Transverse momentum distributions on and off the lattice

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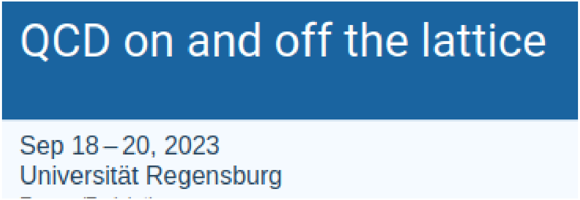
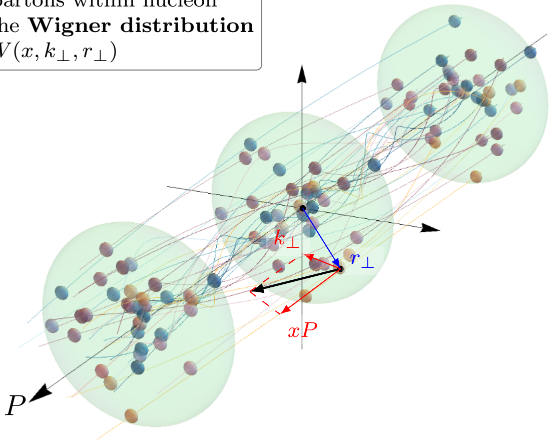


Hadron is a 3D object

Complete information about
motion of partons within nucleon
is encoded in the **Wigner distribution**
$W(x, k_\perp, r_\perp)$



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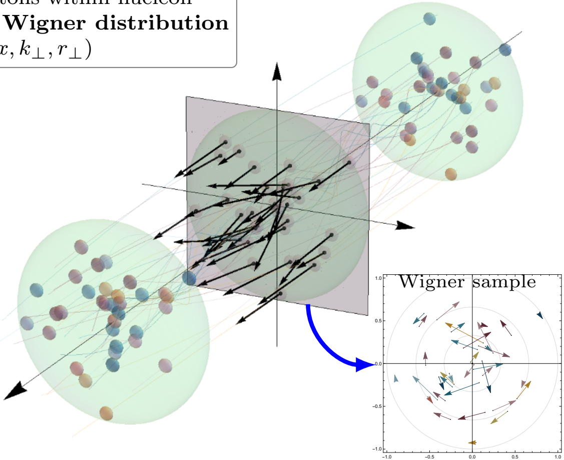
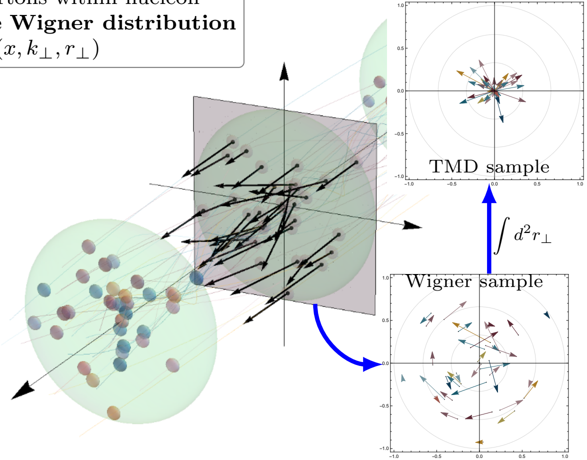


Hadron is a 3D object

Complete information about motion of partons within nucleon is encoded in the **Wigner distribution** $W(x, k_\perp, r_\perp)$

Transverse Momentum Dependent (**TMD**) distribution



Hadron is a 3D object

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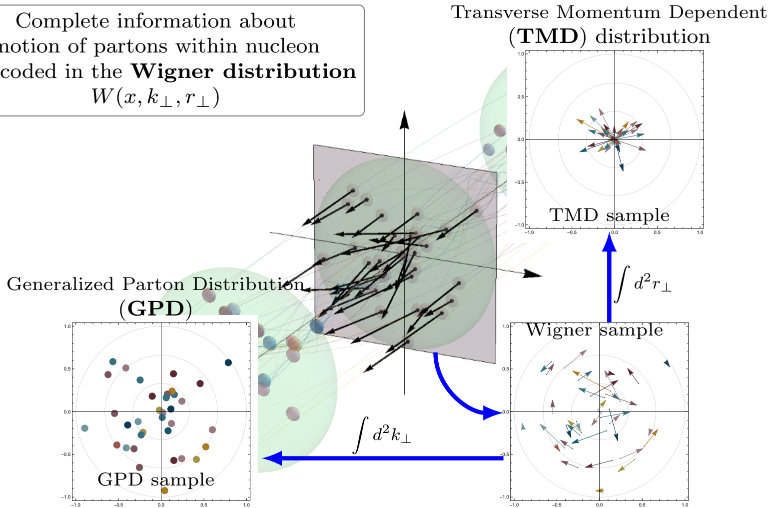
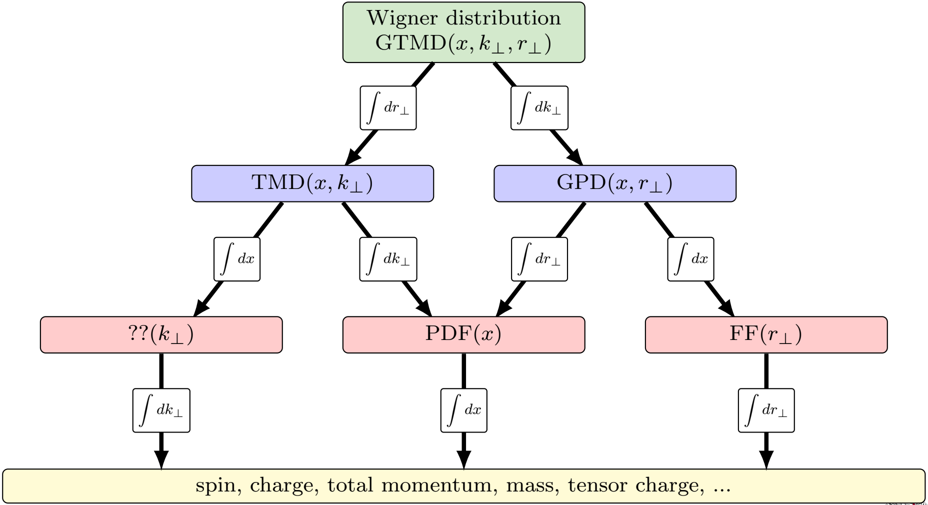
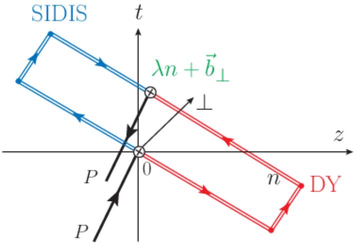



Wigner distribution
GTMD$(x, k_\perp, r_\perp)$
$\int dr_\perp$
$\int dk_\perp$
TMD$(x, k_\perp)$
GPD$(x, r_\perp)$
$\int dx$
$\int dk_\perp$
$\int dr_\perp$
$\int dx$
??$(k_\perp)$
PDF(x)
FF$(r_\perp)$
$\int dk_\perp$
$\int dx$
$\int dr_\perp$
spin, charge, total momentum, mass, tensor charge, ...



