Colored Quantum Chaos

Eigenstate Thermalization in SU(2) Gauge Theory

Berndt Mueller (Duke Univ.)

X. Yao, PRD 108, L031504 (2023)
BM, X. Yao, arXiv: 2307.00045
L. Ebner, BM, A. Schäfer, C. Seidl, X. Yao, arXiv: 2308.16202

QCD on and off the Lattice
19 September 2023
Regensburg (Germany)

Prelude

Z. Phys. A – Atoms and Nuclei 322, 539–545 (1985)

Zeitschrift für Physik A Atoms and Nuclei

Hartree-Fock-calculation of parity-violation in cesium

A. Schäfer, B. Müller & W. Greiner

We present a relativistic Hartree-Fock calculation of the parity violating $E1$-matrixelement of the $6s \leftrightarrow 7s$ transition in cesium. Our result $E1 = -8.4 \cdot 10^{-12}\, iea_0$ for $\sin^2\theta_W = 0.22$ is in good agreement with the experimental value.

PHYSICAL REVIEW A VOLUME 40, NUMBER 12 DECEMBER 15, 1989

Prospects for an atomic parity-violation experiment in U^{90+}

A. Schäfer, G. Soff, P. Indelicato, B. Müller, and W. Greiner

Parity mixing of electron states should be extremely strong for heliumlike uranium. We calculate its size and discuss whether it could be determined experimentally. We analyze one specific scheme for such an experiment. The required laser intensities for two-photon spectroscopy of the $2\,{}^3P_0 - 2\,{}^1S_0$ level splitting is of the order of 10^{17} W/cm^2. A determination of parity mixing would require at least 10^{21} W/cm^2.

We are still waiting for the HITRAP facility at FAIR !

PHYSICAL REVIEW LETTERS

VOLUME 56 | 24 MARCH 1986 | NUMBER 12

Improved Bounds on the Dimension of Space-Time

Berndt Müller and Andreas Schäfer

Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo 188, Japan, and Institut für Theoretische Physik, Johann Wolfgang Goethe Universität, D-6000 Frankfurt, Germany[a]

(Received 26 August 1985)

We treat the perihelion shift of the planetary motion and the Lamb shift in hydrogen in an arbitrary number of space dimensions. Comparison with experimental data shows that the deviation from dimensionality four of space-time is less than 10^{-9} and 3.6×10^{-11}, respectively, on the length scales associated with these phenomena.

Our result was cited in the next edition of the *Review of Particle Properties*

Andreas' interest in hadron structure goes back a long way:

Physics Letters B
Volume 143, Issues 4–6, 16 August 1984, Pages 323-325

Electric and magnetic polarizability of the nucleon in the MIT bag model

A. Schäfer, B. Müller, D. Vasak [1], W. Greiner

The electric and magnetic polarizabilities of the proton and neutron are calculated in the framework of the MIT bag model. Neglecting vacuum-polarization we get $\alpha_p = \alpha_n = 10.8 \times 10^{-4}$ fm^3, $\beta_p = 2.3 \times 10^{-4}$ fm^3 and $\beta_n = 1.5 \times 10^{-4}$ fm^3, in good agreement with experiment. The difficulties in treating the vacuum-polarization consistenly are discussed.

Physics Letters B
Volume 230, Issues 1–2, 26 October 1989, Pages 141-148

The valence and strange-sea quark spin distributions in the nucleon from semi-inclusive deep inelastic lepton scattering

Leonid L. Frankfurt, Mark I. Strikman, Lech Mankiewicz [1], Andreas Schäfer, Ewa Rondio, Andrzej Sandacz, Vassilios Papavassiliou

The insight that model building no longer suffices, if there exists a systematic method to rigorously solve hadron structure in QCD, formed the basis for the hugely successful effort to form a world-class Lattice-QCD group in Regensburg. Congratulations to all who contributed!

How do systems governed by QCD thermalize and how does entropy get created?

Entropy production in high-energy processes

Berndt Muller (Duke U.), Andreas Schafer (Regensburg U.) (Jun, 2003)

e-Print: hep-ph/0306309 [hep-ph]

Unpublished

We calculate the entropy produced in the decoherence of a classical field configuration and compare it with the entropy of a fully thermalized state with the same energy. We find that decoherence alone accounts for a large fraction of the equilibrium entropy

Decoherence and entropy production in relativistic nuclear collisions

Rainer J. Fries, Berndt Müller, and Andreas Schäfer
Phys. Rev. C **79**, 034904 – Published 13 March 2009

Gluon radiation

Towards a Theory of Entropy Production in the Little and Big Bang

Teiji Kunihiro, Berndt Müller, Akira Ohnishi, Andreas Schäfer

Progress of Theoretical Physics, Volume 121, Issue 3, March 2009, Pages 555–575,

Husimi-Wehrl entropy

ENTROPY CREATION IN RELATIVISTIC HEAVY ION COLLISIONS

International Journal of Modern Physics E | Vol. 20, No. 11, pp. 2235-2267 (2011)

BERNDT MÜLLER and ANDREAS SCHÄFER

Review

But how to perform rigorous calculations in a QFT when the coupling is not weak ?

The AdS/CFT correspondence came to the rescue:

PHYSICAL REVIEW D **84,** 026010 (2011)

Holographic thermalization

V. Balasubramanian,[1] A. Bernamonti,[2] J. de Boer,[3] N. Copland,[2] B. Craps,[2] E. Keski-Vakkuri,[4,5] B. Müller,[6] A. Schäfer,[7] M. Shigemori,[8] and W. Staessens[2]

Using the AdS/CFT correspondence, we probe the scale dependence of thermalization in strongly coupled field theories following a sudden injection of energy via calculations of two-point functions, Wilson loops, and entanglement entropy in $d = 2, 3, 4$. ……….. For homogeneous initial conditions the entanglement entropy thermalizes slowest and sets a timescale for equilibration that saturates a causality bound over the range of scales studied.

Holographic Kolmogorov-Sinai entropy and the quantum Lyapunov spectrum

Georg Maier,[a] Andreas Schäfer[a] and Sebastian Waeber[b,c]

Published in: *JHEP* 01 (2022) 165 • e-Print: 2107.01300 [hep-th]

Quantum Kolmogorov-Sinai entropy saturates the MSS bound ($\lambda_i \leq 2\pi T$)

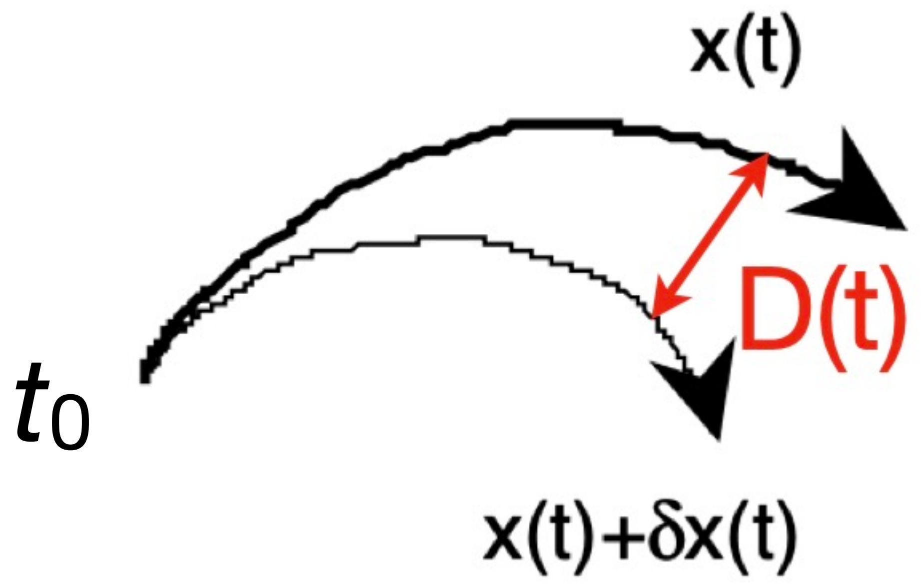
Classical theory of entropy growth:

Lyapunov exponents and KS entropy

Lyapunov exponents - KS entropy

- **A constant growth rate of the observable entropy, i.e. the entropy measured after coarse graining, is a characteristic feature of *chaotic* dynamical systems.**
- Consider two evolutions of such a system starting from slightly different initial conditions $(\vec{x}(t_0), \vec{p}(t_0))$ and $(\vec{x}(t_0) + \delta\vec{x}(t_0), \vec{p}(t_0) + \delta\vec{p}(t_0))$. A dynamical system is *chaotic* if the distance in phase space between the two systems grows exponentially:

$$D(t) = \sqrt{|\delta\vec{x}(t)|^2 + |\delta\vec{p}(t)|^2} = D_0 e^{\lambda t}$$



- λ is called the (largest) *Lyapunov exponent*.
- More generally, one can construct a spectrum of modes around the original trajectory in phase space and obtain the associated spectrum of Lyapunov exponents λ_i. The rate of growth of the coarse grained entropy is known as the Kolmogorov-Sinai (KS) entropy h_{KS}. It is given by

$$dS/dt = h_{\mathrm{KS}} \equiv \sum_{\lambda_i > 0} \lambda_i .$$

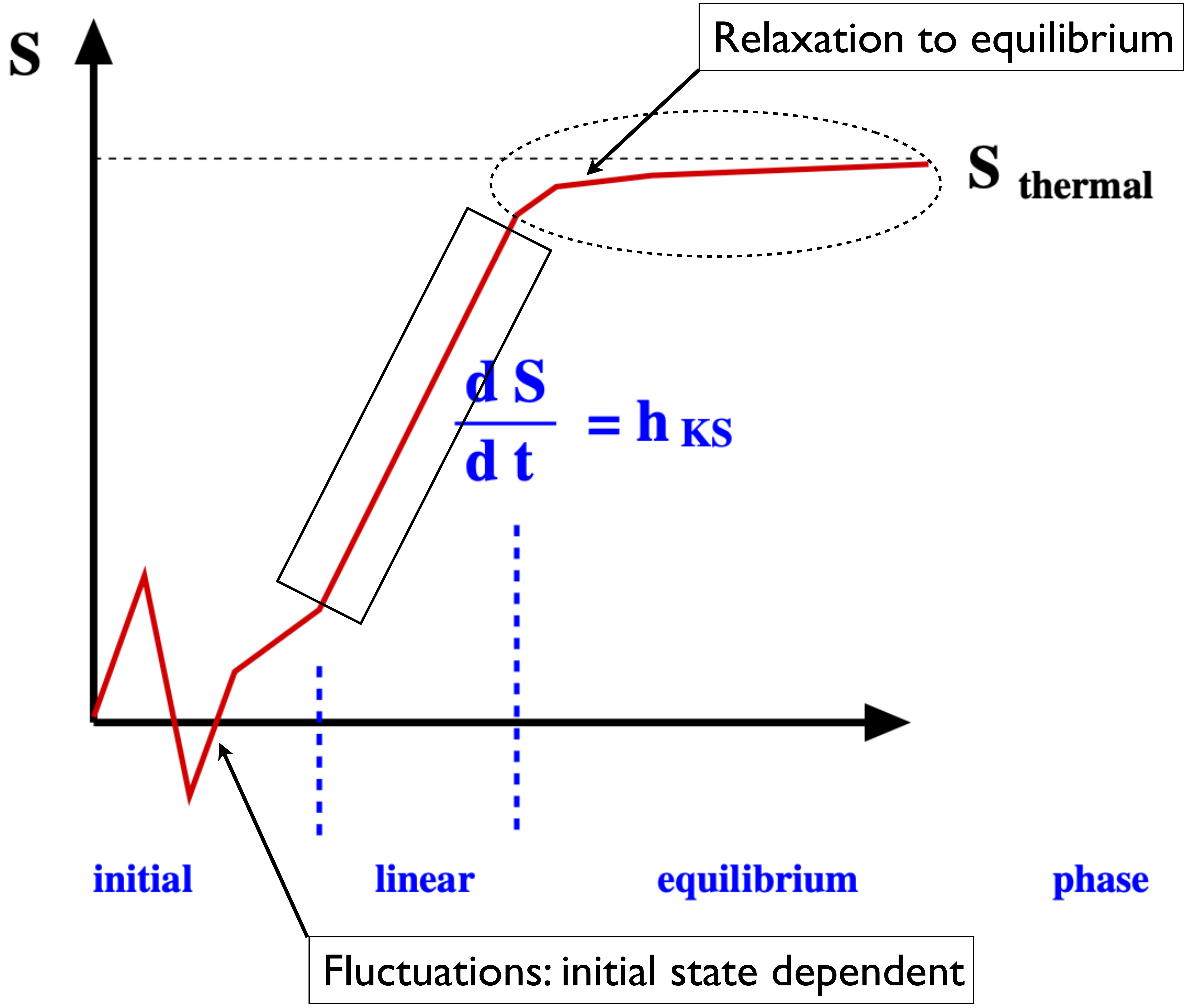
Thermalization of a chaotic system

Depending on the size of initial fluctuations, after some initial period, the measurable entropy of the system grows linearly with time:

