(c_x, c_y, 1)$$

$$\zeta'(c_x, c_y) \approx \zeta'_{\text{bcc}} + 0.25s_{\text{bcc}}^{xyxy} (c_x^2 + c_y^2)$$

$$s_{\text{bcc}}^{xyxy} = 0.74240395$$



In the general case, any arbitrary deformation of a cubic lattice can be determined by three independent elastic moduli:  $s^{xxxx}$ ,  $s^{xyxy}$  and  $s^{xyxy}$ . It is interesting to note that  $s_{\text{bcc}}^{xyxy} = -s_{\text{bcc}}^{xxxx}/2$ .

Define **effective shear modulus**  $\mu_{\text{eff}}$ , as an averaged "over all the rotations of the Cartesian axes" (Ogata & Ichimaru, 1990).

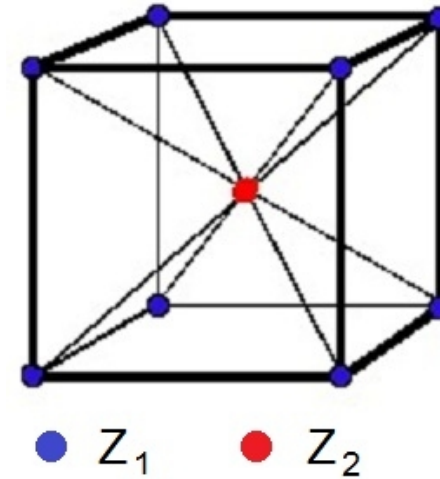
$$\mu_{\text{eff}}^{\text{bcc}} \equiv \frac{a}{10a_1} (S^{xxxx} - S^{xxyy} + 3S^{xyxy} - P') = 0.1194572n \frac{Z^2 e^2}{a}$$

It is in a good agreement with the results (Baiko, 2011), derived from perturbation theory.

Notice that  $0.1194572 = -\frac{2}{15}\zeta_{\text{bcc}}$

Similarly, we can determine the elastic moduli of multicomponent lattices. For example, for **binary body-centered cubic (sc2) lattice**

$$\begin{aligned}
 U_M &= N \frac{e^2}{a} \xi, \\
 \xi &= \frac{a}{2N_{\text{cell}}} \sum_{lpp'} Z_p Z_{p'} (1 - \delta_{pp'} \delta_{\mathbf{R},0}) \frac{\text{erfc}(AY_{lpp'})}{Y_{lpp'}} \\
 &\quad - \frac{Aa}{N_{\text{cell}} \sqrt{\pi}} \sum_p Z_p^2 - \frac{3}{8N_{\text{cell}}^2 A^2 a^2} \sum_{pp'} Z_p Z_{p'} \\
 &\quad + \frac{3}{2N_{\text{cell}}^2 a^2} \sum_{mpp'} Z_p Z_{p'} (1 - \delta_{\mathbf{G}_m,0}) \\
 &\quad \times \frac{1}{G_m^2} \exp \left[ -\frac{G_m^2}{4A^2} + i\mathbf{G}_m(\boldsymbol{\chi}_p - \boldsymbol{\chi}_{p'}) \right].
 \end{aligned}$$



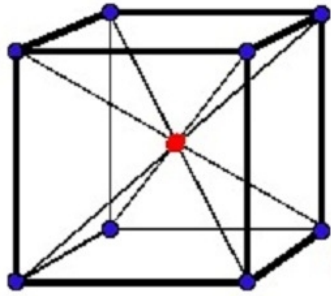
$$\alpha \equiv \frac{Z_2}{Z_1}$$

*Kozhberov & Baiko, 2015*

$$U_M = -N \frac{Z_1^2 e^2}{a} \left[ 0.3492518210505(1 + \alpha^2) + 0.19742561358\alpha \right]$$

$$\zeta'(c_1, c_2, \alpha) \approx \zeta'_{\text{sc2}} - p'_{\text{sc2}} [(c_1 - 1) + (c_2 - 1)]$$

$$+ 0.25s_{\text{sc2}}^{xxxx} [(c_1 - 1)^2 + (c_2 - 1)^2] + 0.5s_{\text{sc2}}^{xyxy} (c_1 - 1)(c_2 - 1)$$



•  $Z_1$     •  $Z_2$

$$\alpha \equiv \frac{Z_2}{Z_1}$$

$$p'_{\text{sc2}} = \zeta'_{\text{sc2}}(\alpha)/3 \approx -0.472882(1 + \alpha^2) - 0.267312\alpha$$

$$s_{\text{sc2}}^{xxxx} = 0.32969383(1 + \alpha^2) - 2.144195558\alpha$$

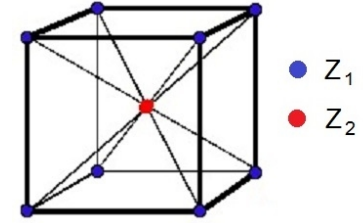
$$s_{\text{sc2}}^{xyxy} = -0.637729828(1 + \alpha^2) + 0.804785789\alpha$$

$$s_{\text{sc2}}^{xyxy} = -0.164846915(1 + \alpha^2) + 1.072097779\alpha.$$

$$\mu_{\text{eff}}^{\text{sc2}} = -\frac{2}{15} \zeta_{\text{sc2}} n \frac{Z_1^2 e^2}{a} = n \frac{Z_1^2 e^2}{a} (0.0465669(1 + \alpha^2) + 0.0263234\alpha)$$