$$\text{TMD}^{[\Gamma]}(x,b) = \int d\lambda e^{-ix\lambda P^+} \langle P,S|\bar{q}(\lambda n + b)[\text{infinite staple link}]\frac{\Gamma}{2}q(0)|P,S\rangle$$

- The gauge link is along light-cone
- Type and properties of TMD depend on Γ

There are 8 TMD distributions
each parametrizes a particular relation between spin, momentum and orbital momentum

		Quark Polarization		
		Unpolarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	$f_1(x,k_T^2)$ Unpolarized		$h_1^\perp(x,k_T^2)$ - Boer-Mulders
	L		$g_1(x,k_T^2)$ - Helicity	$h_{1L}^\perp(x,k_T^2)$ - Kozinian-Mulders, "worm" gear
	T	$f_{1T}^\perp(x,k_T^2)$ - Sivers	$g_{1T}(x,k_T^2)$ - Kozinian-Mulders, "worm" gear	$h_1(x,k_T^2)$ - Transversity $h_{1T}^\perp(x,k_T^2)$ - Pretzelosity

There are 8 TMD distributions
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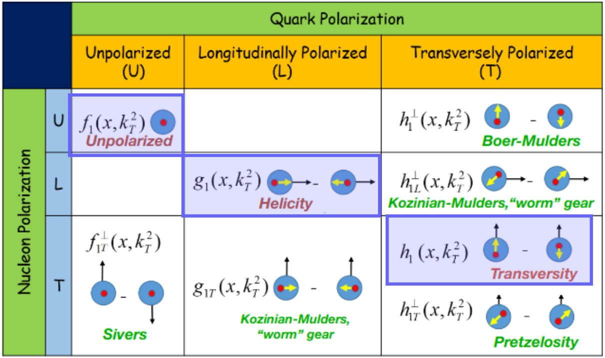
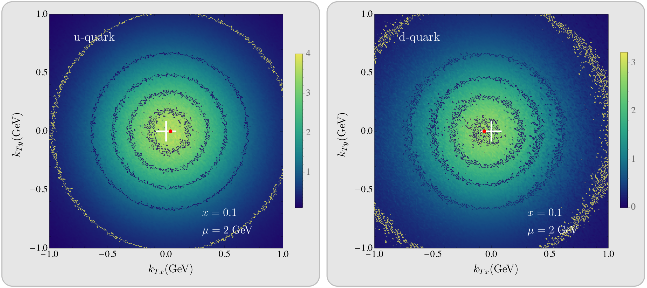


Only a few distributions have 1D analog
Rest are due to non-zero OAM

To get tomographic picture of nucleon one must combine several distributions

$$\rho_{q\leftarrow h^\uparrow}(x,k_T) = f(x,p_T) - \frac{k_T}{M} f_{1T}^{\perp}(x,k_T)$$

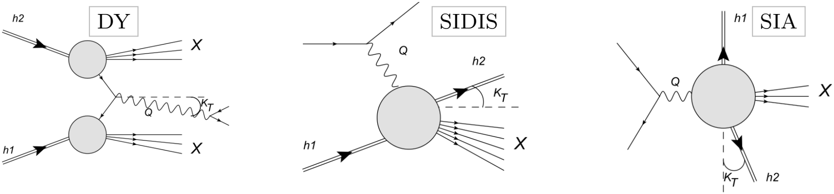


Tomography state-of-the-art [M.Bury, A.Prokudin, AV, Dec. 2020]

- Global fit of ~ 20 experiments
- Multiple processes: $p+p\to\gamma/Z$, $p+p^\uparrow\to\gamma/Z$, $p+\pi\to\gamma$, $p+\gamma\to h$, ...
- N^2LO perturbative accuracy: N^2LO evolution, N^2LO perturbative matching, ...

This picture is already outdated, due to the recent progress...

$$\frac{d\sigma}{dq_T} = \sigma_0 \int \frac{d^2 b}{(2\pi)^2} e^{i(bq_T)} C\left(\frac{Q}{\mu}\right) F(x_1, b; \mu, \zeta) F(x_2, b; \mu, \bar{\zeta})$$



Main scales:
The invariant mass of photon: $|q^2| = Q^2$
Transverse component of photon momentum: q_T

$$Q \gg \Lambda \qquad Q \gg q_T$$

TMD factorization theorem

$$\frac{d\sigma}{dq_T} = \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{i(bq_T)} C\left(\frac{Q}{\mu}\right) F(x_1, b; \mu, \zeta) F(x_2, b; \mu, \bar{\zeta})$$

N^{4}LO
(for all structure functions)

Nucleon Polarization	Quark Polarization: Unpolarized (U)	Quark Polarization: Longitudinally Polarized (L)	Quark Polarization: Transversely Polarized (T)
U	$f_1(x,k_T^2)$ Unpolarized		$h_1^\perp(x,k_T^2)$ Boer-Mulders
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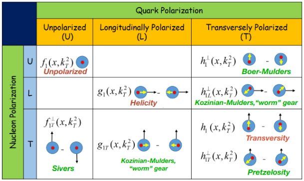
Evolution
N^{4}LO
(for all distributions!)

TMD factorization theorem

$$\frac{d\sigma}{dq_T} = \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{i(bq_T)} C\left(\frac{Q}{\mu}\right) F(x_1, b; \mu, \zeta) F(x_2, b; \mu, \bar{\zeta})$$

N^{4}LO
(for all structure functions)



Double-scale evolution

$$\mu^2 \frac{d}{d\mu^2} \ln F = \gamma_F$$

$$\zeta \frac{d}{d\zeta} \ln F = -\mathcal{D}(b) = \frac{K(b)}{2}$$

$\mathcal{D}$=Collins-Soper kernel
nonperturbative

Evolution
N^{4}LO
(for all distributions!)

TMD factorization theorem

$$\frac{d\sigma}{dq_T} = \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{i(bq_T)} C\left(\frac{Q}{\mu}\right) F(x_1, b; \mu, \zeta) F(x_2, b; \mu, \bar{\zeta})$$

$$\frac{d\sigma}{dq_T} = \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{i(bq_T)} C\left(\frac{Q}{\mu}\right) R[\mathcal{D}](b;\mu) F(x_1, b) F(x_2, b)$$

Double-scale evolution

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$$\zeta \frac{d}{d\zeta} \ln F = -\mathcal{D}(b) = \frac{K(b)}{2}$$