$$dS/dt = h_{\mathrm{KS}}.$$

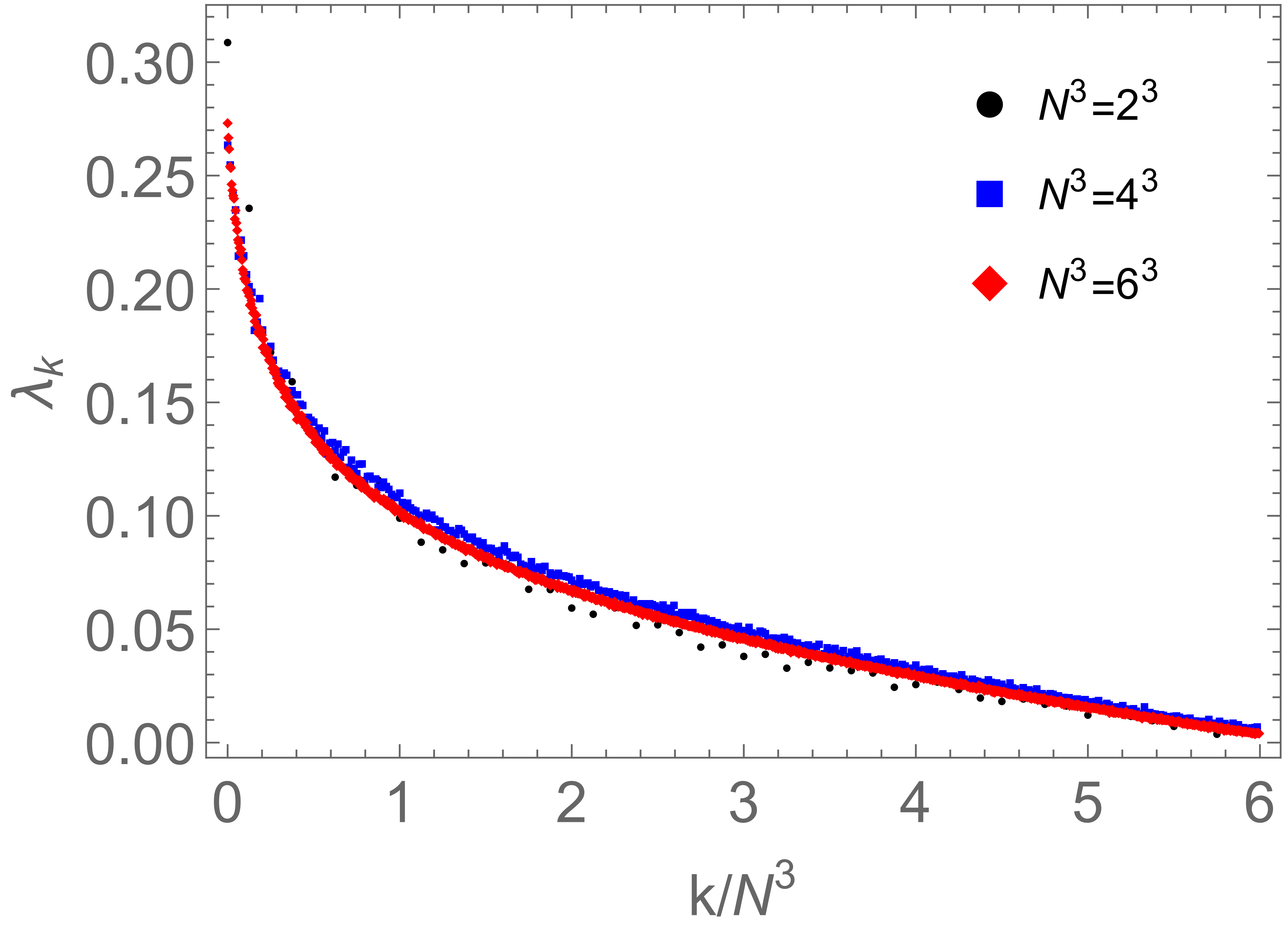
After a time $\tau_{\mathrm{eq}} = S_{\mathrm{eq}}/h_{\mathrm{KS}}$, the entropy of the system approaches the value of the entropy in thermal equilibrium, and further growth is impossible because the volume of accessible phase space at fixed total energy is finite.

This behavior can be calculated numerically in the classical limit of field theory.



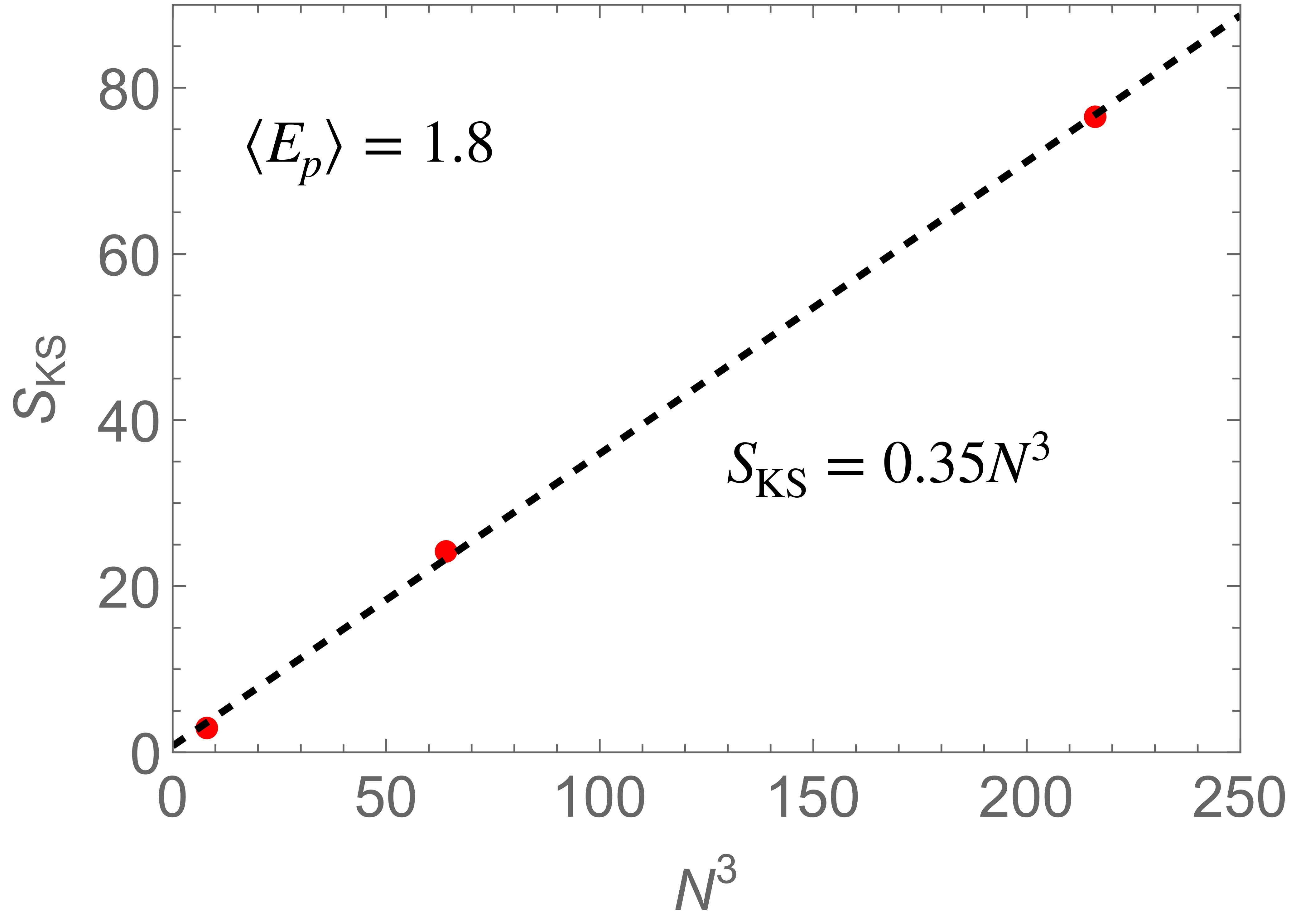
Lyapunov Spectrum SU(2)

Number of unstable modes with positive Lyapunov exponents = number of dynamical modes of the lattice gauge theory



Sum of all positive Lyapunov exponents exhibits volume growth: S_{KS} is extensive

[J. Bolte, BM, A. Schäfer, PRD 61 (2000) 054506]



How chaotic is QCD?

- Maldacena, Shenker, and Stanford [JHEP 08 (2016) 106] argued that there is an upper bound on Lyapunov exponents: $\lambda \leq 2\pi T$, where T is the temperature reached after equilibration.
- Our numerical simulations for the SU(3) gauge theory found [PRD 52 (1995) 1260]

$$\lambda_{\max} = 0.53 g^2 T$$

  Which saturates the MSS bound when $\alpha_s = g^2/4\pi \approx 1$.

- Thermalization in QCD at realistic coupling may thus be about three times slower than at infinitely strong coupling as realized in BH formation or AdS/CFT.
- But this still gives a rather short thermalization time around 1 fm/*c*.

- *Next challenge:* Compute entropy growth in the quantum lattice gauge theory.

Quantum Chaos

SU(2) Lattice Gauge Theory

Thermalization / Hydrodynamization

Current description of rapid thermalization / hydrodynamization uses either semiclassical approximations (kinetic theory) or holographic techniques: One method neglects potentially important quantum effects, the other method describes a quantum field theory that differs from QCD. Can we do better?

How does apparent thermalization happen in a closed quantum system, when energy is conserved?

Time evolution of local operator expectation value in terms of energy eigenstates is:

$$\langle O\rangle(t) = \mathrm{Tr}[O\rho(t)] = \sum_{n,m}\langle n|O|m\rangle\langle m|\rho(0)|n\rangle e^{i(E_n-E_m)t}$$

↓ After some time?

$$\langle O\rangle_{\mathrm{mc}}(E) \qquad E = \mathrm{Tr}(H\rho)$$

Microcanonical ensemble average

Eigenstate Thermalization Hypothesis (ETH)

For most non-integrable systems, matrix elements of "typical" local operators for "typical" energy eigenstates can be represented as

$$\langle n|O|m\rangle = \langle O\rangle_{\rm mc}(E)\delta_{nm} + e^{-S(E)/2} f(E,\omega) R_{nm}$$

$$E = (E_n + E_m)/2$$

$$\omega = E_n - E_m$$

Diagonal part close to microcanonical ensemble average

Correction suppressed exponentially by system size

Gaussian (?) random matrix

Spectral function decays with ω

Deutsch, PRA 43, 2046 (1991)

Srednicki, PRE 50, 888 (1994)

L. D'Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, Adv. Phys. 65 (2016) 239 [1509.06411]

From ETH to Thermalization

For large system and initial state with small energy variation, ETH leads to

(1) Long time average $\overline{O} \approx$ thermal expectation value $\langle O \rangle_T$ —> ergodic

(2) Fluctuations of $\langle O \rangle(t)$ around $\overline{O}$ are exponentially small in system size

(3) Quantum fluctuations $\approx$ thermal fluctuations

(4) Temporal correlation function

$$\langle n|O(t)O(0)|n\rangle - \langle n|O(t)|n\rangle\langle n|O(0)|n\rangle \approx \int d\omega e^{-i\omega t} e^{\beta\omega/2} |f(E,\omega)|^2$$

Where $f(E,\omega)$ is related to the spectral function (depending on operator *O*)

The system, when observed through O, behaves like a system in thermal equilibrium.