The effective shear modulus can also be calculated by the linear mixing rule ( $\mu_{\text{eff}}^{\text{lm}}$ ). A multi-component crystal is represented as the sum of one-component crystals with the same electron number density. For the sc2 lattice

$$\mu_{\text{eff}}^{\text{lm}} = 0.119457n \frac{Z_1^2 e^2}{a} (0.5 + 0.5\alpha^{5/3})(0.5 + 0.5\alpha)^{1/3}$$



Previously, the elastic moduli of binary Coulomb crystals were considered only numerically (Igarashi & Iyetomi, 2003), where for the two-component disordered bcc lattice  $c_{44}^{\text{dis}}$  and  $\mu_{\text{eff}}^{\text{dis}}$  were calculated for several  $\alpha$ :

$\alpha$	$c_{44}$	$c_{44}^{\text{dis}}$	$\mu_{\text{eff}}^{\text{sc2}}$	$\mu_{\text{eff}}^{\text{dis}}$	$\mu_{\text{eff}}^{\text{lm}}$
1	0.18277		0.119457		0.119457
4/3	0.239184	0.241	0.164451	0.164	0.164439
2	0.324956	0.292	0.285482	0.284	0.28544
3	0.385977	0.549	0.544639	0.542	0.544853

$$c_{44} \equiv s_{\text{sc2}}^{xyxy} / (64\pi/3)^{1/3}$$

$$\alpha \equiv \frac{Z_2}{Z_1}$$

Elastic moduli of the sc2 lattice, calculated by the "traditional" method, (in units  $nZ_1^2 e^2/a$ ).

An “alternative” method for the definition the effective shear modulus was proposed in (Kobyakov & Pethick, 2015).  $\mu_{\text{eff},m}$  in this model is determined by equation

$$3\mu_{\text{eff},m}^2 - s^{xyxy}\mu_{\text{eff},m} - (s^{xxxx} - s^{xyyy} - p')s^{xyxy} = 0 .$$

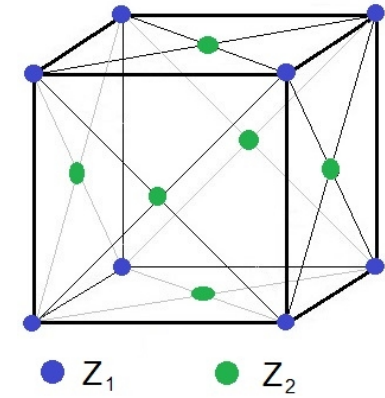
For several  $\alpha$ , the values of  $\mu_{\text{eff},m}^{\text{sc}2}$  are shown in Table (in units of  $nZ_1^2e^2/a$ ). In addition we show  $\mu_{\text{eff},m}^{\text{dis}}$  the effective elastic modulus calculated “alternative” method based on numerical results from (Igarashi & Iyetomi 2003) and  $\mu_{\text{eff},m}^{\text{lm,sc}2}$  the effective modulus of elasticity calculated by the “alternative” method, using the linear mixing rule.

$\alpha$	$\mu_{\text{eff},m}^{\text{sc}2}$	$\mu_{\text{eff},m}^{\text{dis}}$	$\mu_{\text{eff},m}^{\text{lm,sc}2}$
4/3	0.139547	0.138	0.128027
2	0.282088	0.284	0.222235
3	0.517677	0.542	0.424206

$$\mu_{\text{eff},m}^{\text{bcc}} = 0.0930057n \frac{Z^2 e^2}{a}$$

The difference between  $\mu_{\text{eff}}^{\text{sc}2}$  and  $\mu_{\text{eff},m}^{\text{sc}2}$  is significant.

A similar situation with other lattices, for example, for **binary face-centered cubic (fc2) lattice**:



$$\mu_{\text{eff}}^{\text{fcc}} = 0.11944982n \frac{Z_1^2 e^2}{a}$$

$$U_M = -N \frac{Z_1^2 e^2}{a} \zeta_{\text{fc2}} = -N \frac{Z_1^2 e^2}{a} (0.138600677 + 0.1707354535\alpha + 0.5865374846\alpha^2) .$$

$$\mu_{\text{eff}}^{\text{fc2}} = n \frac{Z_1^2 e^2}{a} (0.01848009 + 0.02276473\alpha + 0.07820500\alpha^2)$$

$$\mu_{\text{eff}}^{\text{lm,fc2}} = 0.11944982n \frac{Z_1^2 e^2}{2^{8/3}a} (1 + 3\alpha^{5/3})(1 + 3\alpha)^{1/3} .$$

For  $0.661 \leq \alpha \leq 1.368$

the  $\mu_{\text{eff}}^{\text{lm,fc2}} / \mu_{\text{eff}}^{\text{fc2}}$  ratio always ranges between 1 and 1.002.