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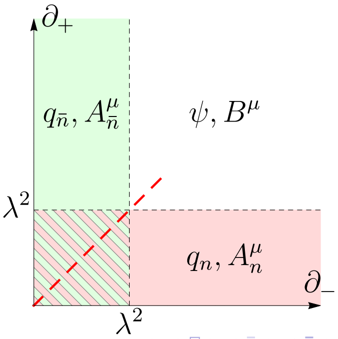
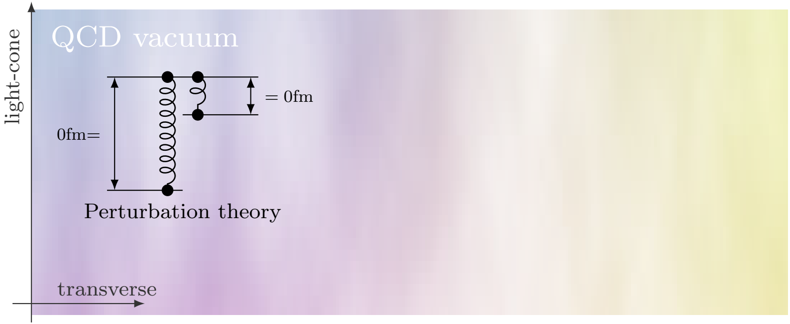
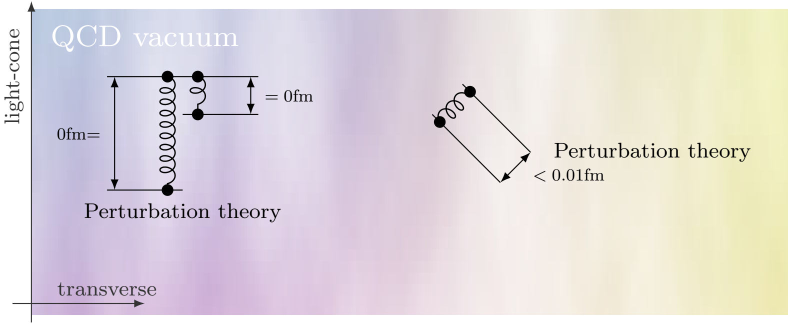
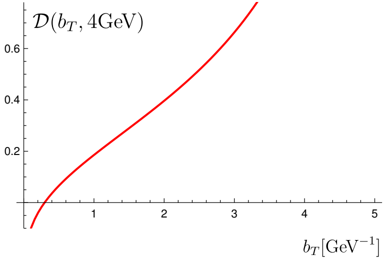
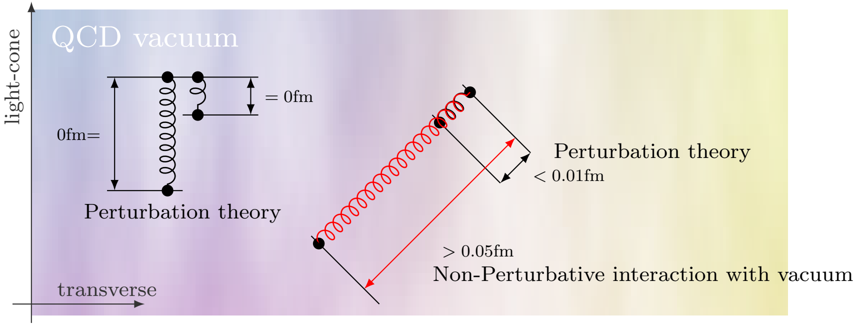
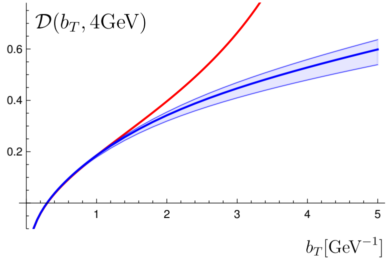



QCD vacuum
light-cone
= 0fm
0fm=
Perturbation theory
transverse


QCD vacuum
light-cone
= 0fm
0fm=
Perturbation theory
Perturbation theory
< 0.01fm
transverse

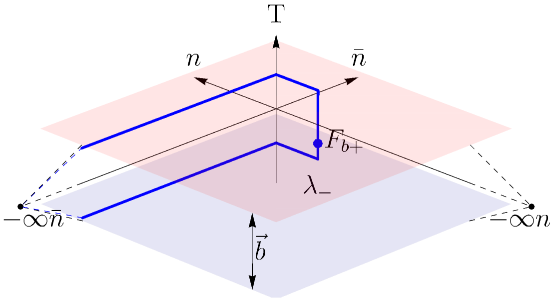

$\mathcal{D}(b_T, 4\text{GeV})$
0.6
0.4
0.2
1
2
3
4
5
$b_T[\text{GeV}^{-1}]$


QCD vacuum
light-cone
0fm=
= 0fm
Perturbation theory
Perturbation theory
< 0.01fm
> 0.05fm
Non-Perturbative interaction with vacuum
transverse


$\mathcal{D}(b_T, 4\text{GeV})$
0.6
0.4
0.2
1
2
3
4
5
$b_T[\text{GeV}^{-1}]$

Collins-Soper kernel ∼ Wilson loop

[AV,19]

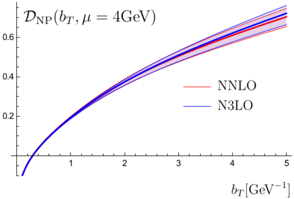


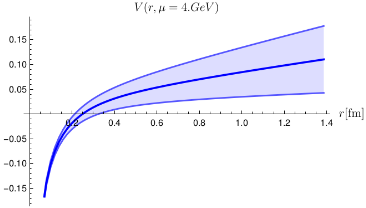
$$\mathcal{D}(b,\mu) = \lambda_- \frac{ig}{2} \frac{\mathrm{Tr} \int_0^1 d\beta \langle 0|F_{b+}(-\lambda_- n + b\beta) W_{C'}|0\rangle}{\mathrm{Tr}\langle 0|W_{C'}|0\rangle} + Z_{\mathcal{D}}(\mu)$$

Relation to the static potential

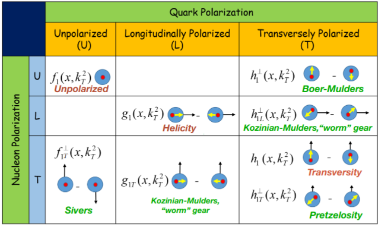
In SVM the potential between two quark sources (confining potential) is [Brambilla,Vairo,hep-ph/9606344]

$$V(\boldsymbol{b}) = 2\int_0^{\boldsymbol{b}} d\boldsymbol{y}(\boldsymbol{b}-\boldsymbol{y})\int_0^\infty d\boldsymbol{r}\Delta(\sqrt{\boldsymbol{r}^2+\boldsymbol{y}^2}) + \int_0^{\boldsymbol{b}} d\boldsymbol{y}\boldsymbol{y}\int_0^\infty d\boldsymbol{r}\Delta_1(\sqrt{\boldsymbol{r}^2+\boldsymbol{y}^2}).$$





$$V(r) = r\frac{\pi}{4}\mathcal{D}''(0) + \frac{\mathcal{D}'(0)}{2} + \frac{r^2}{2}\int_r^\infty \frac{dx}{x^2}\frac{\mathcal{D}'(x)}{\sqrt{x^2-r^2}} + \ldots$$