What must be demonstrated?

Fundamental ETH relation: $\langle n|O|m\rangle = \langle O\rangle_{\rm mc}(E)\delta_{nm} + e^{-S(E)/2} f(E,\omega) R_{nm}$

- Discretize the continuum theory on a spatial lattice, choose boundary conditions
- Show that diagonal part is exponentially close to the microcanonical average
- Show that off-diagonal part is a (Gaussian) random matrix
- Show that the spectral function decays for large ω
- Consider “physical”, i.e. gauge invariant, multiplicatively renormalizable operators
- Operators could be local or sufficiently smeared
- Demonstrate RG behavior for several $g(a)$ when $a \to 0$, to establish the continuum limit
- Demonstrate ETH for several system sizes for fixed $g(a)$, to establish the infinite volume limit

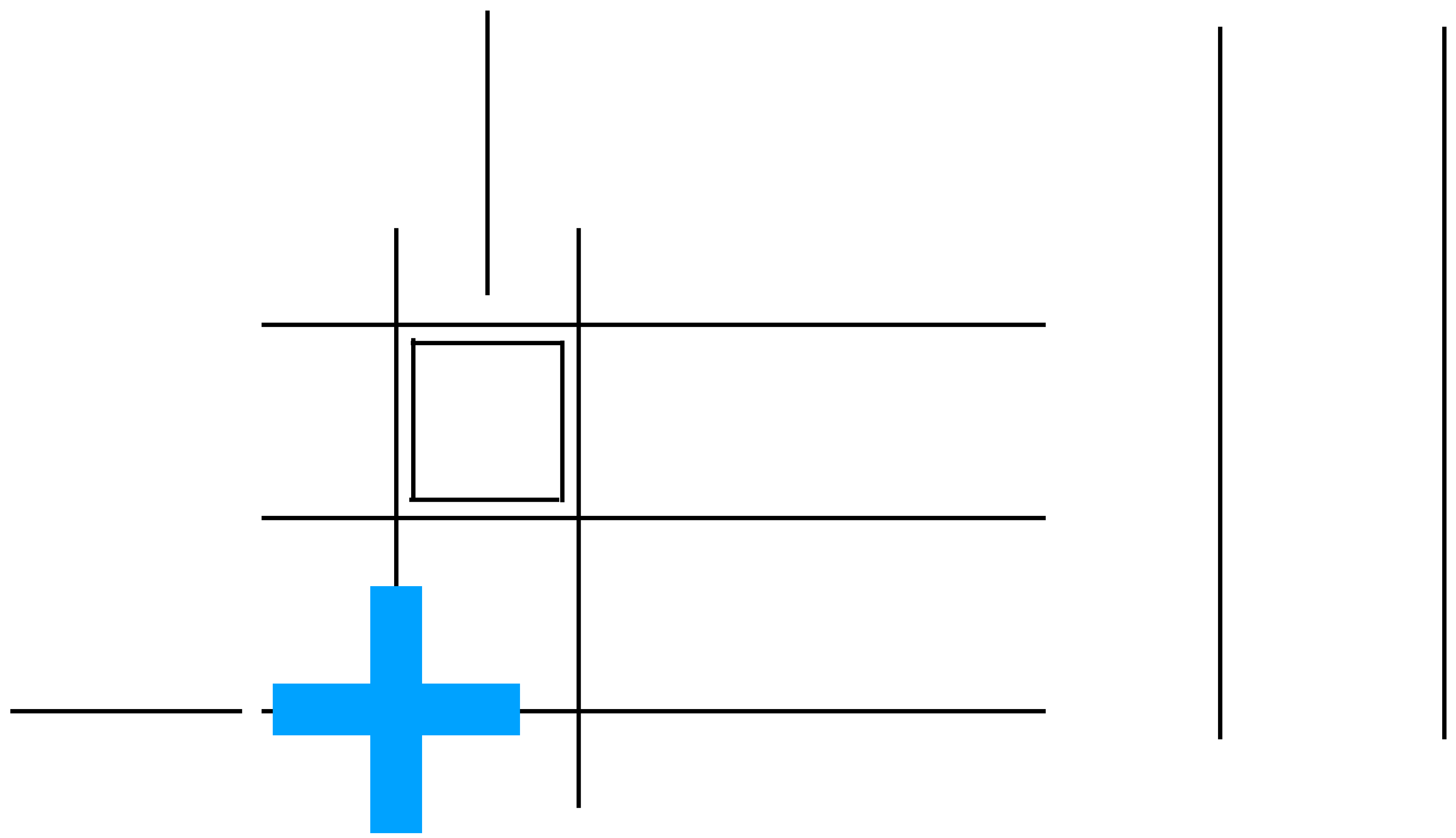
(2+1)-D SU(2) Lattice Gauge Theory

Kogut-Susskind Hamiltonian: $$H = \frac{g^2}{2}\sum_{\text{links}}(E_i^a)^2 - \frac{2}{a^2 g^2}\sum_{\text{plaquettes}}\Box(\boldsymbol{n})$$

$$\Box(\boldsymbol{n}) = \mathrm{Tr}[U^\dagger(\boldsymbol{n},\hat{y})U^\dagger(\boldsymbol{n}+\hat{y},\hat{x})U(\boldsymbol{n}+\hat{x},\hat{y})U(\boldsymbol{n},\hat{x})]$$

$$[E_i^a, U(\boldsymbol{n},\hat{j})] = -\delta_{ij}T^a U(\boldsymbol{n},\hat{j})$$

$$[E_i^a, E_i^b] = i f^{abc} E_i^c$$



Gauss's law: Every vertex transforms as a singlet for a state to be physical

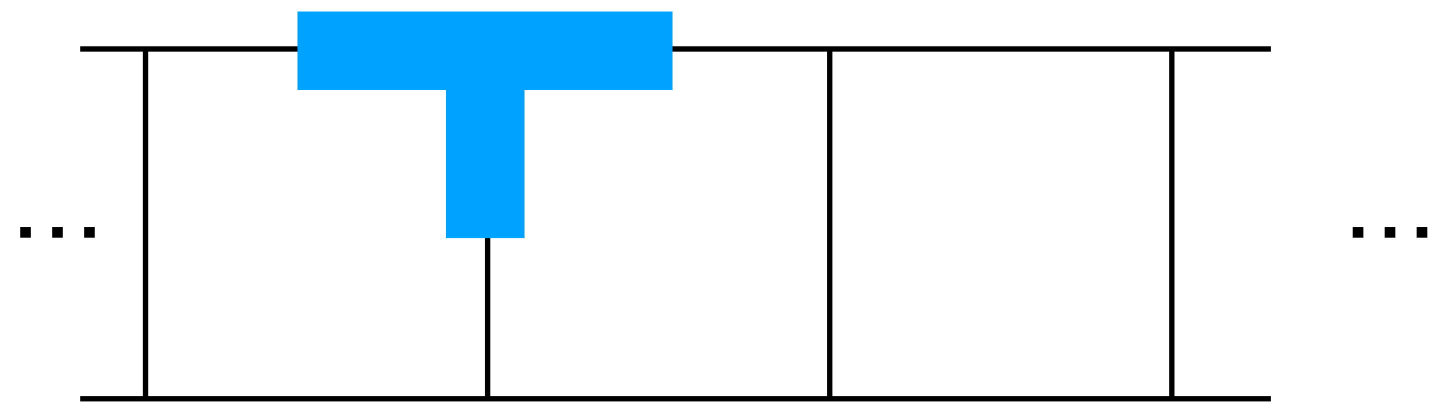
Electric basis on links: $|j\, m_L\, m_R\rangle$

$$E^2|j\, m_L\, m_R\rangle = j(j+1)|j\, m_L\, m_R\rangle$$

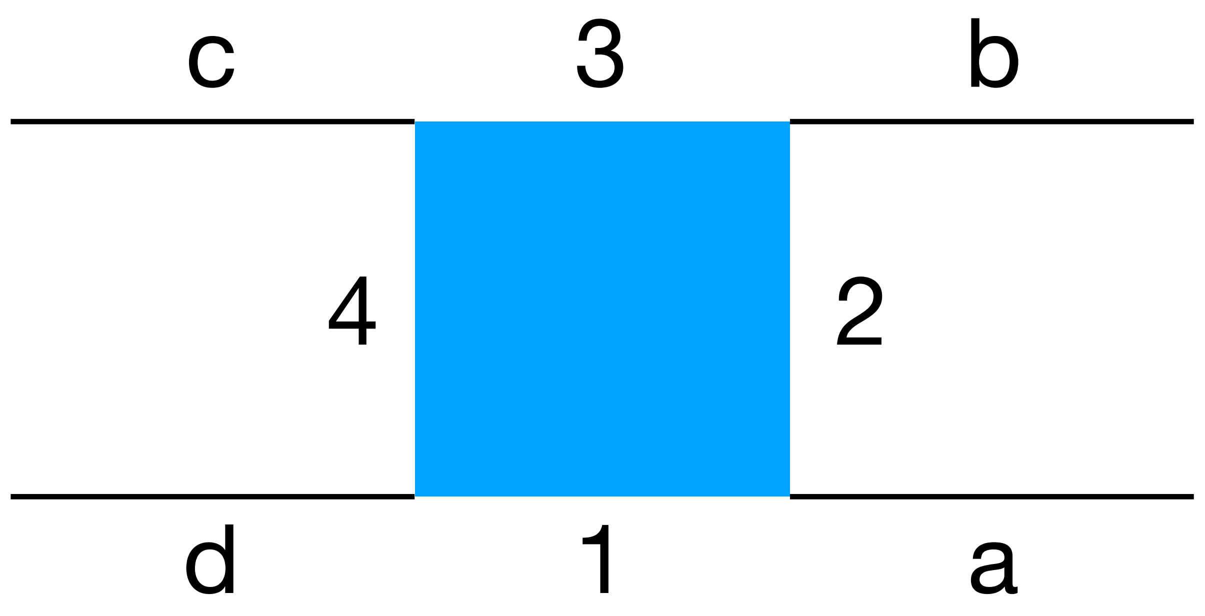
Byrnes, Yamamoto, quant-ph/0510027

(2+1)-D SU(2) on Periodic Plaquette Chain

Each vertex has three links: singlet is uniquely defined by the j values on the three links



Matrix elements between physical states (singlets) expressed in 6j symbols



j: initial

J: final

$$\langle J_1 J_2 J_3 J_4 | \Box | j_1 j_2 j_3 j_4 \rangle = \prod_{\alpha=a,b,c,d} (-1)^{j_\alpha} \prod_{\alpha=1,2,3,4} \left[(-1)^{j_\alpha + J_\alpha} \sqrt{(2j_\alpha+1)(2J_\alpha+1)} \right]$$

$$\left\{ \begin{array}{ccc} j_a & j_1 & j_2 \\ 1/2 & J_2 & J_1 \end{array} \right\} \left\{ \begin{array}{ccc} j_b & j_2 & j_3 \\ 1/2 & J_3 & J_2 \end{array} \right\} \left\{ \begin{array}{ccc} j_c & j_3 & j_4 \\ 1/2 & J_4 & J_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_d & j_4 & j_1 \\ 1/2 & J_1 & J_4 \end{array} \right\}$$

