Potentially observable cylindrical wormholes without exotic matter in general relativity

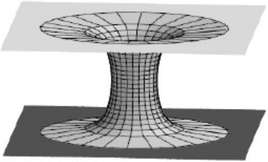
K.A. Bronnikov
(VNIIMS, Moscow; RUDN University, Moscow; MEPhI, Moscow)
Collaboration with
M.V. Skvortsova, S.V. Bolokhov, V.G. Krechet, J.P.S. Lemos

Modern Physics of Compact Stars and Relativistic Gravity 2019
Yerevan, 17-21.09.2019

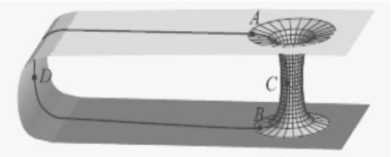
Motivation and plan

- **Wormholes in GR: main problem:**
  Necessity of exotic matter, NEC/WEC violation
- **Notion of a wormhole**: spherical vs. cylindrical
- **Why GR?** Observational status, interest in macroscopic size
- **Why cyl. symmetry?** Attempt to circumvent "topol. censorship"
- **Static cylindrical wormholes**
  - * A no-go theorem: ($\rho < 0$) for twice asympt. regular wormholes
  - * Example: Einstein-Maxwell fields
- **Rotation**.
  - * Structure of the equations, **favorable for WH construction**
  - * Problem: Bad asymptotic behavior
  - * Examples: vacuum, scalar-vacuum solutions, anisotr. fluid etc.
  - * Problem: no flat infinity $\Rightarrow$ no observability
- **Thin shells**: attempt to construct a potentially observable wormhole by joining the throat region with flat space regions
- Examples and concluding remarks

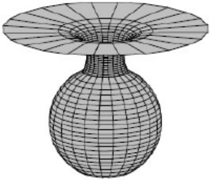
Wormholes (spherical vs. cylindrical)



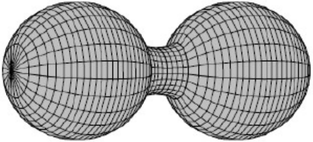
Sph: twice asymptotically flat wormhole (topology $S^2 \times R$).
Cyl: topology $S^1 \times R \times R$



A wormhole as a "handle", a shortcut between remote parts of the Universe (or a time machine if times at A and B are essentially different). The topology depends on that of the ambient world



A "hanging drop" wormhole.
Topology:
R^3 for both Sph and Cyl



A "dumbbell" wormhole
Sph: topology S^3
Cyl: topology $S^2 \times R$

General relativity (A. Einstein, 1915-16): **gravity = curvature**

If space-time is curved, it can be very strongly curved !!!

K. Schwarzschild, 1916 - First exact solution, now termed a **black hole**

L. Flamm, 1916: the Schwarzschild space as a kind of tunnel

A. Einstein, N. Rosen, 1935: attempt to create a particle model based on GR (Einstein-Rosen bridge)

J.A. Wheeler, 1955,... - quanta, space-time foam, charge without charge, mass without mass, the term **"wormhole"**

1973 and so on, **up to now** (H. Ellis, K.B., G. Clement, ...) - numerous exact wormhole solutions in different theories, analysis of their properties, including stability

K. Thorne et al., 1988, V.P. Frolov, I.D. Novikov, 1989........: beginning of a broad discussion of time machines, shortcuts etc, Static, spherically symmetric space-times:

$$ds^2 = A(u)dt^2 - B(u)du^2 - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2)$$

A wormhole throat: a regular minimum of ***r(u):*** $r = r_{th}$

A wormhole: a regular configuration where ***A(u)>0*** and ***B(u)>0*** everywhere (*no horizons*), and, far from the throat, on its both sides, $r >> r_{th}$.

The Universe may also contain structures **infinitely extended** along a certain direction, like cosmic strings. While starlike structures are, in the simplest case, described by spherical symmetry, the simplest stringlike configurations are **cylindrically symmetric.**

$$ds^2 = A(u)dt^2 - B(u)du^2 - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2)$$

At the **throat**: $r' = 0$, $r'' > 0$ =>

for matter of general form compatible with the symmetry, $T_\mu^\nu = \mathrm{diag}(\rho,\ -p_r,\ -p_\perp,\ -p_\perp)$,

these conditions lead to

$$\rho + p_r < 0, \qquad p_r < 0.$$

("Exotic" matter, violation of the Null Energy Condition. But no restriction on sign ρ)

Flat asymptotic: at large r approx.Schwarzschild, with mass m.