+

$$\mathcal{D} = -\frac{\tilde{K}}{2}$$

Collins-Soper kernel

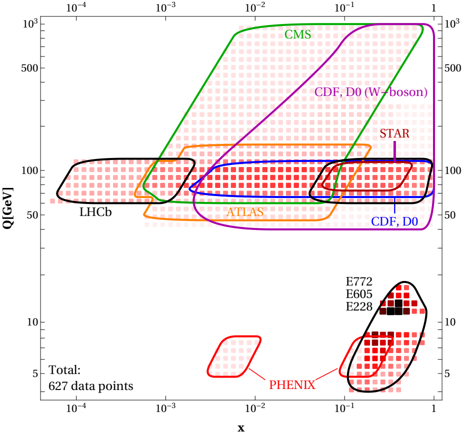
Task: decorrelate 3 functions 2D+2D+1D

- Requires tonns of data (precise and at different energy)
- Requires a perfect understanding of the theory

ART23

[V.Moos, I.Scimemi, AV, P.Zurita, 2305.07473]

First extraction at
N^{4}LL



*** data included for the first time**

- ATLAS
  - Z-boson at 8 (y-diff.)
  - **Z-boson at 13 TeV (0.1% prec.!)**
- CMS
  - Z-boson at 7 and 8 TeV
  - Z-boson at 13 TeV (y-diff.)
  - **Z/γ up to $Q = 1000$GeV**
- LHCb
  - Z-boson at 7 and 8 TeV
  - **Z-boson at 13 TeV (y-diff.)**
- **Further more:**
  - Z-boson at Tevatron
  - **W-boson at Tevatron**
  - **Z-boson at RHIC**
  - DY at PHENIX
  - DY at FERMILAB (fix target)

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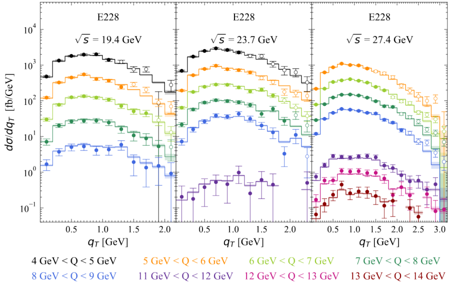
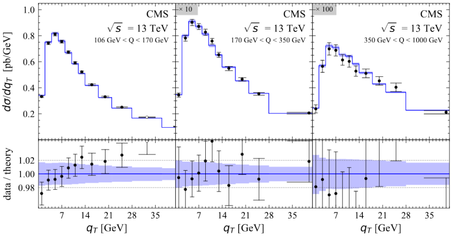
627 data points

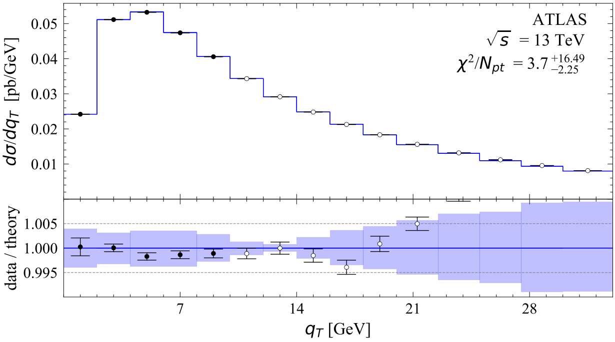
vs. 457 in SV19
vs. 484 in MAP22





4GeV 1000GeV

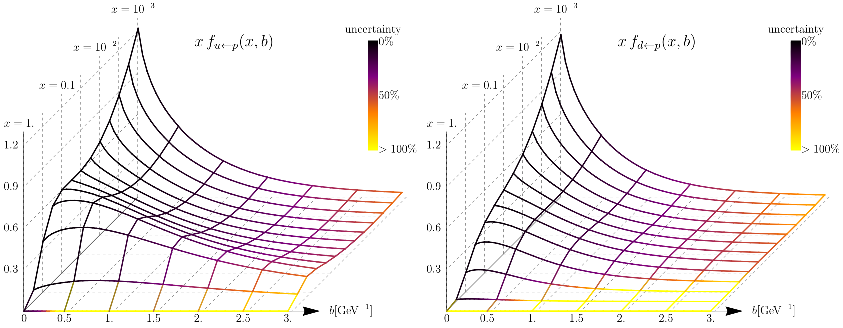
Very presice test of TMD evolution



$$\text{TOTAL } (N_{\text{pt}} = 627): \quad \chi^2/N_{\text{pt}} = 0.96^{+0.09}_{-0.01}$$

ART23

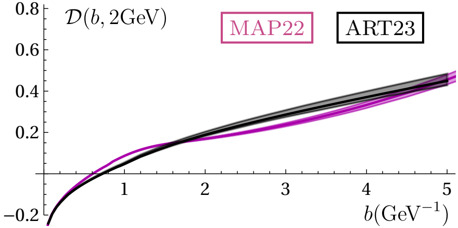
[V.Moos, I.Scimemi, AV, P.Zurita, 2305.07473]



Extra features of analyses:

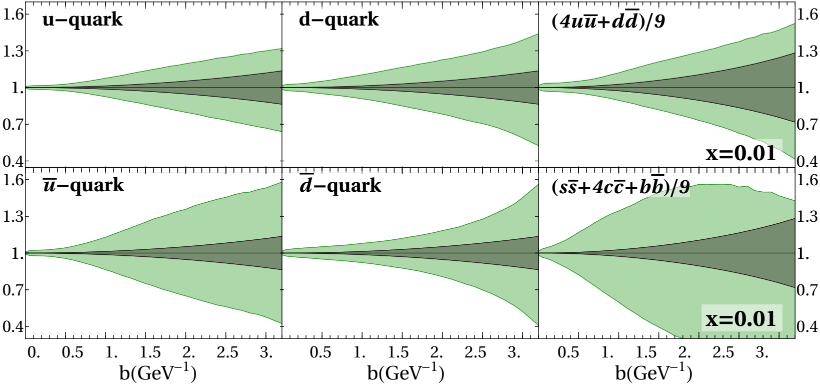
- Flavor dependent NP-ansatz (first time!)
  - 2 parameters per flavor
  - u, d, $\bar{u}$, $\bar{d}$, rest
- New parametrization for Collins-Soper kernel (3 parameters)
- Consistent inclusion of the PDF uncertainty (first time!)
- *artemide*

Collins-Soper kernel

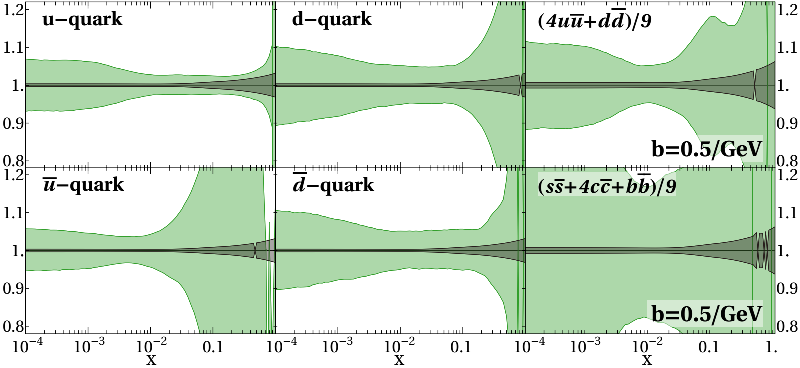


Very small uncertanties
(despite huge uncertanties in TMDPDFs)