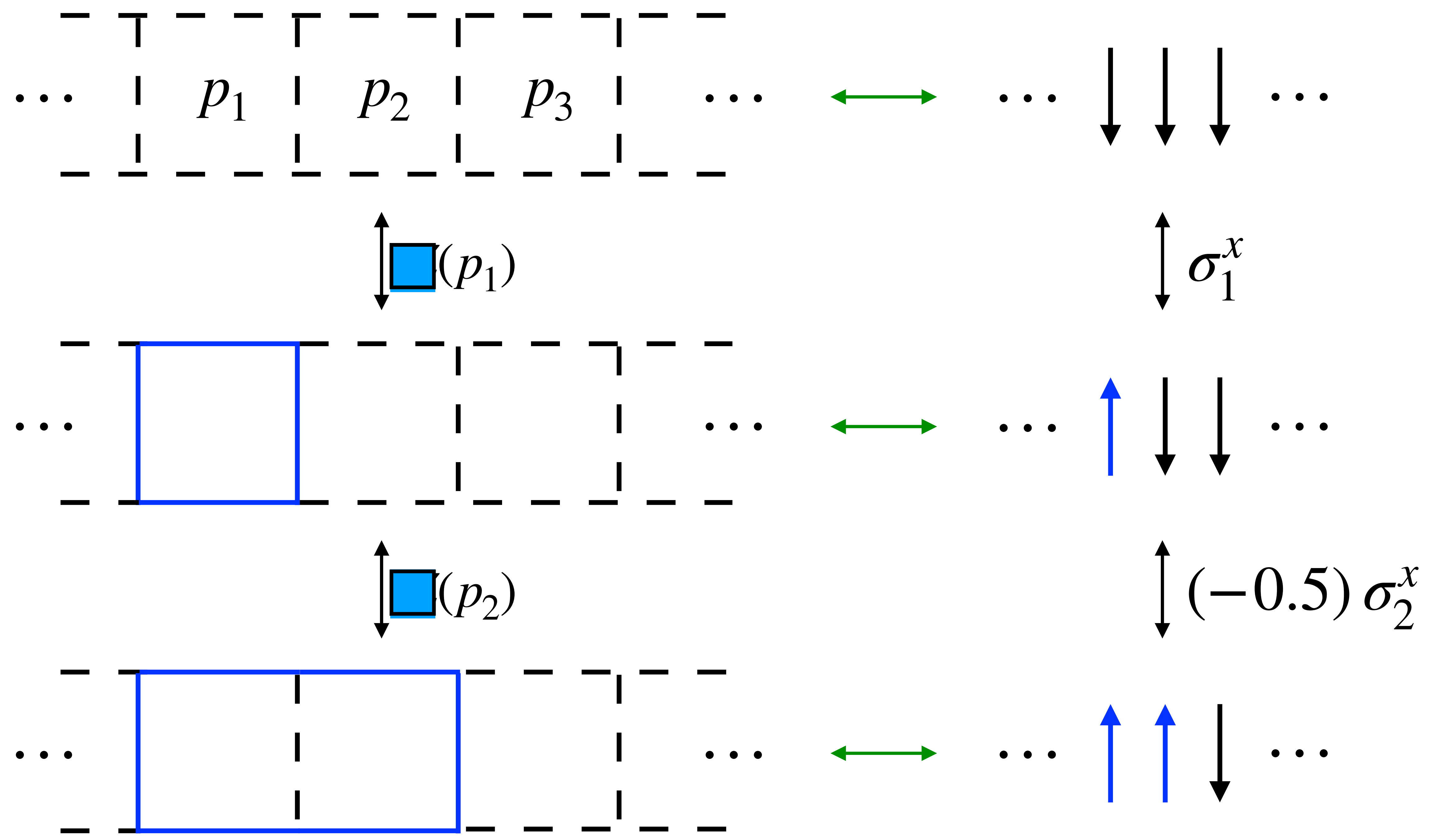
Klco, Stryker, Savage, 1908.06935

Reduced Hilbert space with $j_{\max} = 1/2$

SU(2) with $j_{\max} = 1/2$

can be mapped onto spin chain

[X. Yao, 2303.14264]



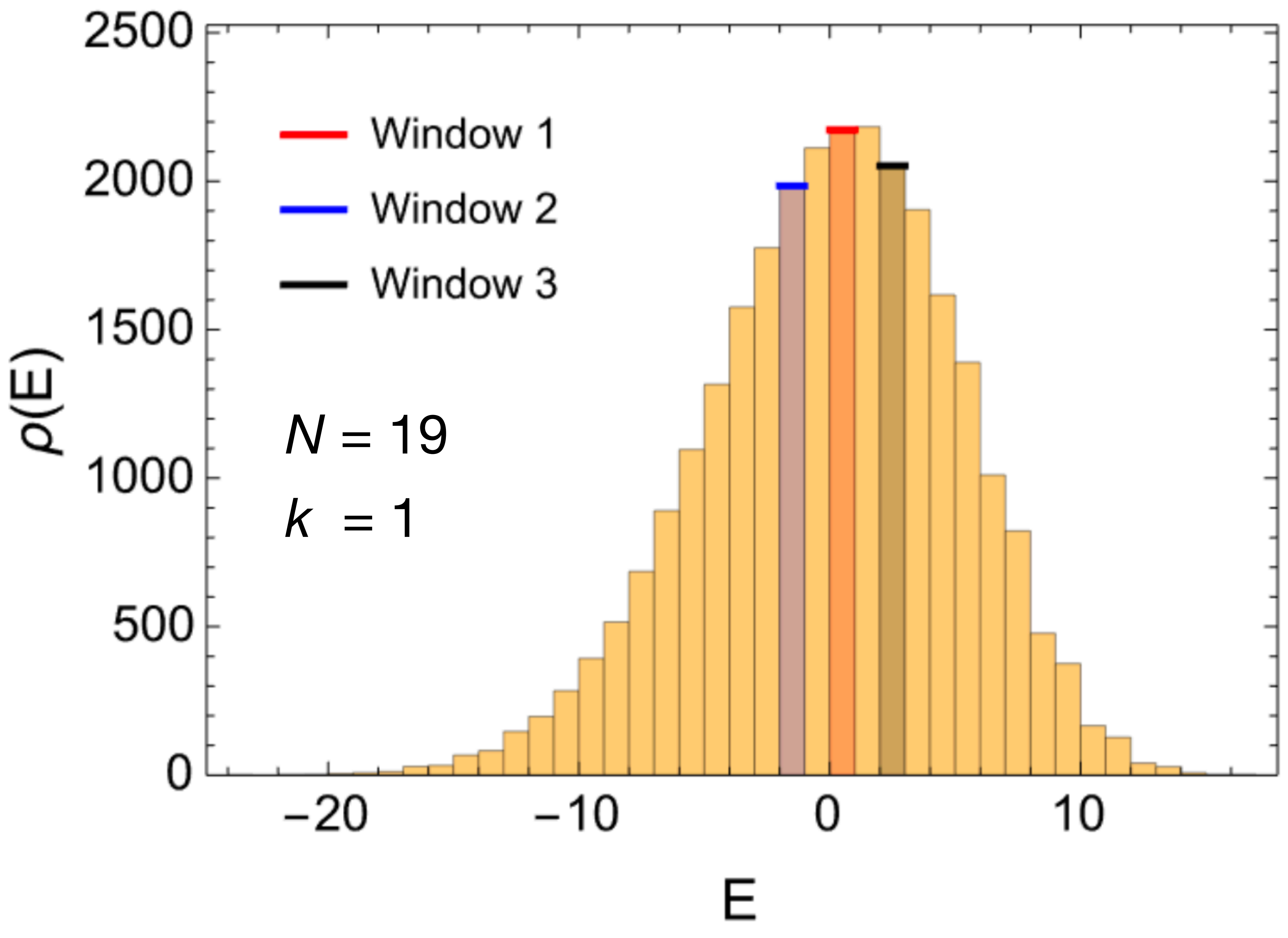
Project onto momentum eigenstates
$-N/2 \leq k \leq N/2$ for N plaquettes

$$aH = J\sum_{i=0}^{N-1}\sigma_i^z\sigma_{i+1}^z + h_z\sum_{i=0}^{N-1}\sigma_i^z + h_x\sum_{i=0}^{N-1}(-0.5)^{(\sigma_{i-1}^z+\sigma_{i+1}^z)/2+1}\sigma_i^x$$

$$J = -3ag^2/16,\ h_z = 3ag^2/8,\ h_x = -2/(ag^2)$$

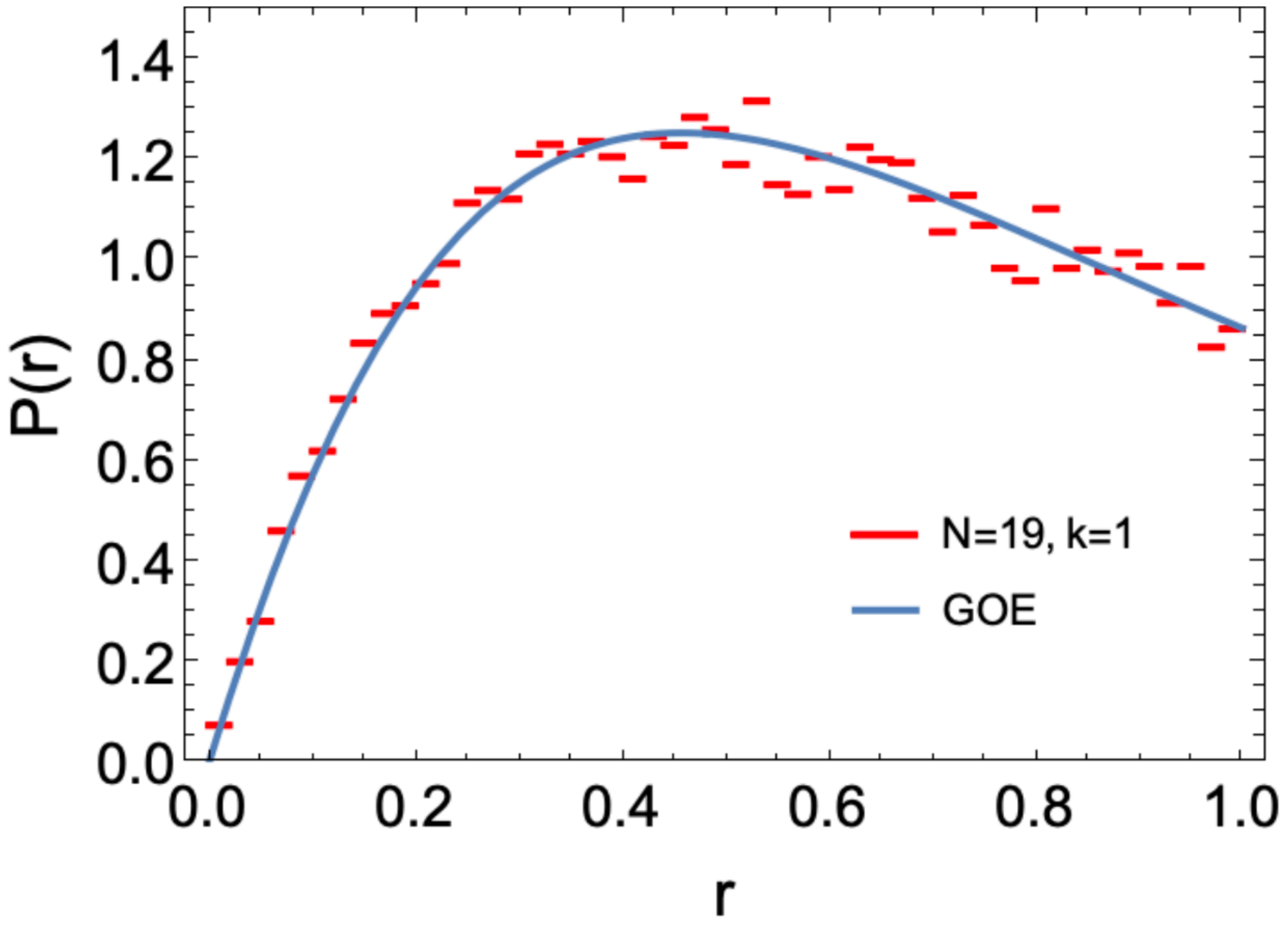
Plaquette Chain with $j_{\max} = 1/2$: Spectrum

Look at matrix elements in 3 energy windows around peak with ca. 2000 eigenstates each