Mass function: $m(r) = 4\pi G \int_{r_0}^{r} \rho r^2 dr,$ r_0 = integration constant.

At the throat, $2m(r) = r$. Integrating from $r_0 = r_{th}$, we obtain $r_{\rm th} = 2m - \varkappa \int_{r_{\rm th}}^{\infty} \rho r^2 dr,$

<u>**This means:**</u> if $\rho > 0$, then $r_{th} < 2m = r$ (Schwarzschild).

a wormhole with a throat of a few meters will have **huge gravity** of, say, Jupiter !!

To avoid that, negative densities are necessary.

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\varphi^2$$

Consider the most natural situation that the wormhole **is observed as a stringlike source of gravity** from an otherwise very weakly curved or even flat environment.

We require: there is a spatial infinity, i.e., at some $u = u_\infty$, $r \equiv e^\beta \to \infty$, the metric is either flat or corresponds to the gravitational field of a cosmic string.

This means: (1) $\gamma \to \text{const}, \quad \xi \to \text{const} \quad \text{as } u \to u_\infty$

(2) at large r, $|\beta'|e^{\beta-\alpha} \to 1 - \mu, \quad \mu = \text{const} < 1 \quad \text{as } u \to u_\infty$

(the parameter **μ** is an **angular defect**). A ***flat asymptotic:*** **μ =0**.

We say ***"a regular asymptotic"*** in the sense ***"a flat or string asymptotic."***

Static cyl. systems: Einstein equations

$$\boxed{ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\varphi^2}$$

Einstein equations: $G_\mu^\nu = -\varkappa T_\mu^\nu, \qquad \varkappa = 8\pi G,$

or

$$R_\mu^\nu = -\varkappa \tilde{T}_\mu^\nu, \qquad \tilde{T}_\mu^\nu = T_\mu^\nu - \frac{1}{2}\delta_\mu^\nu T_\alpha^\alpha$$

We have

$$R_0^0 = -e^{-2\alpha}[\gamma'' + \gamma'(\gamma' - \alpha' + \beta' + \xi')],$$

$$R_1^1 = -e^{-2\alpha}[\gamma'' + \xi'' + \beta'' + \gamma'^2 + \xi'^2 + \beta'^2 - \alpha'(\gamma' + \xi' + \beta')],$$

$$R_2^2 = -e^{-2\alpha}[\xi'' + \xi'(\gamma' - \alpha' + \beta' + \xi')],$$

$$R_3^3 = -e^{-2\alpha}[\beta'' + \beta'(\gamma' - \alpha' + \beta' + \xi')],$$

$$G_1^1 = e^{-2\alpha}(\gamma'\xi' + \beta'\gamma' + \beta'\xi').$$

The most general form of the stress-energy tensor:

$$T_\mu^\nu = \mathrm{diag}(\rho, -p_r, -p_z, -p_\phi),$$

where ρ = energy density,

p_i = pressures of any physical origin in the respective directions.

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\varphi^2$$

We use the harmonic radial coordinate: $\alpha = \beta + \gamma + \mu$.

1. **At a minimum of circular radius *r (u)*, due to β'=0 and β'' > 0,** we have ***R_3^3* < 0**, and from the corresponding component of the Einstein eqs it follows that

$$T_0^0 + T_1^1 + T_2^2 - T_3^3 = \rho - p_r - p_z + p_\phi < 0.$$

If $T_2^2 = T_3^3$, that is, $p_z = p_\phi$, in particular, for **Pascal isotropic fluids** we obtain $p_r > \rho$, violation of ***Dominant Energy Condition*** (if, as usual, ρ > 0).