More accurate estimation of uncertainty

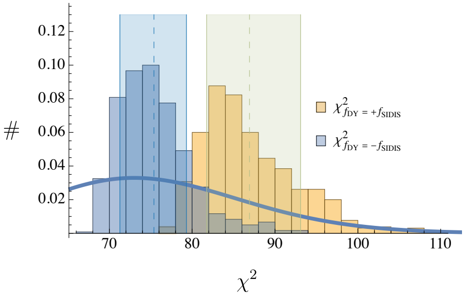


More accurate estimation of uncertainty



Check sign-change

$$f_{1T}^{\perp}(SIDIS) = -f_{1T}^{\perp}(DY)$$



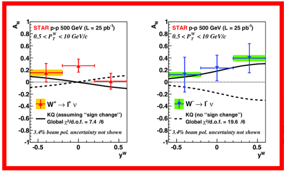
$$f_{1T}^{\perp}(sea) \to -f_{1T}^{\perp}(sea)$$

$\chi^2/N_{pt} = 0.88^{+0.16}_{-0.06}$ **vs.** $\chi^2/N_{pt} = 1.00^{+0.22}_{-0.08}$

Current data does not check sign-change!

If there would be DY at anti-proton...

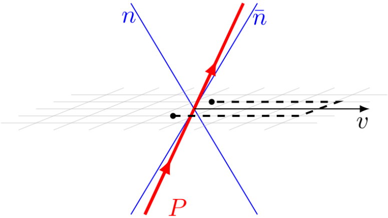
Naive picture

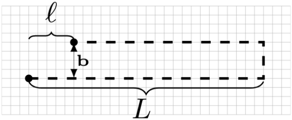


These are only few example of internal problems of TMD phenomenology

- Normalization issues
- Large-q_T tail
- Strong model bias
- Lack of data for polarized TMD distributions
- ...

Lattice computations can help to resolve many of these issues





[B.Musch, P.Hagler, J.Negele, A.Schafer,2011]

$$W^{[\Gamma]}(y) = \langle P,S|\bar{q}(y)[staple]\frac{\Gamma}{2}q(0)|P,S\rangle$$

Theory assumptions

- $L \gg b, \ell$
- $(vP) \gg \Lambda, M$
- $(bv) = (bP) = 0$

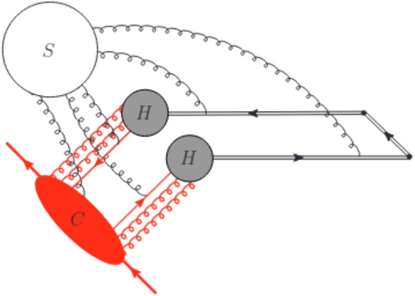
TMD factorization

- $\{\ell, b\} \lesssim \{1, \lambda^{-1}\}$ $\qquad \lambda \sim M/(vP)$

Alike hadron tensor with “instant”-to-light current

$$W^{[\Gamma]}(y; P, S, v; L) = \langle P,S|J^\dagger(y)\frac{\Gamma}{2}J(0)|P,S\rangle$$

$$J_i(y; v, L) = [\infty_T + vL, vL + y][vL + y, y]q_i(y)$$

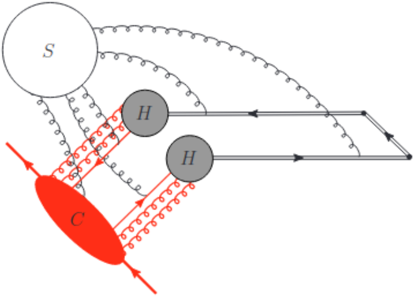


[Ebert, Stewart, Zhao, 19]
[Ji, Liu, Liu, 19]
[A.Schafer, AV, 20]

$$W^{[\Gamma]}(\ell,b;S,P;v,L,\mu) = |C_H|^2 \underbrace{\Phi^{[\Gamma]}(\ell,b;S,P,\mu,\zeta)}_{\text{TMD}} \underbrace{\Psi(b;v,L;\mu,\bar\zeta)}_{\text{``instant-jet''}}$$

$$+\mathcal{O}\Big(\frac{M}{P_+},\frac{1}{bP^+},\frac{b}{L},\frac{\ell}{L},\ell\Lambda_{\text{QCD}}\Big)$$

- Standard TMD factorization (a la Drell-Yan hadron tensor)
  - Collinear modes (hadron)
  - V-modes (for Wilson line)
  - Soft moder (overlap)



[Ebert, Stewart, Zhao, 19]
[Ji, Liu, Liu, 19]
[A.Schafer, AV, 20]

$$W^{[\Gamma]}(\ell, b; S, P; v, L, \mu) = |C_H|^2 \underbrace{\Phi^{[\Gamma]}(\ell, b; S, P, \mu, \zeta)}_{\text{TMD}} \underbrace{\Psi(b; v, L; \mu, \bar{\zeta})}_{\text{“instant-jet”}}$$

$$+\mathcal{O}\Big(\frac{M}{P_+}, \frac{1}{bP^+}, \frac{b}{L}, \frac{\ell}{L}, \ell\Lambda_{\text{QCD}}\Big)$$

- Factorization theorem is confirmed at NNLO [O.del Rio, AV,23]

$$|C_H|^2 = 1 + a_s C_F\Big(-\mathbf{L}^2 + 2\mathbf{L} - 4 + \frac{\pi^2}{6}\Big) + a_s^2(\text{see [2304.14440]}) + \mathcal{O}(a_s^3)$$

- Factorization theorem proven at LP, and NLP [S.Rodini, AV,22]

Factorization theorem for leading-counting qTMDs

$$W^{[\Gamma]}(\ell, b; S, P; v, L, \mu) = |C_H|^2 \Phi^{[\Gamma]}(\ell, b; S, P, \mu, \zeta)\Psi(b; v, L; \mu, \bar{\zeta})$$
$$+\mathcal{O}\Big(\frac{M}{P_+}, \frac{1}{bP^+}, \frac{b}{L}, \frac{\ell}{L}, \ell\Lambda_{\text{QCD}}\Big)$$
$$\zeta\bar{\zeta} = (2(\hat{p}v))^2\mu^2$$

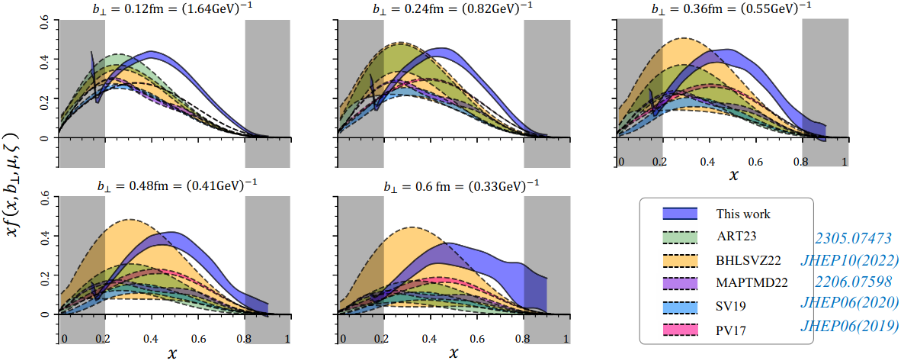
How to use it?