Restricted gap ratio distribution

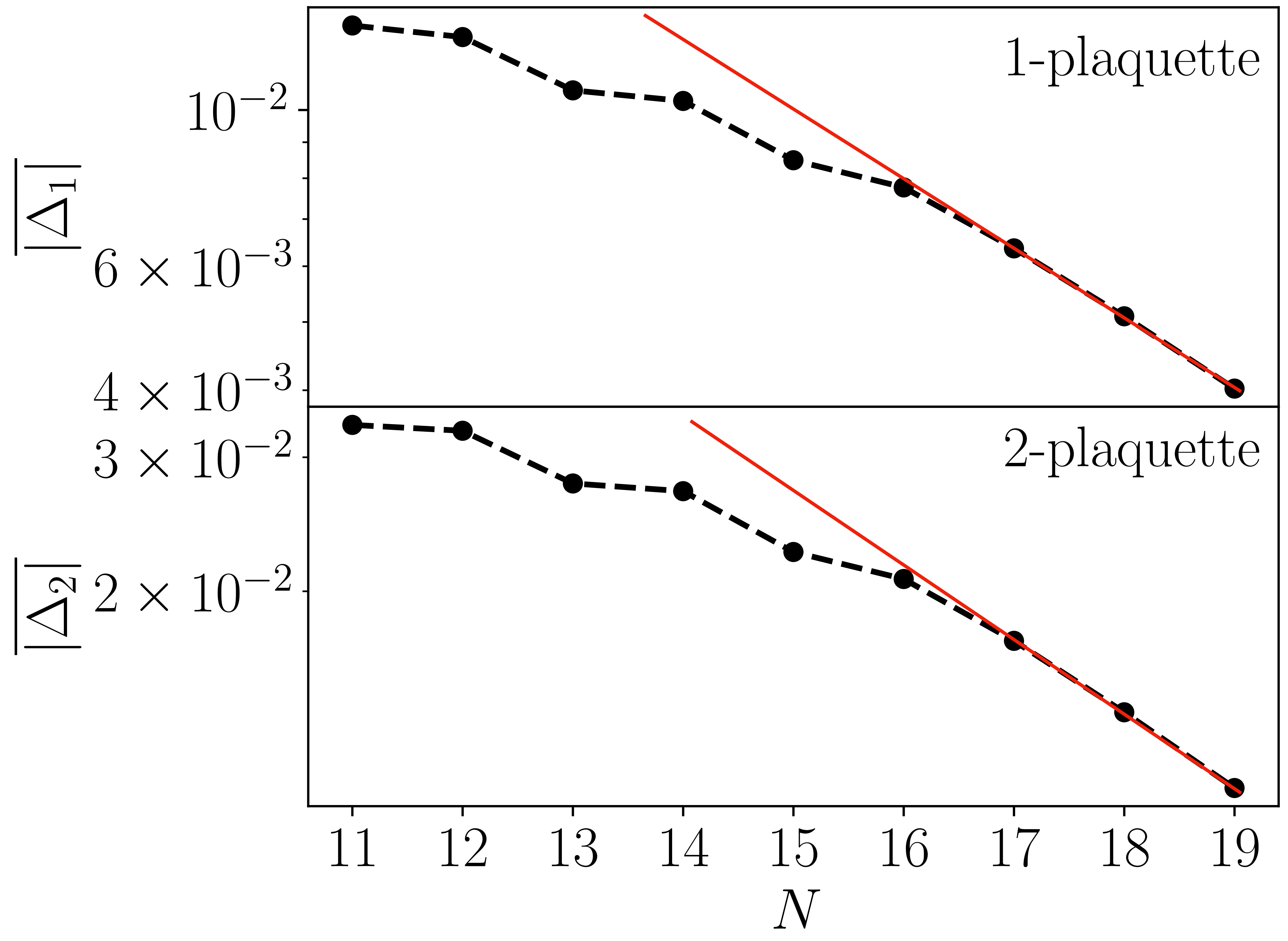
$$0 < r_\alpha = \frac{\min[\delta_\alpha, \delta_{\alpha-1}]}{\max[\delta_\alpha, \delta_{\alpha-1}]} \leq 1$$



Plaquette Chain with $j_{\max} = 1/2$: Diagonal Part

Consider 1-plaquette and 2-plaquette operators with $ag^2 = 1.2$

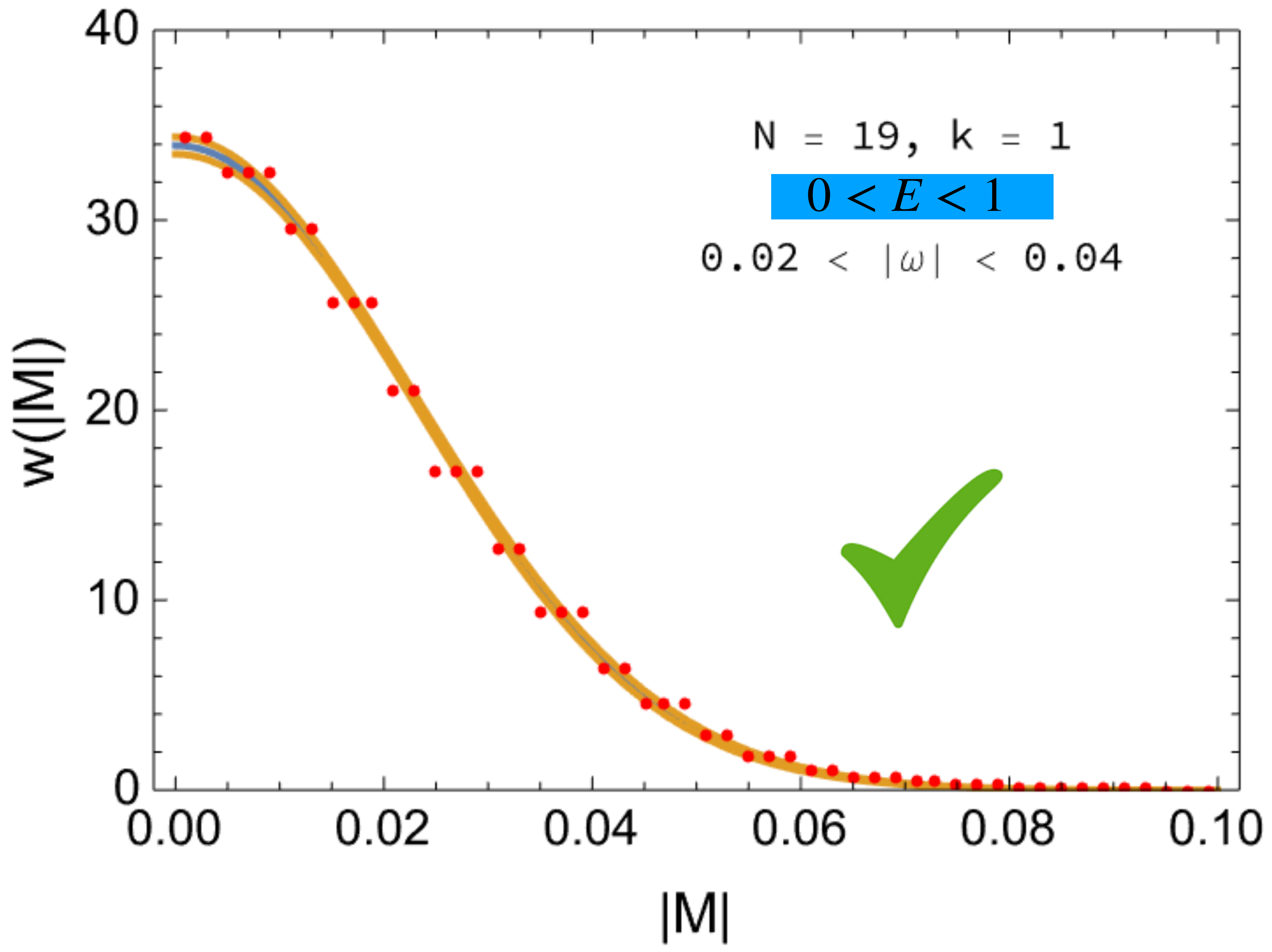
Proxy for microcanonical ensemble: $$\Delta_i(n) = \langle n|O_i|n\rangle - \frac{1}{21}\sum_{m=n-10}^{n+10} \langle m|O_i|m\rangle$$



Exponential decay with system size for $N \geq 16$ ✔

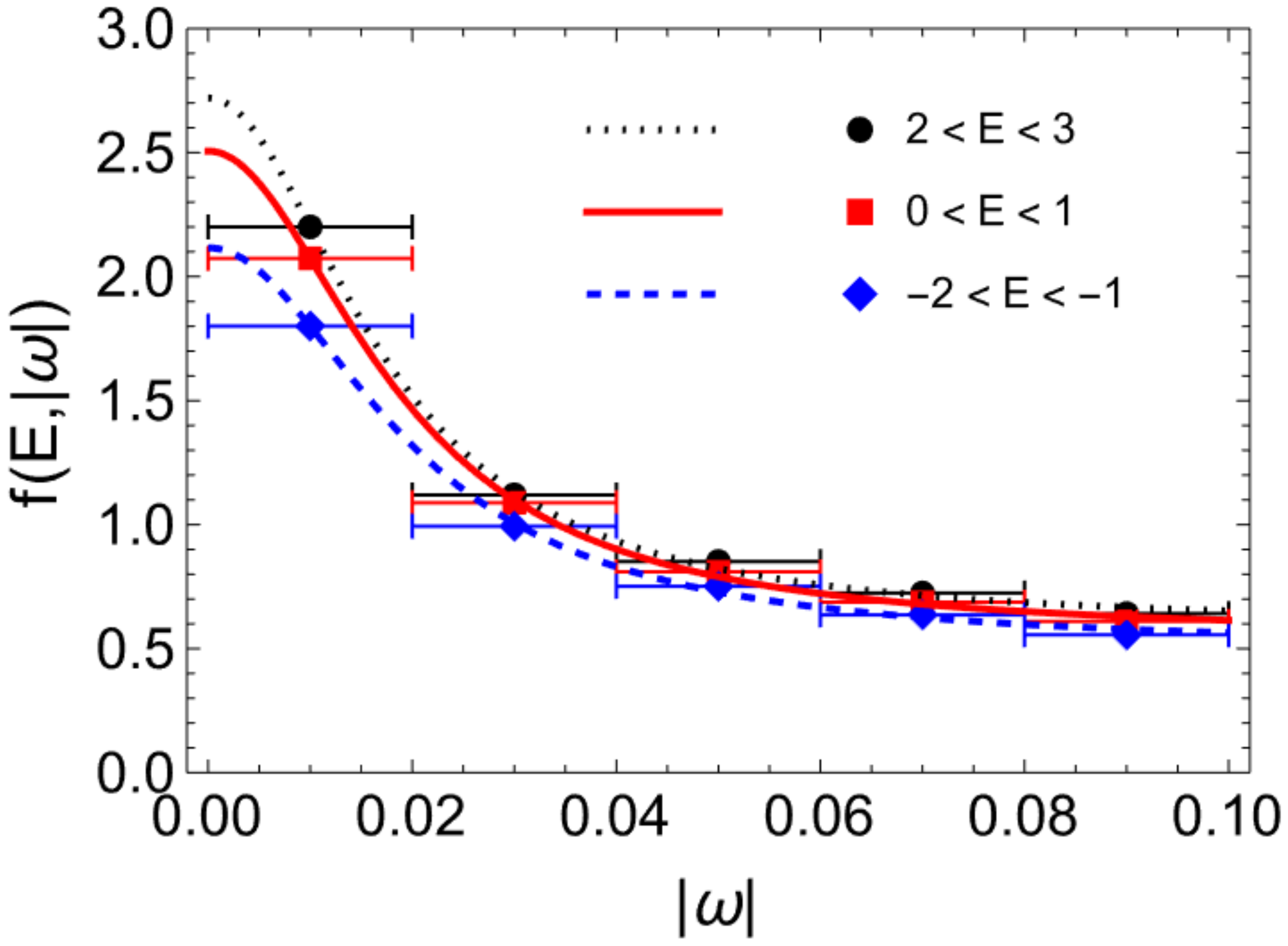
Chain with $j_{\text{max}} = 1/2$: Off-Diagonal Part

Consider off-diagonal part $M_{mn} \equiv \langle m | H_{\text{el}} | n \rangle$



Well described by Gaussian

$$\sigma^2 = \text{Tr}[\boldsymbol{M}^2] = \frac{f_{\text{el}}(E,\omega)^2}{\rho(E)}$$