In the general case of anisotropic pressures, **none of the standard energy conditions are necessarily violated.**

2. However, **if the throat is defined through *the area function a(u) ≡ exp(β+ξ)***, we have there β' + ξ' = 0, β'' + ξ'' > 0, whence $R_2^2 + R_3^3 < 0 \Rightarrow T_0^0 + T_1^1 = \rho - p_r < 0.$

In addition, substituting β' + ξ' = 0 into the Einstein equation $G_1^1 = -\varkappa T_1^1$, we find

$-T_1^1 = p_r \leq 0.$ Combining these two conditions, we see that

$$\rho < p_r \leq 0$$

on the throat, i.e., **there is necessarily a region with negative energy density !!!**

Electromagnetic fields with cyl. symmetry:

Radial (R): $F_{01}(u)\ (E^2 = F_{01}F^{10})$, $F_{23}(u)\ (B^2 = F_{23}F^{23})$.

Azimuthal (A): $F_{03}(u)\ (E^2 = F_{03}F^{30})$, $F_{12}(u)\ (B^2 = F_{12}F^{12})$.

Longitudinal (L): $F_{02}(u)\ \ (E^2 = F_{02}F^{20})$, $F_{13}(u)\ (B^2 = F_{13}F^{13})$.

(E and B = abs. values of electric field strength and magnetic induction, resp.)

$$L_e = -\Phi(F)/(16\pi), \qquad F := F^{\mu\nu}F_{\mu\nu} \qquad \text{Maxwell:} \qquad \Phi(F) \equiv F.$$

Solutions are known: for Maxwell fields [K.B., 1979]
and NED [K. B, G. N. Shikin, and E. N. Sibileva, 2003].

Wormhole solutions: easily obtained with **A-fields only** (Maxwell ED)
or **preferably** (NED) . Example (Maxwell ED):

$$ds^2 = \frac{\cosh^2(hu)}{Kh^2}[e^{2au}dt^2 - e^{2(a+b)u}du^2 - e^{2bu}d\varphi^2] - \frac{Kh^2}{\cosh^2(hu)}dz^2,$$

$$F_{03} = i_m = \text{const}; \qquad F^{12} = i_e e^{-2\alpha}, \qquad i_e = \text{const},$$

$$K = [G(i_e^2 + i_m^2)]^{-1}, h^2 = ab, a, b = \text{const} \qquad a > 0, \qquad b > 0$$

Conclusion on static cyl. wormholes

- Cyl wormhole geometries (with r-throats) can exist without WEC or NEC violation.
- At an a-throat, negative density is required.
- Significant difficulty (as always with cyl. symmetry): obtaining desirable asymptotics.
- Twice asymptotically regular wormholes necessarily involve negative density.

Can rotation help?

Rotation: Ricci and Einstein tensors

$$\boxed{ds^2 = e^{2\gamma(u)}[dt - E\,e^{-2\gamma}d\varphi]^2 - e^{2\alpha}dx^2 - e^{2\mu}dz^2 - e^{2\beta}d\varphi^2}$$

In the comoving reference frame (arbitrary gauge):

$$R_1^1 = -e^{-2\alpha}[\beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'^2 - \alpha'(\beta' + \gamma' + \mu')] + 2\omega^2;$$

$$R_2^2 = \Box_1 \mu;$$

$$R_3^3 = \Box_1 \beta + 2\omega^2 \qquad \omega = \tfrac{1}{2}(E\,e^{-2\gamma})'\,e^{\gamma-\beta-\alpha}$$

$$R_4^4 = \Box_1 \gamma - 2\omega^2 \qquad \omega = \omega_0 e^{-\mu-2\gamma}, \quad \omega_0 = \text{const.}$$

where $\Box_1 f = -g^{-1/2}[\sqrt{g}g^{11}f']' = -e^{-2\alpha}[f'' + f'(\beta' + \gamma' + \mu' - \alpha')]$.

so that $R_\mu^\nu = {}_sR_\mu^\nu + {}_\omega R_\mu^\nu, \quad {}_\omega R_\mu^\nu = \omega^2\,\mathrm{diag}(-2, 2, 0, 2),$

$$G_\mu^\nu = {}_sG_\mu^\nu + {}_\omega G_\mu^\nu, \quad {}_\omega G_\mu^\nu = \omega^2\,\mathrm{diag}(-3, 1, -1, 1)$$

$\underset{s}{G_\mu^\nu}$ and $\underset{\omega}{G_\mu^\nu}$ (each separately) satisfy the "conservation law"