$$\frac{1}{r_{ij}} \rightarrow \frac{\exp(-\kappa_{\text{TF}} r_{ij})}{r_{ij}}$$

$$\begin{aligned} \frac{\Delta U_{\text{TF}}}{NZ^2e^2} = & \left\{ \frac{1}{N_{\text{cell}}} \sum_{l,p,p'} (1 - \delta_{\mathbf{R}_l 0} \delta_{pp'}) \frac{E_- + E_+}{4Y_l} - \frac{\kappa_{\text{TF}}}{2} \text{erf} \left( \frac{\kappa_{\text{TF}}}{2A} \right) - \frac{A}{\sqrt{\pi}} e^{-\frac{\kappa_{\text{TF}}^2}{4A^2}} \right. \\ & \left. + \frac{1}{N_{\text{cell}}^2} \sum_{m,p,p'} \frac{2\pi n}{G_m^2 + \kappa_{\text{TF}}^2} e^{-\frac{G_m^2 + \kappa_{\text{TF}}^2}{4A^2}} e^{-i\mathbf{G}_m(\chi_p - \chi_{p'})} - \frac{2\pi n}{\kappa_{\text{TF}}^2} \right\} - \frac{U_{\text{M}}}{NZ^2e^2}, \end{aligned}$$

$$E_{\pm} = e^{\pm\kappa_{\text{TF}} Y_l} \text{erfc} \left( AY_l \pm \frac{\kappa_{\text{TF}}}{2A} \right)$$

Corrections to the elastic moduli due to polarization of the electron background (scr) can be also obtained from the expansion of the electrostatic energy. If the polarization is small and is described as  $\kappa_{\text{TF}} a \approx 0.185 Z^{1/3} (1 + x_r^2)^{1/4} / x_r^{1/2}$ , where  $x_r$  is the electron relativistic parameter, then the corrections can be written as

$$p^{\text{scr}} = \frac{\eta}{3} \frac{x_r^2}{1 + x_r^2} (\kappa_{\text{TF}} a)^2 n \frac{Z^2 e^2}{a} \quad c_{44}^{\text{scr}} = -0.041198 (\kappa_{\text{TF}} a)^2 n \frac{Z^2 e^2}{a}$$

$$\mu_{\text{eff}}^{\text{scr}} = \frac{4\eta}{15} (\kappa_{\text{TF}} a)^2 n \frac{Z^2 e^2}{a} \approx -0.027662 (\kappa_{\text{TF}} a)^2 n \frac{Z^2 e^2}{a}$$

where  $\eta = -0.1037323337$  is correction to the electrostatic energy obtained in (Baiko 2002).

$$\mu_{\text{eff}}^{\text{scr}} = \frac{1}{5} \left( S_{1111}^{\text{scr}} - S_{1122}^{\text{scr}} - S_{1221}^{\text{scr}} + 4S_{1212}^{\text{scr}} \right)$$

$$= -0.027 (\kappa a)^2 S_0.$$

**Baiko, 2015**

$$\frac{Z^2 e^2}{2} \left[ R_\beta R_\lambda \frac{\partial^2}{\partial R_\alpha \partial R_\gamma} - 2\delta_{\gamma\lambda} n R_\beta \frac{\partial^2}{\partial n \partial R_\alpha} \right.$$

$$\left. + (\delta_{\alpha\beta} \delta_{\gamma\lambda} + \delta_{\alpha\lambda} \delta_{\beta\gamma}) n \frac{\partial}{\partial n} + \delta_{\alpha\beta} \delta_{\gamma\lambda} n^2 \frac{\partial^2}{\partial n^2} \right]$$

$$\times \frac{1}{R} [\exp(-\kappa R) - 1].$$

$$S_{\alpha\beta\gamma\lambda}^{\text{scr}} = \frac{nZ^2 e^2}{2} \sum_{i=1}^5 S_{\alpha\beta\gamma\lambda}^{(i)},$$

$$S_{\alpha\beta\gamma\lambda}^{(1)} = -\delta_{\alpha\beta} \delta_{\gamma\lambda} (\kappa_1^2 + \kappa_2^2) - \delta_{\alpha\lambda} \delta_{\beta\gamma} \kappa_1^2,$$

$$S_{\alpha\beta\gamma\lambda}^{(2)} = -\frac{\pi n}{A^2} \left[ \delta_{\alpha\beta} \delta_{\gamma\lambda} \left( 1 + 3n \frac{d}{dn} + n^2 \frac{d^2}{dn^2} \right) \right.$$

$$\left. + \delta_{\alpha\lambda} \delta_{\beta\gamma} \left( 1 + n \frac{d}{dn} \right) \right] F(u),$$

$$S_{\alpha\beta\gamma\lambda}^{(3)} = \sum_R' \left[ R_\beta R_\lambda \frac{\partial^2}{\partial R_\alpha \partial R_\gamma} - 2\delta_{\gamma\lambda} n R_\beta \frac{\partial^2}{\partial n \partial R_\alpha} \right.$$

$$\left. + (\delta_{\alpha\beta} \delta_{\gamma\lambda} + \delta_{\alpha\lambda} \delta_{\beta\gamma}) n \frac{\partial}{\partial n} + \delta_{\alpha\beta} \delta_{\gamma\lambda} n^2 \frac{\partial^2}{\partial n^2} \right]$$

$$\times \left\{ \frac{A}{2x} [E_+(x, y) + E_-(x, y)] - \frac{A}{x} \text{erfc}(x) \right\},$$

$$S_{\alpha\beta\gamma\lambda}^{(4)} = \frac{\delta_{\alpha\beta} \delta_{\gamma\lambda}}{2\kappa} \left[ E_+(0, y) \left( \kappa_1^2 + \kappa_2^2 - \frac{\kappa_1^2 \kappa_2^2}{\kappa} \right) \right.$$