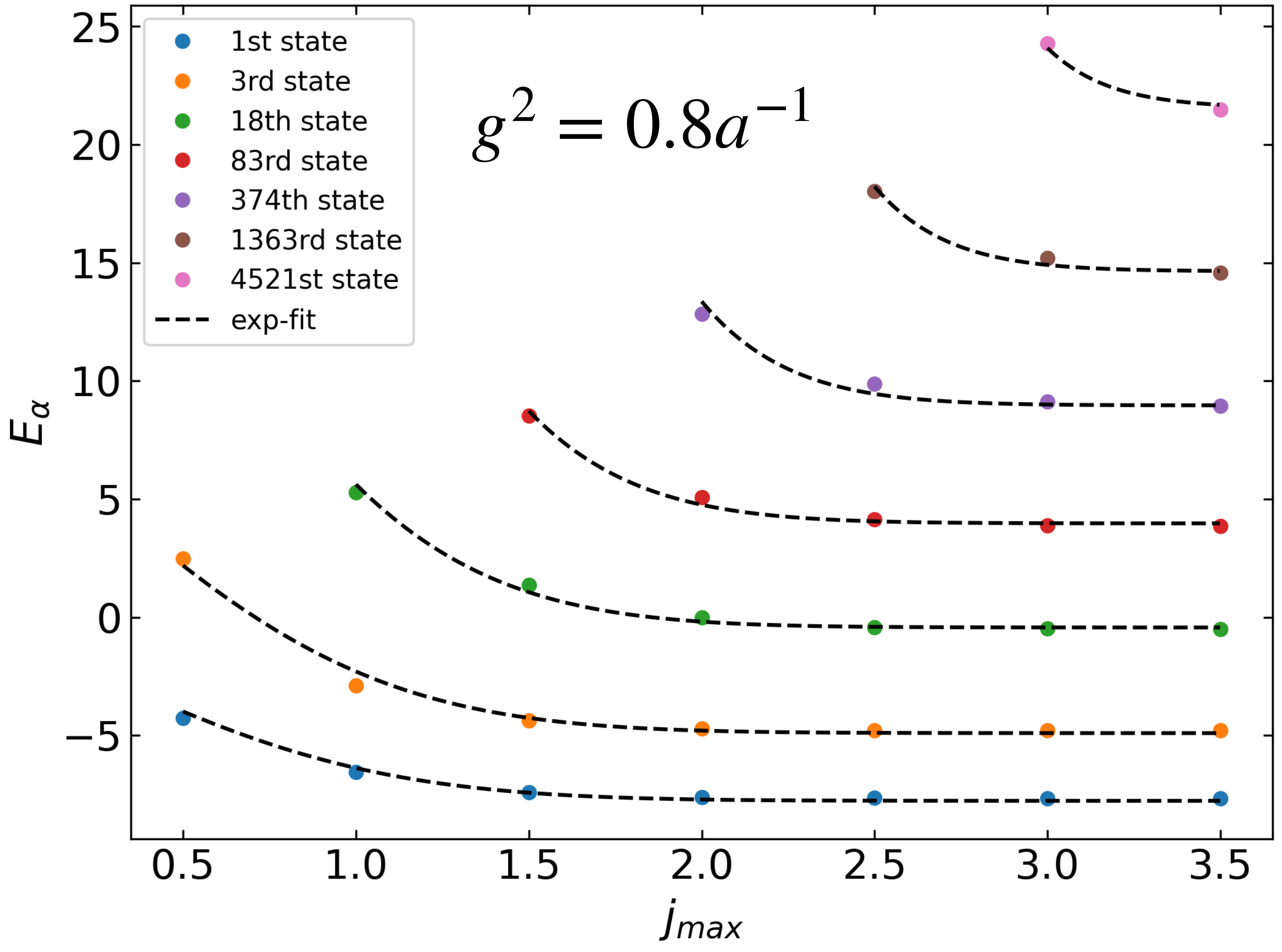


Spectral function at small $|\omega|$ is well described by a diffusive transport peak

$$f(E,\omega) = \frac{a}{\omega^2 + b^2} + c$$

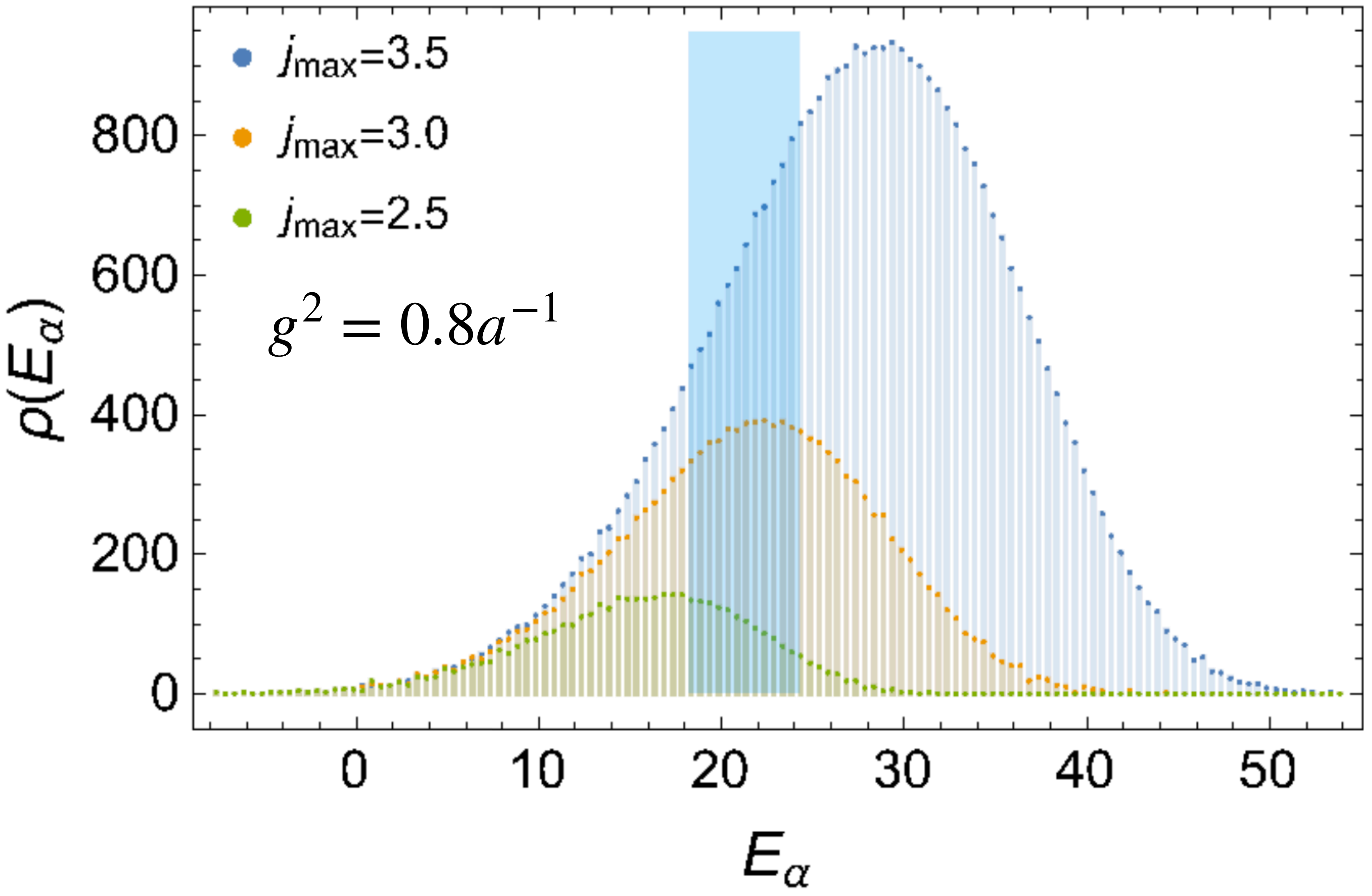
$j_{\max}$ Cutoff Dependence and Convergence

Energy eigenvalues on $N = 3$ chain vs. $j_{\max}$



Take $j_{\max} = 3.5$, only use states within 5% error from asymptotic eigenenergy values