$\nabla_\alpha G_\mu^\alpha = 0$ with respect to the static metric with $E \equiv 0$

The rotational part of the Einstein tensor behaves in the Einstein equations as **an additional SET with very exotic properties:**
the energy density is $-3\omega^2/\varkappa < 0$

$$\boxed{ds^2 = e^{2\gamma(u)}[dt - E\, e^{-2\gamma} d\varphi]^2 - e^{2\alpha} dx^2 - e^{2\mu} dz^2 - e^{2\beta} d\varphi^2}$$

Definitions of throats: the same as in the static case:

r-throat: a regular minimum of the circular radius $r(x) = \exp(\beta)$.
r-wormhole: a regular configuration with $r \gg r_{\min}$ on both sides.

a-throat: a regular minimum of the area function $a(x) = \exp(\beta + \mu)$.
a-wormhole: a regular configuration with $a \gg a_{\min}$ on both sides.

Due to rotation, it is much easier to obtain wormholes than with $\omega = 0$.

Main problem: **bad asymptotic behavior**,

e.g., we do not have $\omega \to 0$ where $\gamma, \mu \to \mathrm{const}$ since

$$\omega = \omega_0\, e^{-\mu - 2\gamma}, \qquad \omega_0 = \mathrm{const},$$

at least in this comoving reference frame.

Example 1: a massless scalar field

$$L_s = \frac{1}{2}\varepsilon \partial_\alpha\phi\partial^\alpha\phi$$

Harmonic radial coordinate: ***x* = *u***, $\alpha = \beta + \gamma + \mu$

$$R_2^2 = 0 \quad\Rightarrow\quad \mu'' = 0,$$
$$R_3^3 = 0 \quad\Rightarrow\quad \beta'' - 2\omega^2\, e^{2\alpha} = 0,$$
$$R_4^4 = 0 \quad\Rightarrow\quad \gamma'' + 2\omega^2\, e^{2\alpha} = 0,$$

$$\mu = -mu \qquad \text{[with a certain choice of } z \text{ scale]},$$
$$\beta + \gamma = 2hu \qquad \text{[with a certain choice of } t \text{ scale]},$$
$$\beta'' - \gamma'' = 4\omega_0^2\, e^{2\beta - 2\gamma}.$$

Solution:

$$\phi = Cu$$

$$\omega = \frac{e^{mu-2hu}}{2s(k,u)},$$

$$e^{2\beta} = \frac{e^{2hu}}{2\omega_0 s(k,u)},$$
$$e^{2\gamma} = 2\omega_0 s(k,u)\, e^{2hu},$$
$$e^{2\mu} = e^{-2mu},$$

$$E = e^{2hu} s(k,u) \int du\, \frac{e^{2hu}}{s(k,u)}.$$

$$s(k,u) = \begin{cases} k^{-1}\sinh ku, & k > 0, \quad u \in \mathbb{R}_+; \\ u, & k = 0, \quad u \in \mathbb{R}_+; \\ k^{-1}\sin ku, & k < 0, \quad 0 < u < \pi/|k|. \end{cases}$$

$$k^2\, \mathrm{sign}\, k = 4(h^2 - 2hm) - 2\varkappa\varepsilon C^2.$$

Parameters: ω_0, *C*, *h*, *m*, *k*. (All inessential constants have been absorbed.)

Wormhole solutions: all solutions with $k < 0$, and many with $k \geq 0$.

Asymptotic behavior: in all cases, either $\exp(\gamma) \to 0$, or $\exp(\gamma) \to \infty$.

Building asymptotically flat models

Trying to build a wormhole model with two flat asymptotics

$X < x_- < 0$	$x_- < x < x_+$	$x > x_+ > 0$
M- (flat, Ω_-)	**V(internal WH metric)**	**M+ (flat, Ω_+)**
Σ-: $x = x_-$	r-throat a-throat	**Σ+: $x = x_+$**

Metrics in M+ and M-: $$ds_{\rm M}^2 = dt^2 - dX^2 - dz^2 - X^2(d\varphi + \Omega dt)^2.$$

Relevant quantities:

$$e^{2\gamma} = 1 - \Omega^2 X^2, \qquad e^{2\beta} = \frac{X^2}{1 - \Omega^2 X^2},$$

$$E = \Omega X^2, \qquad \omega = \frac{\Omega}{1 - \Omega^2 X^2}.$$

(arbitrary scales along the z and t axes can be chosen)