$$\left. - \frac{\kappa_1^2 \kappa_2^2}{A\sqrt{\pi}} \exp(-u) \right] + \frac{\delta_{\alpha\lambda} \delta_{\beta\gamma}}{2\kappa} E_+(0, y) \kappa_1^2,$$

$$S_{\alpha\beta\gamma\lambda}^{(5)} = \frac{\pi n}{A^2} \sum_G' \left[ \frac{\partial^2}{\partial G_\beta \partial G_\lambda} G_\alpha G_\gamma + 2\delta_{\gamma\lambda} n G_\alpha \frac{\partial^2}{\partial n \partial G_\beta} \right.$$

$$\left. + (3\delta_{\alpha\beta} \delta_{\gamma\lambda} + \delta_{\alpha\lambda} \delta_{\beta\gamma}) n \frac{\partial}{\partial n} + \delta_{\alpha\beta} \delta_{\gamma\lambda} n^2 \frac{\partial^2}{\partial n^2} \right]$$

$$\times [D(g, u) - D(g, 0)], \quad (\text{A1})$$

where primes mean that the  $R = 0$  and  $G = 0$  terms in the lattice sums must be omitted. Furthermore,

$$F(u) = \frac{1}{u} [1 - \exp(-u)] - 1,$$

$$E_\pm(x, y) = \exp(\pm 2xy) \text{erfc}(x \pm y),$$

$$\text{erfc}(x) = E_\pm(x, 0),$$

$$D(g, u) = \frac{1}{g+u} \exp(-g-u),$$

$$u = y^2, x = \mathcal{A}R,$$

$$y = \frac{\kappa}{2\mathcal{A}}, g = \frac{G^2}{4\mathcal{A}^2},$$

$$n \frac{dF}{dn} = \frac{dF}{du} \frac{\kappa_1^2}{4\mathcal{A}^2},$$

$$n \frac{d^2 F}{dn^2} = \frac{d^2 F}{du^2} \frac{\kappa_1^2}{16\mathcal{A}^4} + \frac{dF}{du} \frac{\kappa_2^2}{4\mathcal{A}^2},$$

$$\frac{\partial^2}{\partial R_\alpha \partial R_\gamma} \frac{E_\pm}{x} = \delta_{\alpha\gamma} \frac{\mathcal{A}^2}{x^2} \left( \frac{\partial E_\pm}{\partial x} - \frac{E_\pm}{x} \right)$$

$$+ R_\alpha R_\gamma \frac{\mathcal{A}^4}{x^3} \left( \frac{\partial^2 E_\pm}{\partial x^2} - \frac{3}{x} \frac{\partial E_\pm}{\partial x} + \frac{3}{x^2} E_\pm \right),$$

$$\frac{\partial^2 E_\pm}{\partial x^2} = 4y^2 E_\pm \mp 2y E_R + 2x E_R,$$

$$\frac{\partial^2 E_\pm}{\partial y^2} = 4x^2 E_\pm - 2x E_R \pm 2y E_R,$$

$$\frac{\partial^2 E_\pm}{\partial x \partial y} = E_\pm (4xy \pm 2),$$

$$E_R = \frac{2}{\sqrt{\pi}} \exp(-x^2 - y^2),$$

$$\frac{\partial^2 (G_\alpha G_\gamma D)}{\partial G_\beta \partial G_\lambda} = (\delta_{\alpha\beta} \delta_{\gamma\lambda} + \delta_{\alpha\lambda} \delta_{\beta\gamma}) D + G_\alpha G_\gamma \frac{\partial^2 D}{\partial G_\beta \partial G_\lambda}$$

$$+ \delta_{\alpha\beta} G_\gamma \frac{\partial D}{\partial G_\lambda} + \delta_{\alpha\lambda} G_\gamma \frac{\partial D}{\partial G_\beta}$$

$$+ \delta_{\gamma\beta} G_\alpha \frac{\partial D}{\partial G_\lambda} + \delta_{\gamma\lambda} G_\alpha \frac{\partial D}{\partial G_\beta},$$

$$n \frac{\partial^2}{\partial n \partial R_\alpha} \frac{E_\pm}{x} = \frac{\mathcal{A} R_\alpha \kappa_1^2}{2x^2} \left( \frac{\partial^2 E_\pm}{\partial x \partial y} - \frac{1}{x} \frac{\partial E_\pm}{\partial y} \right),$$

$$n \frac{\partial E_\pm}{\partial n} \frac{1}{x} = \frac{\partial E_\pm}{\partial y} \frac{\kappa_1^2}{2\mathcal{A}x},$$

$$n^2 \frac{\partial^2 E_\pm}{\partial n^2} \frac{1}{x} = \frac{\partial^2 E_\pm}{\partial y^2} \frac{\kappa_1^2}{4\mathcal{A}^2 x} + \frac{\partial E_\pm}{\partial y} \frac{\kappa_2^2}{2\mathcal{A}x},$$

with all  $n$  derivatives of  $E_\pm(x, 0)$  being 0,

$$\frac{\partial E_\pm}{\partial x} = \pm 2y E_\pm - E_R,$$

$$\frac{\partial E_\pm}{\partial y} = \pm 2x E_\pm \mp E_R,$$

$$\frac{\partial D}{\partial G_\beta} = \frac{G_\beta}{2\mathcal{A}^2} \frac{\partial D}{\partial g},$$