Energy level spectrum for different j_{max}

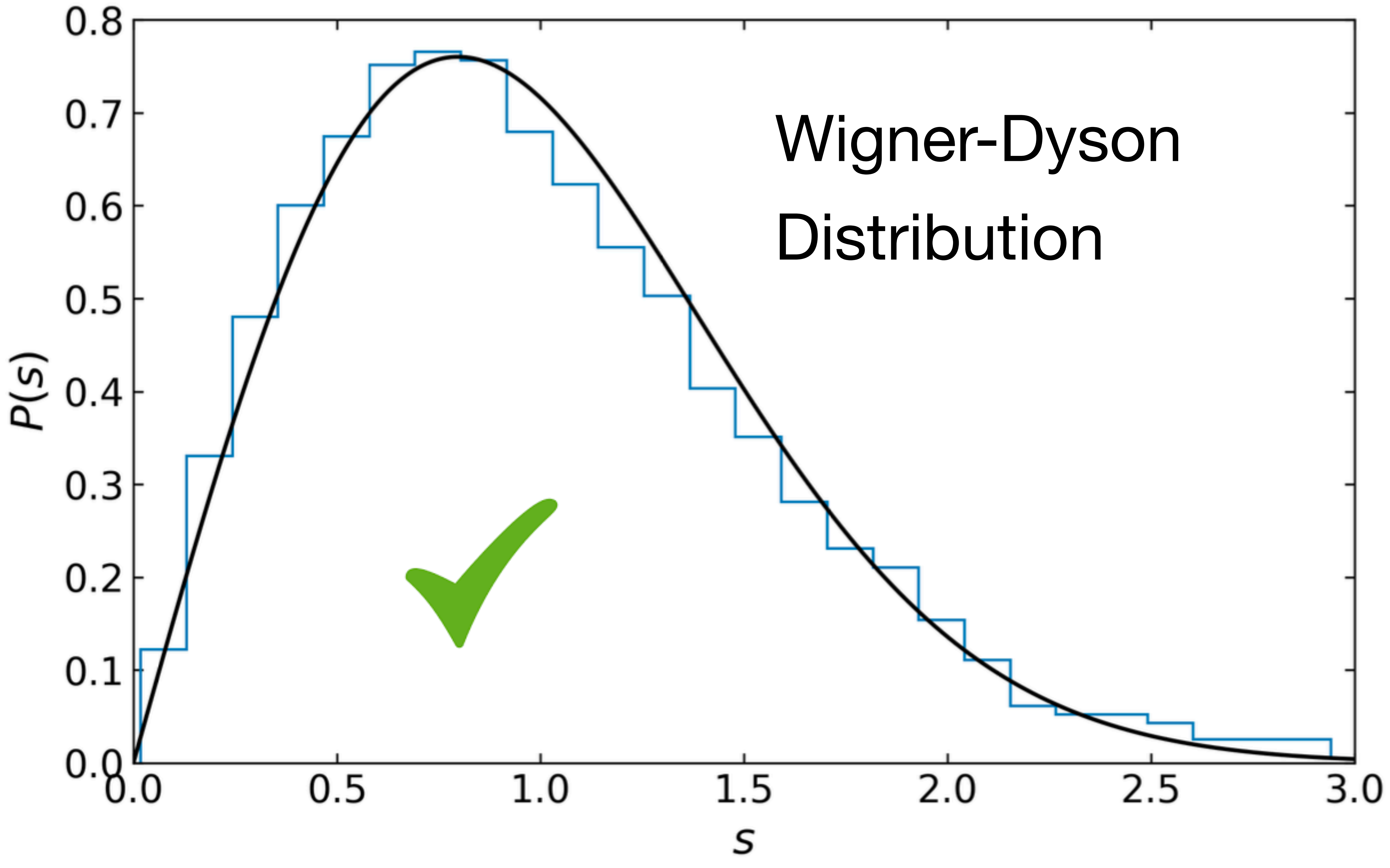


Select converged region 18 < E < 24

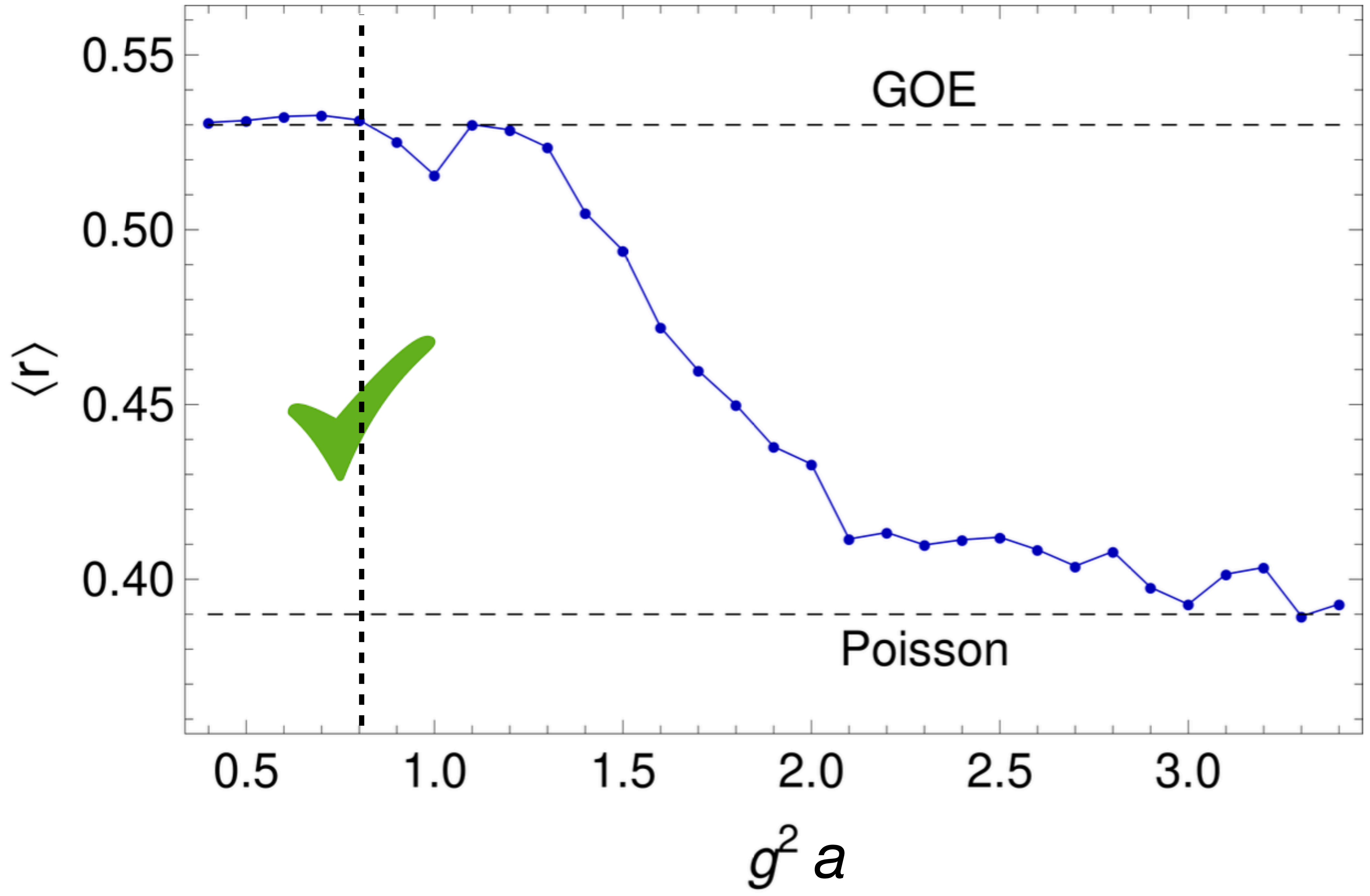
Ebner, BM, Schäfer, Seidl, Yao, 2308.16202

N = 3 Chain with $j_{\max} = 7/2$: Spectrum

Nearest-neighbor level statistics exhibits GOE characteristics at $g^2a = 0.8$

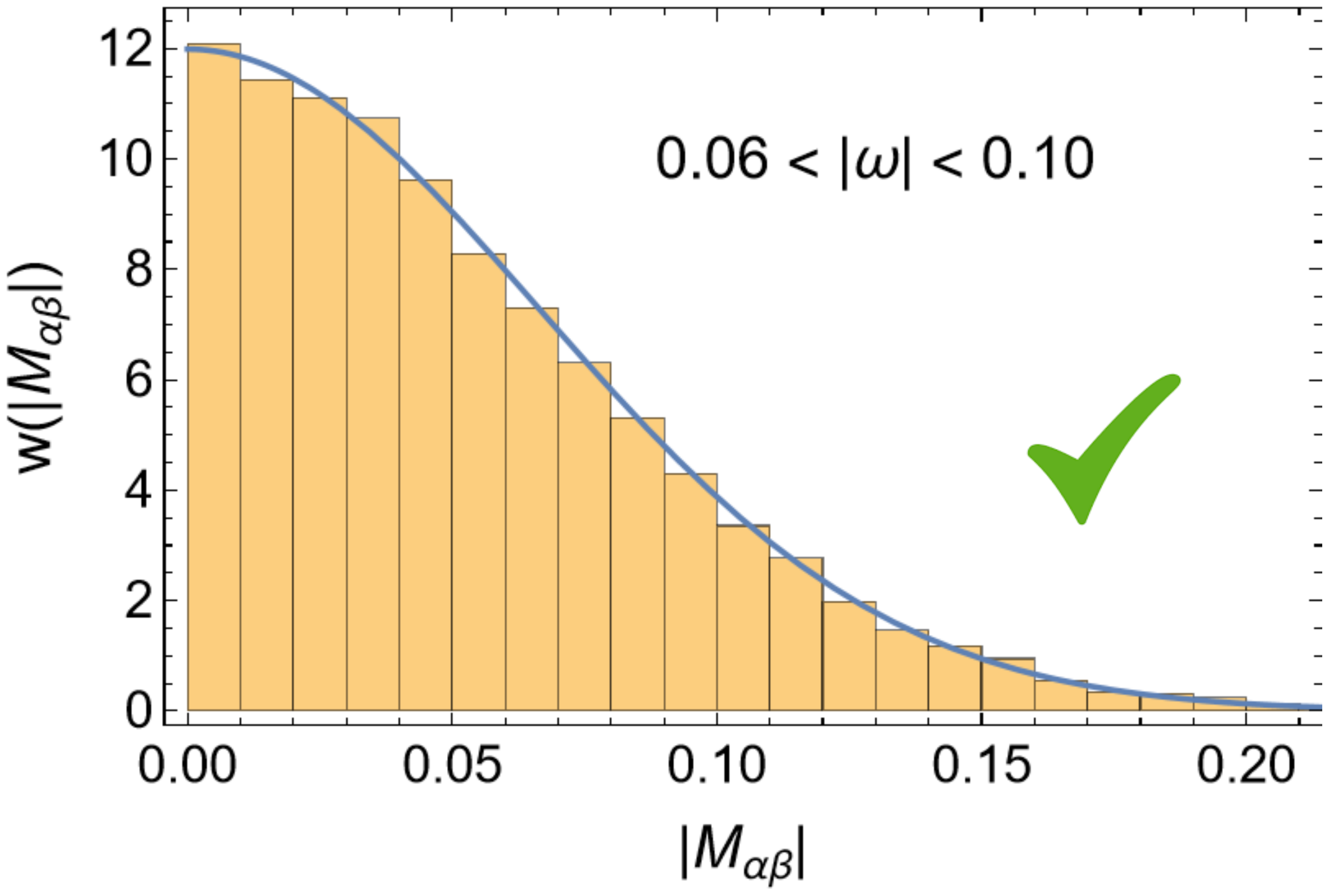


Mean restricted gap ratio shows GOE behavior at weak coupling and Poisson at strong coupling

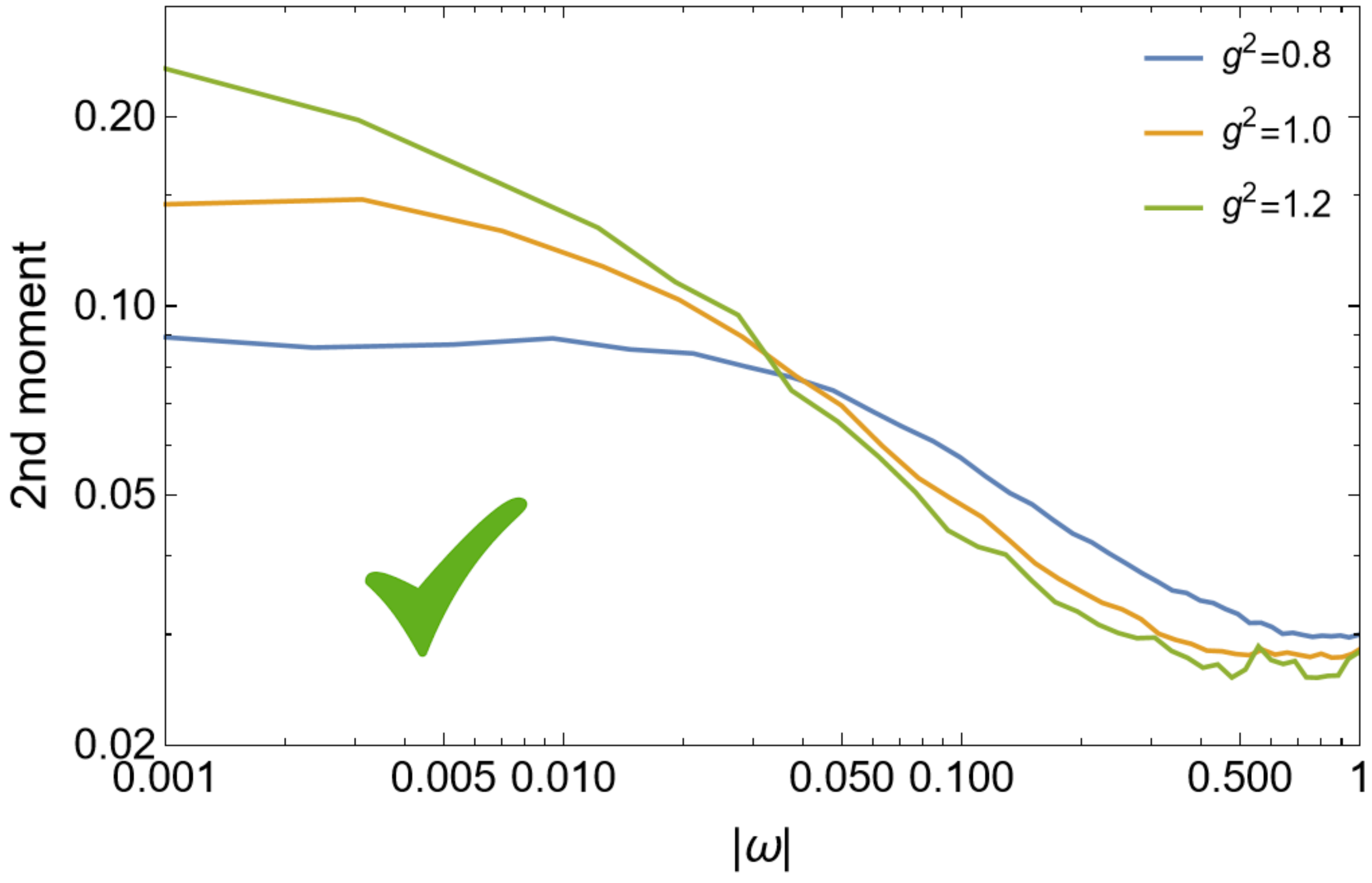


$N = 3$ Chain with $j_{\mathrm{max}} = 7/2$: Off-Diagonal Part

Off-diagonal elements of H_{el} are Gaussian distributed



Spectral function at small |ω| shows a diffusive transport peak with plateau



Plateau disappears when system is non-chaotic

Test GOE Behavior: *N* = 3, $j_{\max} = 7/2$

Construct band matrix by dropping deciphered matrix elements at time *T*

$$O^T_{mn} = \begin{cases} \langle m|O|n\rangle, & |E_m - E_n| \leq \frac{2\pi}{T} \\ 0, & |E_m - E_n| > \frac{2\pi}{T} \end{cases}$$