Matching: $$[\beta] = 0, \qquad [\mu] = 0, \qquad [\gamma] = 0, \qquad [E] = 0,$$

Next step:
find the surface stress-energy tensors on Σ+ and Σ-

This is done in Darmois-Israel formalism in terms of the extrinsic curvature:

$$S_a^b = -\frac{1}{8\pi}[\tilde{K}_a^b], \qquad \tilde{K}_a^b := K_a^b - \delta_a^b K, \qquad K = K_a^a, \qquad \textbf{[f] := f(+) - f(-).}$$

With natural parametrization of **Σ+** and **Σ-** $K_{ab} = -\,e^{\alpha(u)}\Gamma^1_{ab} = \frac{1}{2}e^{-\alpha(u)}\frac{\partial g_{ab}}{\partial x^1}.$

$$\Rightarrow \quad \tilde{K}_{00} = -e^{-\alpha+2\gamma}(\beta' + \mu'),$$
$$\tilde{K}_{03} = -\frac{1}{2}e^{-\alpha}E' + Ee^{-\alpha}(\beta' + \gamma' + \mu'),$$
$$\tilde{K}_{22} = e^{-\alpha+2\mu}(\beta' + \gamma'),$$
$$\tilde{K}_{33} = e^{-\alpha+2\beta}(\gamma' + \mu') + e^{-\alpha-2\gamma}[EE' - E^2(\beta' + 2\gamma' + \mu')].$$

S^0_0 is the surface density while $-S^2_2 = p_z$ and $-S^3_3 = p_\varphi$ are pressures in the respective directions.

(No need to adjust coordinates in different regions since all relevant quantities are reparametrization-independent.)

Can both surface SETs be physically plausible and non-exotic under some choice of the system parameters?

Criterion: the WEC (including the NEC):

$$S_{00}/g_{00} = \sigma \geq 0, \qquad S_{ab}\xi^a\xi^b \geq 0, \tag{1}$$

where ξ^a is any null vector ($\xi^a\xi_a = 0$) on $\Sigma = \Sigma_\pm$; the second inequality comprises the NEC as part of the WEC.

Instead of working with arbitrary ξ^a, it is sufficient to find σ and principal pressures p_z, p_φ as eigenvalues of the surface SET in local Minkowski (tangent) space, or, equivalently, the quantities $\widetilde{K}_{mn}$. To do that, we can use any orthonormal triad on $\Sigma_\pm$.

Building phantom-free models 2

Let us find the tangent-space (triad) components of $\widetilde{K}_{ab}$ using the following orthonormal triad on Σ:

$$e^a_{(0)} = (\mathrm{e}^{-\gamma}, 0, 0); \qquad e^a_{(2)} = (0, \mathrm{e}^{-\mu}, 0); \qquad e^a_{(3)} = (E\, \mathrm{e}^{-\beta-2\gamma}, 0, \mathrm{e}^{-\beta}) \tag{19}$$

(the parentheses mark triad indices). The triad components $\widetilde{K}_{(mn)} = e^a_{(m)} e^b_{(n)} \widetilde{K}_{ab}$ turn out to be surprisingly simple and may be represented by the matrix

$$(\widetilde{K}_{(mn)}) = \begin{pmatrix} -\mathrm{e}^{-\alpha}(\beta' + \mu') & 0 & -\omega \\ 0 & \mathrm{e}^{-\alpha}(\beta' + \gamma') & 0 \\ -\omega & 0 & \mathrm{e}^{-\alpha}(\gamma' + \mu') \end{pmatrix}. \tag{20}$$

The shell matter SET consists of discontinuities of these matrix elements divided by $\varkappa$. The matrix of these discontinuities has the same structure as (20):

$$([\widetilde{K}_{(mn)}]) = \begin{pmatrix} a & 0 & d \\ 0 & b & 0 \\ d & 0 & c \end{pmatrix}, \tag{21}$$