$$\frac{\partial^2 D}{\partial G_\beta \partial G_\lambda} = \frac{\delta_{\beta\lambda}}{2\mathcal{A}^2} \frac{\partial D}{\partial g} + \frac{G_\beta G_\lambda}{4\mathcal{A}^4} \frac{\partial^2 D}{\partial g^2},$$

$$n \frac{\partial^2 D}{\partial n \partial G_\beta} = \frac{G_\beta \kappa_1^2}{8\mathcal{A}^4} \frac{\partial^2 D}{\partial g^2},$$

$$n \frac{\partial D}{\partial n} = \frac{\kappa_1^2}{4\mathcal{A}^2} \frac{\partial D}{\partial g},$$

$$n^2 \frac{\partial D}{\partial n^2} = \frac{\kappa_2^2}{4\mathcal{A}^2} \frac{\partial D}{\partial g} + \frac{(\kappa_1^2)^2}{16\mathcal{A}^4} \frac{\partial^2 D}{\partial g^2},$$

with all  $n$  derivatives of  $D(g, 0)$  being 0,

$$\kappa_1^2 = n \frac{d\kappa}{dn} = \frac{\kappa(1 + \beta^2)}{6},$$

$$\kappa_2^2 = n^2 \frac{d^2 \kappa}{dn^2} = -\frac{\kappa(5 + 3\beta^4)}{36},$$

$$\kappa_1^2 = n \frac{d\kappa^2}{dn} = \frac{\kappa^2(1 + \beta^2)}{3},$$

$$\kappa_2^2 = n^2 \frac{d^2 \kappa^2}{dn^2} = -\frac{\kappa^2(2 - \beta^2 + \beta^4)}{9}.$$

Consider stability of a one-component bcc lattice with a polarized electronic background at  $T = 0$

$$\det \left( D_{pp'}^{\mu\lambda}(\mathbf{k}) - \omega^2(\mathbf{k}) \delta_{pp'} \delta^{\mu\lambda} \right) = 0$$

$$D_{\text{TF}}^{\alpha\beta}(\mathbf{q}) \frac{M}{Z^2 e^2} = 4\pi n \sum_{\mathbf{G}} \left\{ \frac{(G^\alpha + q^\alpha)(G^\beta + q^\beta)}{(\mathbf{G} + \mathbf{q})^2 + \kappa_{\text{TF}}^2} e^{-A[(\mathbf{G} + \mathbf{q})^2 + \kappa_{\text{TF}}^2]} - \frac{G^\alpha G^\beta}{G^2 + \kappa_{\text{TF}}^2} e^{-A(G^2 + \kappa_{\text{TF}}^2)} \right\} \\ + \frac{1}{2} \sum_{\mathbf{R}}' [1 - \cos(\mathbf{q} \cdot \mathbf{R})] \left\{ \left[ (\kappa_{\text{TF}} R - 1) E_+ - (\kappa_{\text{TF}} R + 1) E_- - \frac{2R}{\sqrt{\pi A}} e^{-A\kappa_{\text{TF}}^2 - R^2/4A} \right] \right. \\ \left. \times \left( \frac{\delta^{\alpha\beta}}{R^3} - \frac{3R^\alpha R^\beta}{R^5} \right) + \left( \kappa_{\text{TF}}^2 E_+ + \kappa_{\text{TF}}^2 E_- + \frac{R}{A\sqrt{\pi A}} e^{-A\kappa_{\text{TF}}^2 - R^2/4A} \right) \frac{R^\alpha R^\beta}{R^3} \right\}.$$

$$E_{\pm} = e^{\pm \kappa_{\text{TF}} R} \operatorname{erfc} \left( \frac{R}{2\sqrt{A}} \pm \kappa_{\text{TF}} \sqrt{A} \right)$$

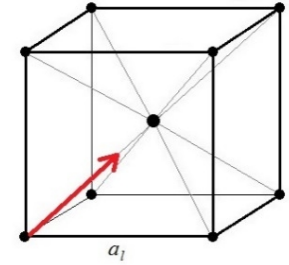
**Baiko, 2002**

$\omega^2(\mathbf{k})$  are the eigenvalues of the dynamic matrix  $D_{pp'}^{\mu\lambda}(\mathbf{k})$ . For the first time for a one-component crystal with a polarized electron background, this expression was obtained in (Baiko 2002).

When ions are in equilibrium positions, the potential energy of the lattice is minimal, so that the dynamic matrix should be positively defined. It is believed that **a lattice is unstable if at some wave vector  $\mathbf{k}$ , one or several its frequencies are complex-valued ( $\omega^2(\mathbf{k}) < 0$ ).**

Consider a deformation

$$0.5a_1(n_1, n_2, n_3) \longrightarrow 0.5a_1(n_1 + n_2\epsilon/2, n_2 + n_1\epsilon/2, n_3/(1 - \epsilon^2/4))$$



For  $\kappa_{\text{TF}}a \geq 4.76$  the bcc lattice is not stable.

For any  $\kappa_{\text{TF}}a < 4.76$  complex-valued modes appear at some  $\epsilon_{\text{max}}$ .