1. Direct usage

- Fourier transform to x-space
- Function Ψ ("reduced soft factor")

[Ji, NPB955(2020); Zhang, PRL125(2020); Li, PRL128(2022); Chu, 2302.09961]

from Qi-An Zhang talk at LaMet23 [hep-lat/2211.02340]



Difficult to estimate systematics...

How to use it?

2. Extraction of evolution only

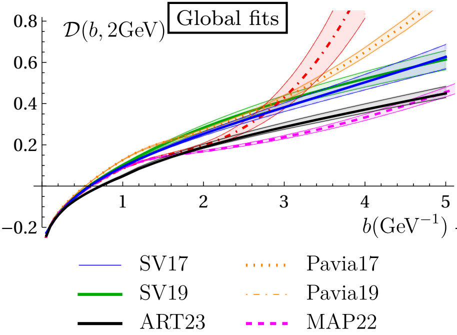
- Fourier transform to x-space
- ~~Function Ψ ("reduced soft factor")~~

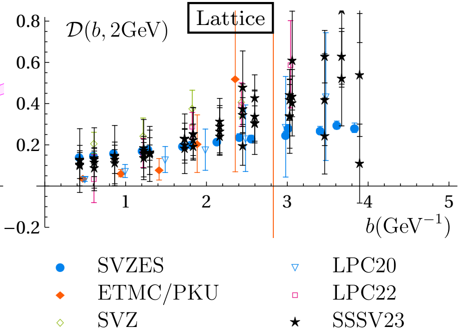
$$\frac{W(x,b;P_1,...)}{W(x,b;P_2,...)} = \frac{|C(P_1)|^2\Psi(x,b;\zeta_1,..)\cancel{\Phi(b,..)}}{|C(P_2)|^2\Psi(x,b;\zeta_2,..)\cancel{\Phi(b,..)}}$$

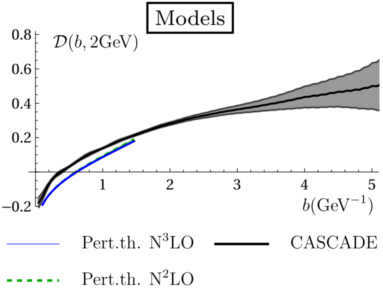
$$= R[\zeta_1 \to \zeta_2]\frac{|C(P_1)|^2\cancel{\Psi(x,b;\zeta_2,..)}}{|C(P_2)|^2\cancel{\Psi(x,b;\zeta_2,..)}} = \left(\frac{(vP_1)}{(vP_2)}\right)^{\mathcal{D}(b,\mu)}\frac{|C(P_1,\mu)|^2}{|C(P_2,\mu)|^2}$$

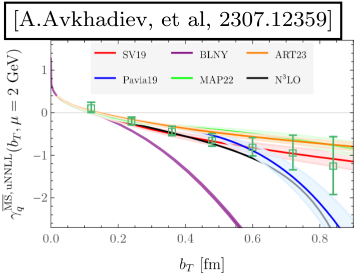
Very direct access to the CS kernel

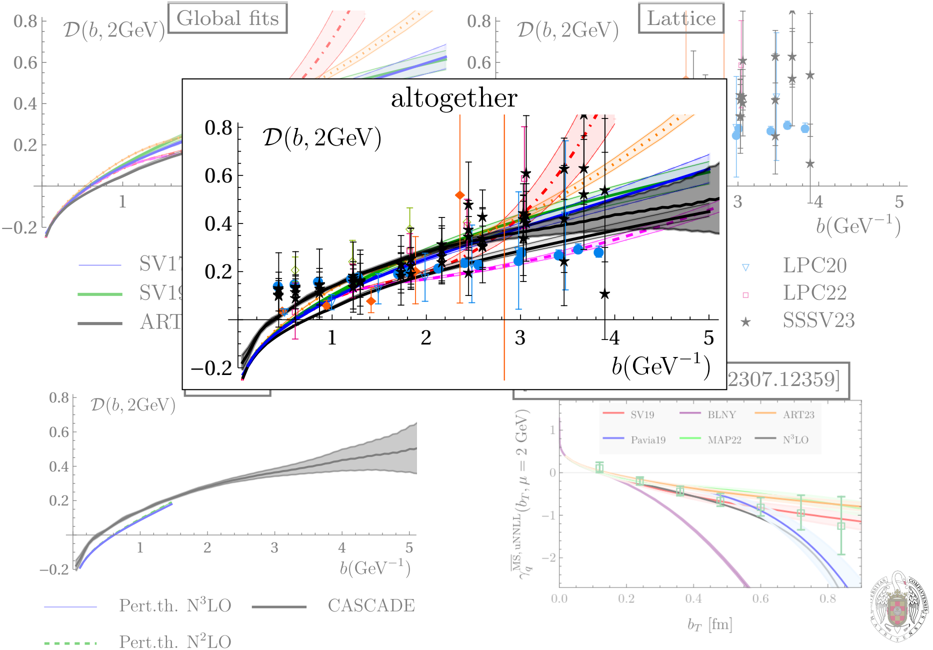
* There is an alternative approach without Fourier transform (directly in ℓ-space), but it requires extra hypothesis about some ratios [A.Schafer, AV, 20]

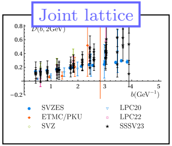

Global fits
$\mathcal{D}(b, 2\text{GeV})$
0.8
0.6
0.4
0.2
−0.2
1
2
3
4
5
$b(\text{GeV}^{-1})$
SV17
SV19
ART23
Pavia17
Pavia19
MAP22

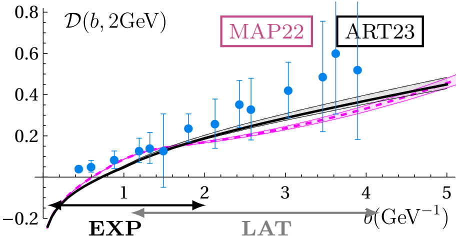

Lattice
$\mathcal{D}(b, 2\text{GeV})$
0.8
0.6
0.4
0.2
−0.2
1
2
3
4
5
$b(\text{GeV}^{-1})$
SVZES
ETMC/PKU
SVZ
LPC20
LPC22
SSSV23


Models
$\mathcal{D}(b, 2\text{GeV})$
0.8
0.6
0.4
0.2
−0.2
1
2
3
4
5
$b(\text{GeV}^{-1})$
Pert.th. N^{3}LO
CASCADE
Pert.th. N^{2}LO


[A.Avkhadiev, et al, 2307.12359]
SV19
BLNY
ART23
Pavia19
MAP22
N^{3}LO
$\gamma_q^{\overline{\text{MS}},\text{uNNLL}}(b_T, \mu = 2\text{ GeV})$
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
b_T [fm]


Global fits
Lattice
altogether
$\mathcal{D}(b, 2\text{GeV})$
$b(\text{GeV}^{-1})$
SV17
SV19
ART
LPC20
LPC22
SSSV23
2307.12359]
SV19
BLNY
ART23
Pavia19
MAP22
N³LO
b_T [fm]
Pert.th. N³LO
CASCADE
Pert.th. N²LO





PRO

- Can access large-b
- Can study "exotic" sources
- Directly in b-space

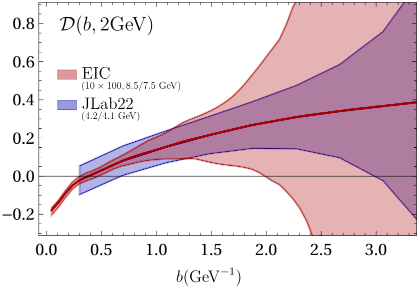
CONTRA

- Large power corrections
- Lattice artifacts
- Unknown scheme factor

Similar ratio can be extracted from colliders

- Requires Fourier transform to b-space
- Requires measurement in specially prepared bins

Direct measurement of CS kernel from collider data



In future lattice will be preciser, but experiment will be also preciser.