As a result, the WEC requirements read

$$a + c + \sqrt{(a-c)^2 + 4d^2} \geq 0,$$

$$a + c + \sqrt{(a-c)^2 + 4d^2} + 2b \geq 0,$$

$$a + c \geq 0.$$

No-go theorem

Many internal wormhole solutions are unable to meet these requirements, In particular, those in which

$$T_t^t = T_\varphi^\varphi$$

This includes, for example, field systems with

$$L = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi) - P(\phi)F^{\mu\nu}F_{\mu\nu},$$

i.e., **scalar fields** with an arbitrary potential ***V(ϕ)*** and an arbitrary function ***P(ϕ)*** characterizing the scalar-electromagnetic interaction, assuming that ***ϕ = ϕ(x)*** and that the **Maxwell tensor** describes a stationary azimuthal magnetic field ($F_{21} = -F_{12}$) or its electric analog.

It has been proven that for all such systems the NEC is inevitably violated either on Σ+ or on Σ- or on both

[K.B., arXiv: 1509.06924].

Equations for perfect fluids

For perfect fluids with $p = w\rho, \quad w = \mathrm{const}$, we have $T_\mu^\nu = \rho\,\mathrm{diag}(1, -w, -w, -w)$. The conservation law $\nabla_\nu T_\mu^\nu = 0$ has the form

$$p' + (\rho + p)\gamma' = 0 \quad \Rightarrow \quad w\rho' + (1 + w)\gamma' = 0.$$

(the same as in the static case). For $w \neq 0$ it gives

$$\rho = \rho_0\, \mathrm{e}^{-\gamma(w+1)/w}, \qquad \rho_0 = \mathrm{const},$$

and, in terms of the harmonic coordinate x ($\alpha = \beta + \gamma + \mu$), the Einstein equations for our metric read

$$\begin{aligned}
&\mathrm{e}^{-2\alpha}\gamma'' + 2\omega^2 = \tfrac{1}{2}(3w + 1)\varkappa\rho, \\
&\mathrm{e}^{-2\alpha}\mu'' = \tfrac{1}{2}(w - 1)\varkappa\rho, \\
&\mathrm{e}^{-2\alpha}\beta'' - 2\omega^2 = \tfrac{1}{2}(w - 1)\varkappa\rho, \\
&\mathrm{e}^{-2\alpha}(\beta'\gamma' + \beta'\mu' + \gamma'\mu') + \omega^2 = w\varkappa\rho.
\end{aligned}$$

Also recall that $\omega = \omega_0\, \mathrm{e}^{-\mu - 2\gamma}$.
These equations are completely analytically solved only for $w = -1$ (a cosmological constant) - a case of no interest due to the above no-go theorem.
In other cases we can seek special solutions.

Example 2: Stiff perfect fluid. Solution

Assume $w = 1$. Then there is a special exact wormhole solution:

$$\mu, \gamma = \mathrm{const}, \quad e^{\beta} = \frac{k}{\sqrt{2\omega_0^2}\cos(kx)}, \quad k = \mathrm{const} > 0, \quad x \in (-\pi/2, \pi/2),$$

$$E = \frac{1}{\omega_0}\int \frac{k^2 dx}{\cos^2(kx)} = \frac{k}{\omega_0}\tan(kx),$$

where the integration constants are chosen to make the system symmetric with respect to the throat surface $x = 0$. Thus the metric is known completely.
In a more convenient radial variable, $y = k\tan(kx)$, the metric reads

$$ds^2 = \left(dt - \frac{y}{\omega_0}d\varphi\right)^2 - \frac{dy^2}{2\omega_0^2(k^2+y^2)} - dz^2 - (k^2+y^2)\frac{d\varphi^2}{2\omega_0^2}.$$

The solution is regular in the whole range $y \in \mathbb{R}$, but at $y^2 > k^2$ we have $g_{33} > 0$, hence the φ-circles are timelike and violate causality.
To match this metric to the flat one at some $y = y_0$ identified with some $X = X_0$ in Minkowski space, we rescale time assuming

$$dt = \sqrt{P}d\tau, \quad P = \mathrm{const} < 1.$$

Then we can identify the time τ on $\Sigma_\pm$ with t from the external metric and provide $[\gamma] = 0$. Now we can implement matching and try to satisfy the WEC.