The breaking stress:

$$\sigma_{\text{max}} = \left. \frac{\partial (U_{\text{M}}/V)}{\partial \epsilon} \right|_{\epsilon_{\text{max}}}$$

At  $\kappa_{\text{TF}}a = 4/7$

$$\sigma_{\text{max}} = 0.019376n \frac{Z^2 e^2}{a}$$

$\kappa_{\text{TF}}a$	$\epsilon_{\text{max}}$	$\frac{\sigma_{\text{max}}}{nZ^2 e^2/a}$
0.0	0.1109	0.02007
0.4	0.1099	0.01972
0.71	0.1079	0.01901
0.87	0.1059	0.01841
1.04	0.1038	0.01773
1.19	0.1018	0.01708
1.3	0.0998	0.01651
1.42	0.0979	0.01591
1.53	0.0957	0.01529
1.62	0.0938	0.01477
1.73	0.0918	0.01417
1.81	0.0898	0.01366

which is in good agreement with the breaking stress value, obtained from molecular dynamics simulations in (Chugunov & Horowitz 2010).

At  $T \rightarrow 0$  parameter  $\Gamma \equiv Z^2 e^2 / (aT) \rightarrow \infty$

$$\sigma_{\text{b}}^{\text{max}} = \left( 0.0195 - \frac{1.27}{\Gamma - 71} \right) n_i \frac{Z^2 e^2}{a}.$$

## What is the next ???

(after Davide De Grandis talk)

still  $T = 0$

the uniform magnetic field

$$B = 0$$



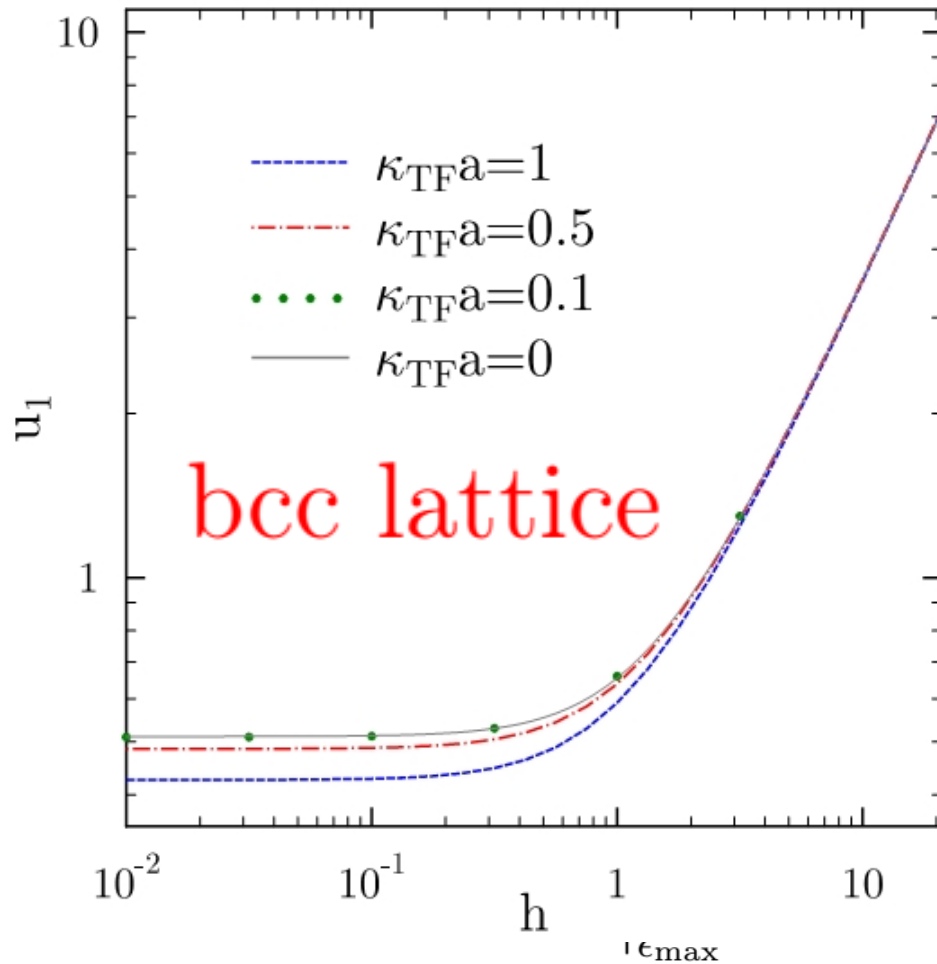
$$B \neq 0$$

$$\sigma_{\max}(\kappa_{\text{TF}}a) = \left. \frac{\partial (U_{\text{M}}/V)}{\partial \epsilon} \right|_{\epsilon_{\max}}$$

$$\sigma_{\max}(\kappa_{\text{TF}}a, h, \mathbf{n}) = \left. \frac{\partial ((U_{\text{M}} + E_0)/V)}{\partial \epsilon} \right|_{\epsilon_{\max}}$$