The true power of lattice simulations is access to “difficult” or impossible for experiment channels

- x-moments of TMDs
- Gluon CS-kernel
- Gluon TMDs
- Meson TMDs
- Higher-twist TMDs
- .....

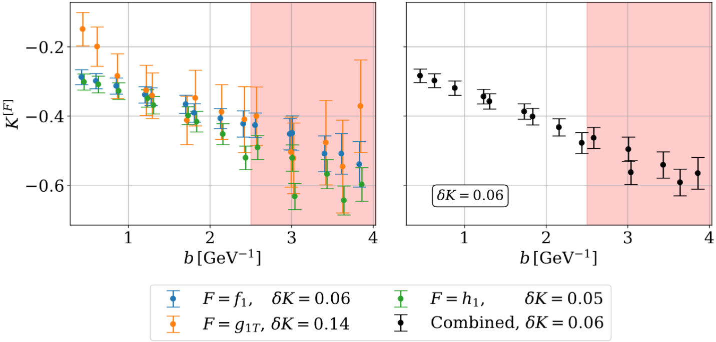
Latest example:
test of of universality of CS kernel
[Hai-Tao Shu, M.Schlemmer, T.Sizmann, A.Schafer, et al: 2302.06502]

Collins-Soper kernel is the evolution kernel for TMDs
and it universal for

- All TMDPDFs/TMDFFs of twist-2 (all types and hadrons)
- All TMDPDFs/TMDFFs of twist-3 (all types and hadrons) [AV, et al, 2008.01744],[Ebert,at al, 2112.09771]
- All quasi-partonic TMDPDFs/TMDFFs of twist-3 (all types and hadrons) [AV, et al, 2008.01744]

Check of universality for $\{f_1, g_{1T}, h_1\}$

[M.Schlemmer,A.Schafer, et al,2103.16991]



$$K = -2\mathcal{D}$$

NLP TMD factorization is very complicated!

[Rodini,AV:2211.04494]

$$F(x,b;\mu)=\frac{1}{x}\left(\frac{(2|x|(vP))^2}{\zeta}\right)^{-\mathcal{D}(b,\mu)}\Big\{\mathbb{C}_{11}(\mathbf{L}_p,\mu)A(x,b;\mu,\zeta)$$
$$+\mathbb{C}_{11}(\mathbf{L}_p,\mu)\left(\mathbf{\Psi}_2(b)+\mathring{\mathcal{D}}(b)\ln\left(\frac{\mu(2|x|(vP))}{\zeta}\right)\right)B(x,b;\mu,\zeta)$$
$$+\int_{-1}^{1}dx_2\left(\mathbb{C}_R(\mathbf{L}_p,x,x_2)C(\tilde{x},b;\mu,\zeta)+s\pi\mathbb{C}_I(\mathbf{L}_p,x,x_2)D(\tilde{x},b;\mu,\zeta)\right)\Big\}$$

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Twist-3 qTMD

twist-2 TMD

Derivative twist-2 TMD

twist-3 "reduced SF"

Twist-3 TMDs $\langle\bar{q}Gq\rangle$

Derivative CS-kernel

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Derivative CS-kernel

Γ	qTMD	A	B	C	D
$\mathbb{1}$	E			$2\mathbf{h}_{\oplus}$	$2\mathbf{h}_{\ominus}$
	$E_T^{\perp}$			$2\mathbf{h}_{\oplus T}^{A\perp}$	$2\mathbf{h}_{\ominus T}^{A\perp}$
$i\gamma^5$	E_L			$2\mathbf{h}_{\oplus L}$	$2\mathbf{h}_{\ominus L}$
	E_T			$2\mathbf{h}_{\oplus T}^{D\perp}$	$2\mathbf{h}_{\ominus T}^{D\perp}$
γ^{α}	F_T	$-f_{1T}^{\perp}-\frac{b^2M^2}{2}\mathring{f}_{1T}^{\perp}$	$\frac{b^2M^2}{2}f_{1T}^{\perp}$	$\mathbf{f}_{\ominus T}-\mathbf{g}_{\oplus T}$	$-\mathbf{f}_{\oplus T}-\mathbf{g}_{\ominus T}$
	$F_L^{\perp}$			$-\mathbf{f}_{\ominus L}^{\perp}+\mathbf{g}_{\oplus L}^{\perp}$	$\mathbf{f}_{\oplus L}^{\perp}+\mathbf{g}_{\ominus L}^{\perp}$
	$F^{\perp}$	$\mathring{f}_1$	$-f_1$	$\mathbf{f}_{\ominus}^{\perp}-\mathbf{g}_{\oplus}^{\perp}$	$-\mathbf{f}_{\oplus}^{\perp}-\mathbf{g}_{\ominus}^{\perp}$
	$F_T^{\perp}$	$\mathring{f}_{1T}^{\perp}$	$-f_{1T}^{\perp}$	$-\mathbf{f}_{\ominus T}^{\perp}+\mathbf{g}_{\oplus T}^{\perp}$	$\mathbf{f}_{\oplus T}^{\perp}+\mathbf{g}_{\ominus T}^{\perp}$
$\gamma^{\alpha}\gamma^5$	G_T	$g_{1T}+\frac{b^2M^2}{2}\mathring{g}_{1T}$	$-\frac{b^2M^2}{2}g_{1T}$	$-\mathbf{f}_{\oplus T}-\mathbf{g}_{\ominus T}$	$-\mathbf{f}_{\ominus T}+\mathbf{g}_{\oplus T}$
	$G_L^{\perp}$	$\mathring{g}_1$	$-g_1$	$\mathbf{f}_{\oplus L}^{\perp}+\mathbf{g}_{\ominus L}^{\perp}$	$\mathbf{f}_{\ominus L}^{\perp}-\mathbf{g}_{\oplus L}^{\perp}$
	$G^{\perp}$			$\mathbf{f}_{\oplus}^{\perp}+\mathbf{g}_{\ominus}^{\perp}$	$\mathbf{f}_{\ominus}^{\perp}-\mathbf{g}_{\oplus}^{\perp}$
	$G_T^{\perp}$	$\mathring{g}_{1T}$	$-g_{1T}$	$\mathbf{f}_{\oplus T}^{\perp}+\mathbf{g}_{\ominus T}^{\perp}$	$\mathbf{f}_{\ominus T}^{\perp}-\mathbf{g}_{\oplus T}^{\perp}$
$i\sigma^{\alpha\beta}\gamma^5$	$H_T^{\perp}$	$-h_{1T}^{\perp}+\mathring{h}_1-\frac{b^2M^2}{4}\mathring{h}_{1T}^{\perp}$	$-h_1+\frac{b^2M^2}{4}h_{1T}^{\perp}$	$2\mathbf{h}_{\ominus T}^{A\perp}$	$-2\mathbf{h}_{\oplus T}^{A\perp}$
	H	$-2h_1^{\perp}$		$-2\mathbf{h}_{\ominus}$	$2\mathbf{h}_{\oplus}$
$i\sigma^{+-}\gamma^5$	$H_L^{\perp}$	$-2h_{1L}^{\perp}-b^2M^2\mathring{h}_{1L}^{\perp}$	$b^2M^2h_{1L}^{\perp}$	$-2\mathbf{h}_{\ominus L}$	$2\mathbf{h}_{\oplus L}$
	H_T	$-h_{1T}^{\perp}-\mathring{h}_1-\frac{b^2M^2}{4}\mathring{h}_{1T}^{\perp}$	$h_1+\frac{b^2M^2}{4}h_{1T}^{\perp}$	$-2\mathbf{h}_{\ominus T}^{D\perp}$	$2\mathbf{h}_{\oplus T}^{D\perp}$