Consider $\Sigma_\pm$ $(y = \pm y_0 > 0,\ X = \pm X_0,\ X_0 > 0)$.
The **matching conditions** $[\gamma] = 0,\ [E] = 0.,\ [\beta] = 0$ give, respectively,

$$P = 1 - \Omega^2 X^2, \qquad \Omega X^2 = \frac{y\sqrt{P}}{\omega_0}, \qquad \frac{k^2 + y^2}{2\omega_0^2} = \frac{X^2}{P};$$

we assume $\omega_0 > 0$ and omit the index "zero" at X and y.
Independent parameters: P, k, ω_0, y (internal metric), Ω, X (external).
These six parameters are connected by the above three equalities. We choose as as **independent parameters** $X = \pm X_0,\ y = \pm y_0,\ P < 1$, so that

$$\Omega = \frac{\sqrt{1-P}}{X}, \qquad \omega_0 = \frac{\sqrt{P}y}{\sqrt{1-P}X}, \qquad k^2 = y^2 \frac{1+P}{1-P},$$

The quantities a, b, c, d from the above **WEC conditions** are expressed as

$$a = \frac{P^{3/2}|y| - 1}{P|X|}, \qquad b = \frac{1 - |y|\sqrt{P}}{|X|}, \qquad c = \frac{P-1}{P|X|}, \qquad d = \frac{\sqrt{P}y \mp 1 \pm P}{XP\sqrt{1-P}}.$$

It is directly verified that the WEC holds on both $\Sigma_\pm$ if $\boxed{y \geq \frac{2-P}{P^{3/2}}.}$

Important: We have $y_0^2 < k^2$, hence $y^2 < k^2$ in the whole internal region, and there are **no closed timelike curves**.

Example 3: Special anisotropic fluid. Solution

Consider an anisotropic fluid with

$$T_\mu^\nu = \rho \ \mathrm{diag}(1, -1, 1, -1) \ \oplus T_3^0 = -2\rho E\, \mathrm{e}^{-2\gamma}$$

(similar to a z-directed magnetic field in the static case, not directly extended to $E \neq 0$). From $\nabla_\mu T_1^\mu = 0$ it follows

$$\rho = \rho_0\, \mathrm{e}^{-2\gamma - 2\mu}, \qquad \rho_0 = \mathrm{const} > 0,$$

The Einstein equations are solved using again the harmonic radial coordinate x:

$$r^2 \equiv \mathrm{e}^{2\beta} = \frac{r_0^2}{Q^2(x_0^2 - x^2)}, \qquad \mathrm{e}^{2\gamma} = Q^2(x_0^2 - x^2),$$

$$\mathrm{e}^{2\mu} = \mathrm{e}^{2mx}(x_0 - x)^{1 - x/x_0}(x_0 + x)^{1 + x/x_0},$$

$$E = \frac{r_0}{2x_0^2}\left[2x_0 x + (x_0^2 - x^2) \ln \frac{x_0 + x}{x_0 - x}\right], \qquad x_0 := \frac{|\omega_0|}{\varkappa \rho_0 r_0}, \qquad Q^2 := \varkappa \rho_0 r_0^2,$$

The coordinate x ranges from $-x_0$ to x_0. Integration constants: ω_0, ρ_0, r_0, m, plus introduced dimensionless.constants x_0, Q.

The circular radius $r \to \infty$ as $x \to \pm x_0$, $\Rightarrow$ wormhole nature of the geometry, but $x = \pm x_0$ are singularities: the Kretschmann invariant behaves as $|x_0 - x|^{-4}$.

Assume $m = 0$ (symmetry w.r.t. the throat $x = 0$). Matching at some $x = \pm x_s < x_0$ to Minkowski regions at $X = \pm X_s \quad \Rightarrow$

$$X = r_0, \qquad Q^2(x_0^2 - x^2) = 1 - \Omega^2 X^2 =: P,$$

$$2x_0^2\sqrt{1-P} = 2xx_0 + (x_0^2 - x^2)\ln\frac{x_0 + x}{x_0 - x}, \qquad (x = x_s).$$