$E_0 = 1.5N\hbar\langle\omega_i(\mathbf{k})\rangle = 1.5N\hbar\omega_{\text{p}}u_1(h, \mathbf{n})$  is the zero-point energy  
usually  $|U_{\text{M}}| \gg E_0$

$$h \equiv \omega_B/\omega_{\text{p}} \approx B_{15}/\sqrt{\rho_8}$$



$$\sigma_b^{\max} = \left( 0.0195 - \frac{1.27}{\Gamma - 71} \right) n_i \frac{Z^2 e^2}{a}.$$

at  $\kappa_{\text{TF}} a = 4/7$  and  $B = 0$

$B \neq 0$

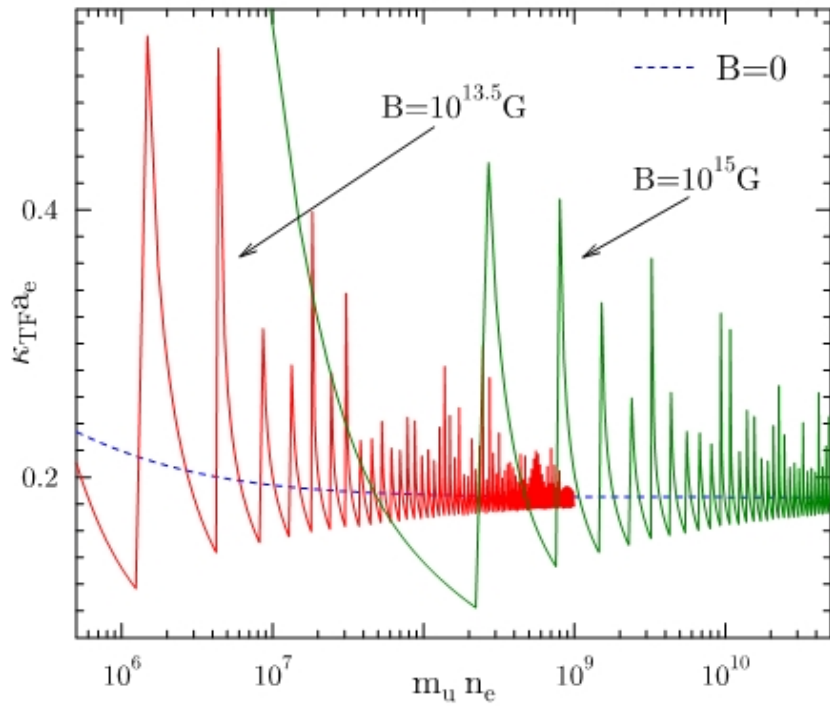
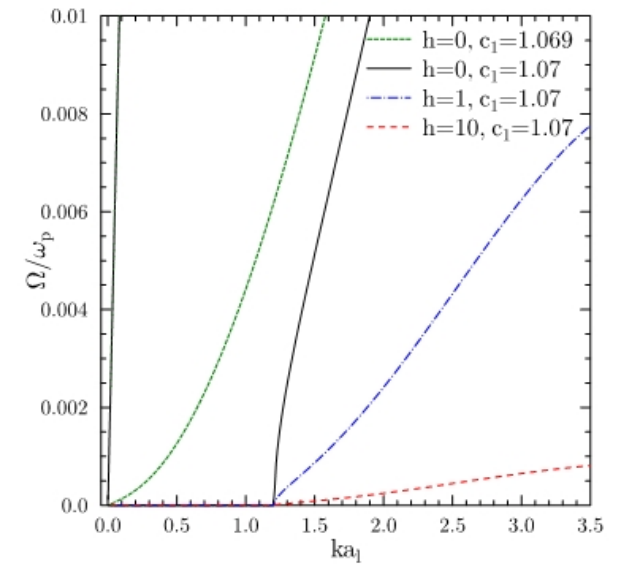
$$\sigma_{\max}(\kappa_{\text{TF}} a, h, \mathbf{n}) = \left. \frac{\partial ((U_{\text{M}} + E_0)/V)}{\partial \epsilon} \right|_{\epsilon_{\max}}$$

$E_0 = 1.5N\hbar\langle\omega_i(\mathbf{k})\rangle = 1.5N\hbar\omega_p u_1(h, \mathbf{n})$  is the zero-point energy  
usually  $|U_{\text{M}}| \gg E_0$

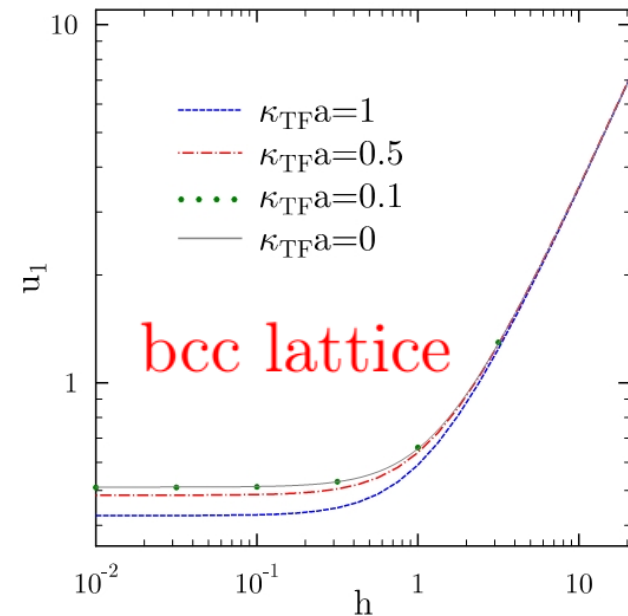
$$h \equiv \omega_B/\omega_p \approx B_{15}/\sqrt{\rho_8}$$

$$\sigma_{\max}(\kappa_{\text{TF}}a, h, \mathbf{n}) = \frac{\partial((U_M + E_0)/V)}{\partial \epsilon} \Big|_{\epsilon_{\max}}$$

$$\kappa_{\text{TF}}^2 = \frac{2Be^3\mu_e}{\pi c^2 \hbar^2} \sum_{l=0}^{l_{\max}} \frac{2 - \delta_{l,0}}{\sqrt{\mu_e^2 - (m_e c^2)^2 - 2\hbar c e B l}}$$