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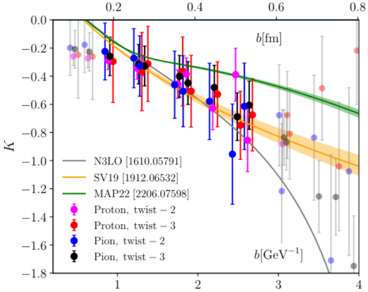
$i\gamma^5$	E_L			$2\mathbf{h}_{\oplus L}$	$2\mathbf{h}_{\ominus L}$
	E_T			$2\mathbf{h}^{D\perp}_{\oplus T}$	$2\mathbf{h}^{D\perp}_{\ominus T}$
γ^{α}	F_T	$-f_{1T}^{\perp}-\frac{b^2M^2}{2}\mathring{f}_{1T}^{\perp}$	$\frac{b^2M^2}{2}f_{1T}^{\perp}$	$\mathbf{f}_{\ominus T}-\mathbf{g}_{\oplus T}$	$-\mathbf{f}_{\oplus T}-\mathbf{g}_{\ominus T}$
	$F_L^{\perp}$			$-\mathbf{f}^{\perp}_{\ominus L}+\mathbf{g}^{\perp}_{\oplus L}$	$\mathbf{f}^{\perp}_{\oplus L}+\mathbf{g}^{\perp}_{\ominus L}$
	$F^{\perp}$	$\mathring{f}_1$	$-f_1$	$\mathbf{f}^{\perp}_{\ominus}-\mathbf{g}^{\perp}_{\oplus}$	$-\mathbf{f}^{\perp}_{\oplus}-\mathbf{g}^{\perp}_{\ominus}$
	$F_T^{\perp}$	$\mathring{f}_{1T}^{\perp}$	$-f_{1T}^{\perp}$	$-\mathbf{f}^{\perp}_{\ominus T}+\mathbf{g}^{\perp}_{\oplus T}$	$\mathbf{f}^{\perp}_{\oplus T}+\mathbf{g}^{\perp}_{\ominus T}$
$\gamma^{\alpha}\gamma^5$	G_T	$g_{1T}+\frac{b^2M^2}{2}\mathring{g}_{1T}$	$-\frac{b^2M^2}{2}g_{1T}$	$-\mathbf{f}_{\oplus T}-\mathbf{g}_{\ominus T}$	$-\mathbf{f}_{\ominus T}+\mathbf{g}_{\oplus T}$
	$G_L^{\perp}$	$\mathring{g}_1$	$-g_1$	$\mathbf{f}^{\perp}_{\oplus L}+\mathbf{g}^{\perp}_{\ominus L}$	$\mathbf{f}^{\perp}_{\ominus L}-\mathbf{g}^{\perp}_{\oplus L}$
	$G^{\perp}$			$\mathbf{f}^{\perp}_{\oplus}+\mathbf{g}^{\perp}_{\ominus}$	$\mathbf{f}^{\perp}_{\ominus}-\mathbf{g}^{\perp}_{\oplus}$
	$G_T^{\perp}$	$\mathring{g}_{1T}$	$-g_{1T}$	$\mathbf{f}^{\perp}_{\oplus T}+\mathbf{g}^{\perp}_{\ominus T}$	$\mathbf{f}^{\perp}_{\ominus T}-\mathbf{g}^{\perp}_{\oplus T}$
$i\sigma^{\alpha\beta}\gamma^5$	$H_T^{\perp}$	$-h_{1T}^{\perp}+\mathring{h}_1-\frac{b^2M^2}{4}\mathring{h}_{1T}^{\perp}$	$-h_1+\frac{b^2M^2}{4}h_{1T}^{\perp}$	$2\mathbf{h}^{A\perp}_{\ominus T}$	$-2\mathbf{h}^{A\perp}_{\oplus T}$
	H	$-2h_1^{\perp}$		$-2\mathbf{h}_{\ominus}$	$2\mathbf{h}_{\oplus}$
$i\sigma^{+-}\gamma^5$	$H_L^{\perp}$	$-2h_{1L}^{\perp}-b^2M^2\mathring{h}_{1L}^{\perp}$	$b^2M^2h_{1L}^{\perp}$	$-2\mathbf{h}_{\ominus L}$	$2\mathbf{h}_{\oplus L}$
	H_T	$-h_{1T}^{\perp}-\mathring{h}_1-\frac{b^2M^2}{4}\mathring{h}_{1T}^{\perp}$	$h_1+\frac{b^2M^2}{4}h_{1T}^{\perp}$	$-2\mathbf{h}^{D\perp}_{\ominus T}$	$2\mathbf{h}^{D\perp}_{\oplus T}$

6 qTMDs (out of 16)
can be used to determine
CS-kernel
(alike in twist-2 case)

Check of universality for $\{f_1(proton), f_1(pion), e(proton), e(pion)\}$

[Hai-Tao Shu, et al,2302.06502]



$$K = -2\mathcal{D}$$

Conclusion

The synergy in the phenomenology of lattice and collider data
is in their complementarity

b-space $\longleftrightarrow$ k_T-space

low-energy $\longleftrightarrow$ high-energy

low-statistic many channels $\longleftrightarrow$ high-statistic few channels

... $\longleftrightarrow$ ...

Outline of talk:

- ART23 extraction
  - N^{4}LL
  - Larger data set (mainly due to LHC data)
  - (more) Accurate determination of uncertanties
  - *artemide:* https://github.com/VladimirovAlexey/artemide-public
- Universality of CS kernel
  - Evolution for different polarizations is the same
  - Evolution for twist-2 and twist-3 TMDs is the same
  - Evolution for pion and proton TMDs is the same