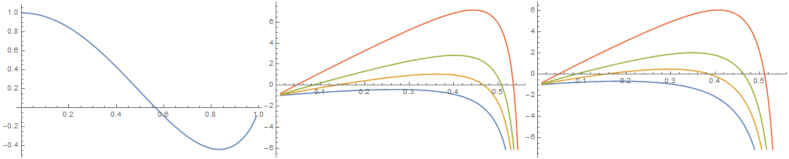
For $[\mu] = 0$ we adjust the z scale $\Rightarrow \quad -g_{zz} = M^2 := \mathrm{e}^{2\mu(x_s)}$ from the internal metric. Denoting $y = x_s/x_0$ and $L(y) = \ln[(1+y)/(1-y)]$, we have

$$M = M(y) = (1-y)^{-(1-y)/2}(1+y)^{-(1+y)/2},$$
$$P = P(y) = (1-y^2)\Big[1 - yL(y) - \tfrac{1}{4}(1-y^2)L^2(y)\Big].$$

Then the quantities a, b, c, d in the WEC requirements are

$$a = [-\mathrm{e}^{-\alpha}(\beta' + \mu')] = -\frac{1}{P(y)} + \frac{M(y)}{x_0^2}\left(\frac{y}{1-y^2} + \frac{1}{2}L(y)\right),$$
$$b = [\mathrm{e}^{-\alpha}(\beta' + \gamma')] = 1,$$
$$c = [\mathrm{e}^{-\alpha}(\gamma' + \mu')] = -\frac{1}{P(y)} + 1 + \frac{M(y)}{x_0^2}\left(\frac{y}{1-y^2} - \frac{1}{2}L(y)\right),$$
$$d = -[\omega] = -\frac{\sqrt{1-P(y)}}{P(y)} \pm \frac{M(y)}{x_0^2(1-y^2)}.$$

Example 3: Special anisotropic fluid. The WEC — 2



Left: $P(y)$. Middle: $a(y)$ for $x_0 = 0.3,\ 0:4,\ 0:5, 0:75$ (upside down).
Right: $a(y) + c(y)$ for $x_0 = 0.3,\ 0.4,\ 0.5,\ 0:75$ (upside down).

The expressions for a, b, c, d depend on two parameters, x_0 and y, and a, b, c are the same on Σ_+ and Σ_-, while d does not affect the results. Actually, if $a > 0$ and $a + c > 0$, then WEC holds. It can be found that (see the figures)

- The condition $0 < P(y) < 1$, required by construction, holds for $0 < y < 0.564$ (all numerical estimates are approximate);
- $a > 0$ and $a > c$ hold in a large range of x_0 and y, for example, $x_0 = 0.5,\ y \in (0.15, 0.47)$ and $x_0 = 0.3,\ y \in (0.05, 0.53)$.
- $a + c > 0$ in almost the same range of the parameters, e.g, $x_0 = 0.5,\ y \in (0.15, 0.38)$ and $x_0 = 0.3,\ y \in (0.05, 0.51)$.

Thus in a significant range in the parameter space (x_0, y) our asymptotically flat wormhole model completely satisfies the WEC. Also, $g_{\varphi\varphi} < 0$ between Σ_- and Σ_+, hence the model does not contain closed timelike curves.

Conclusion on rotating wormholes in GR

Stationary cylindrical configurations:

The vortex grav. field is singled out and behaves like matter with exotic properties. Exact solutions have been found for scalar fields and some kinds of anisotropic perfect fluid.

Is it possible to have WH geometry without exotic matter?

YES (a vortex gravitational field instead of WEC violating matter)

Can they have two flat (or string) asymptotic regions?

NO if we consider pure solutions with rotation

YES with Minkowski regions and rotating thin shells, under a proper choice of matter in the internal region

---

K.B., J.P.S. Lemos, *Cylindrical wormholes*, PRD 79, 104089 (2009); arXiv: 0902.2360

K.B., V. G. Krechet, J.P.S. Lemos, *Rotating cylindrical wormholes*, Phys. Rev. D 87, 084060 (2013); arXiv: 1303.2993

K.B., V. G. Krechet, *Potentially observable cylindrical wormholes without exotic matter in GR*, Phys. Rev. D 99, 084051 (2019); arXiv: 1807.03641

K.B., S,V. Bolokhov, M.V. Skvortsova, *Cylindrical wormholes. A search for viable models In GR.* IJMPD.{\bf 28}, 1941008 (2019), arXiv: 1903.09862

THANK YOU!