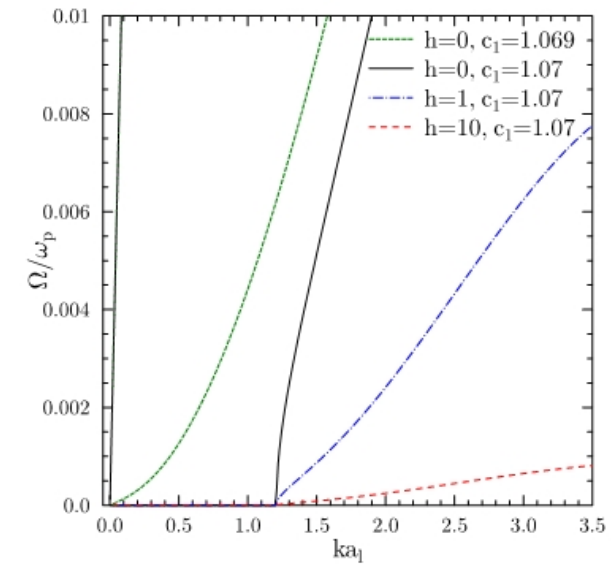
$$1.5 N \hbar \omega_p u_1(h, \mathbf{n})$$



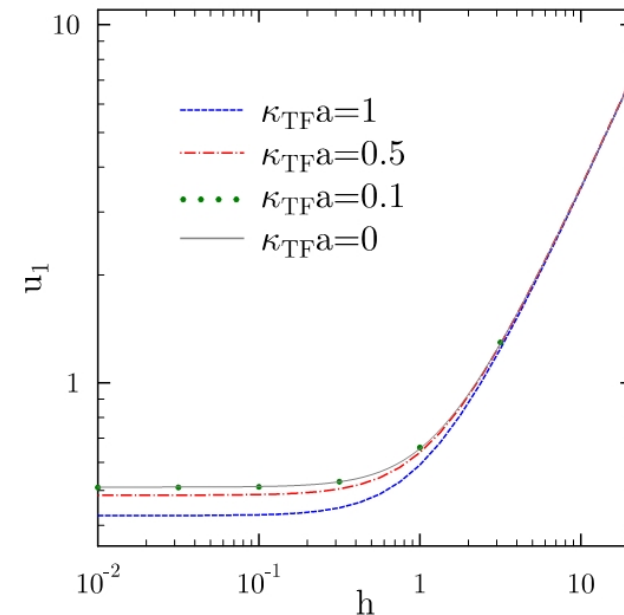
$$h \equiv \omega_B/\omega_p \approx B_{15}/\sqrt{\rho_8}$$

$$\sigma_{\max}(\kappa_{\text{TF}}a, h, \mathbf{n}) = \left. \frac{\partial ((U_M + E_0)/V)}{\partial \epsilon} \right|_{\epsilon_{\max}}$$

$$\kappa_{\text{TF}}^2 = \frac{2Be^3\mu_e}{\pi c^2\hbar^2} \sum_{l=0}^{l_{\max}} \frac{2 - \delta_{l,0}}{\sqrt{\mu_e^2 - (m_e c^2)^2 - 2\hbar c e B l}}$$



$$1.5N\hbar\omega_p u_1(h, \mathbf{n})$$



We need to calculate the zero-point energy of the deformed bcc lattice in the magnetic field.

$$h \equiv \omega_B/\omega_p \approx B_{15}/\sqrt{\rho_8}$$

## Results and conclusions:

- The elastic moduli and electrostatic pressure can be found from the expansion of the electrostatic energy of the deformed lattice.
- The elastic moduli of multi-component crystal mixtures were analytically calculated, for the first time. Corrections due to the polarization of the electron background were refined.
- It seems to us that it is more convenient to use the “traditional” definition of the effective shear modulus (Ogata, 1990). Since in this case, the results of numerical simulations are consistent with analytical.
- You can use the “linear mixing rule” to find the effective shear modulus, while it does not work for individual elastic moduli.

$$\mu_{\text{eff}}^{\text{lm}} = 0.119457 \frac{Z_1^2 e^2 n}{a} \left( \frac{N_1}{N} + \frac{N_2}{N} \alpha \right)^{1/3} \left( \frac{N_1}{N} + \frac{N_2}{N} \alpha^{5/3} \right)$$

$$N_1 \Rightarrow Z_1, \quad N_2 \Rightarrow Z_2, \quad N = N_1 + N_2, \quad \alpha \equiv Z_2/Z_1$$

# Physics of Neutron Stars 2020

## Invited Speakers:

- Matteo Bachetti
- Konstantinos N. Gourgouliatos
- Alice K. Harding (TBC)
- Jason W. T. Hessels
- James M. Lattimer
- Sandro Mereghetti
- Samaya M. Nissanke
- Evan P. O'Connor
- Alessandro Papitto
  - Emily Petroff
- Alexander A. Philippov
  - Bettina Posselt
- Sergey S. Tsygankov
- Roberto Turolla
  - Anna L. Watts

July 20 - July 24

St. Petersburg, Russia



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The registration will be closed on the 16th of March, 2020