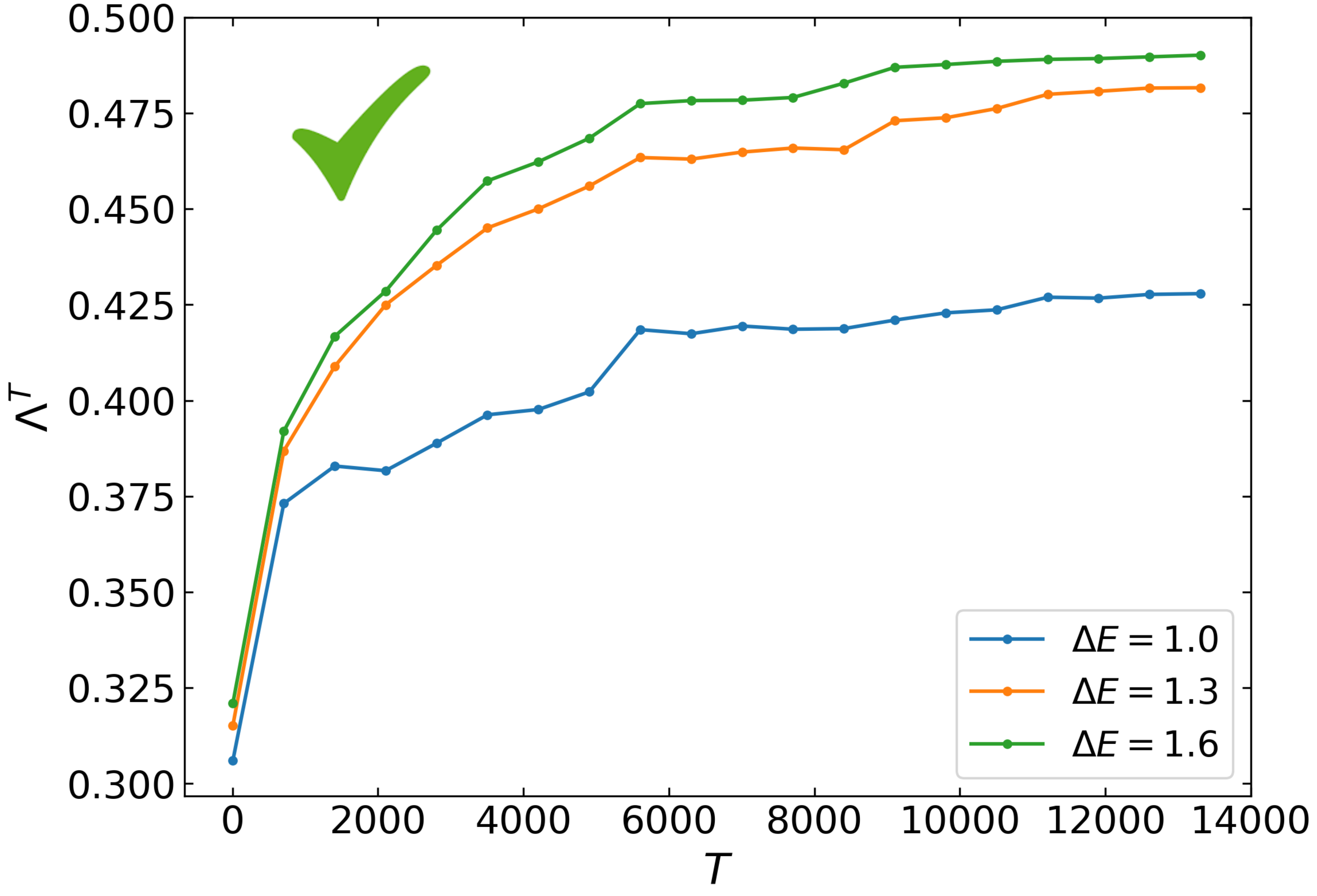
GOE measure Λ^T

$$O^T_c = O^T - \mathrm{Tr}[O^T]/d$$

$$\Lambda^T = \frac{\left(\mathrm{Tr}[(O^T_c)^2]\right)^2}{d\ \left(\mathrm{Tr}[(O^T_c)^4]\right)}$$

For Gaussian Orthogonal Ensemble (GOE):

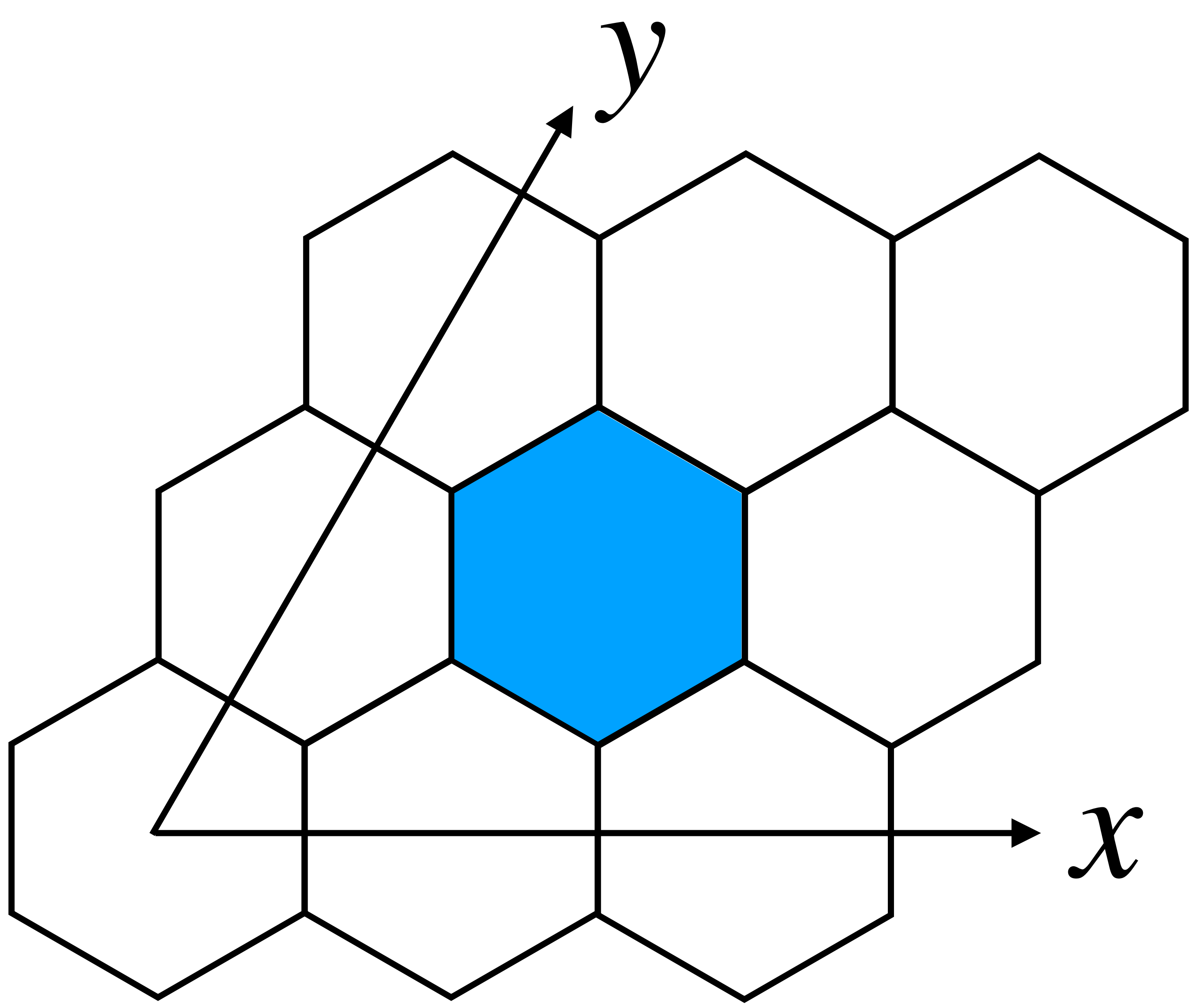
$$\Lambda^T = 0.5$$



(2+1)-D SU(2) on Honeycomb Lattice

Problem: on square lattice each vertex has four links and singlet is not uniquely defined by four j values

Solution: use honeycomb lattice



$$H_{\rm el} = \frac{g^2}{2}\frac{3\sqrt{3}}{2}\sum_{\boldsymbol{n}}\sum_{i=1}^{3} E_i^2(\boldsymbol{n})$$

$$H_{\rm mag} = -\frac{4\sqrt{3}}{9a^2g^2}\sum_{\boldsymbol{n}} \hexagon(\boldsymbol{n})$$

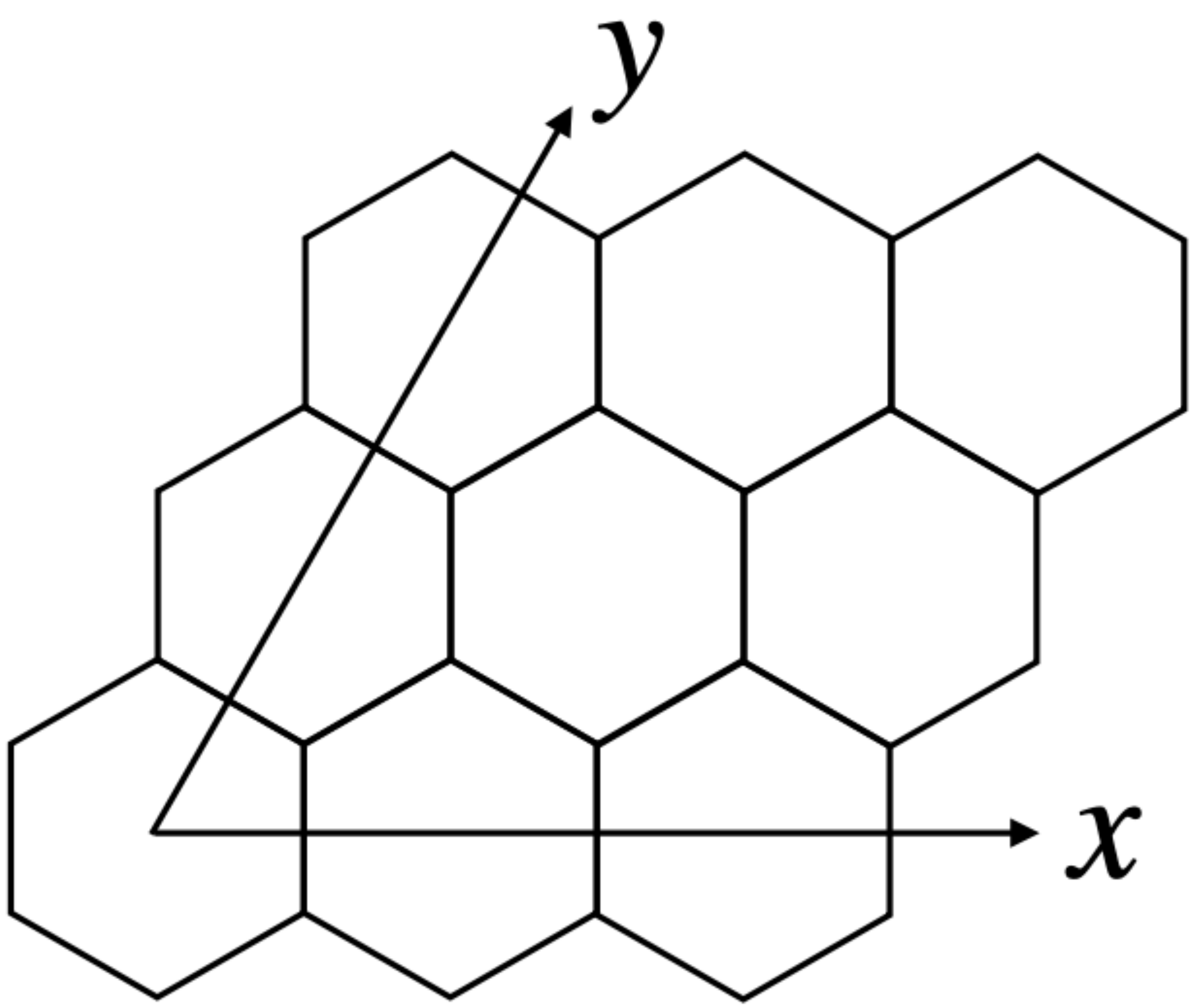
$\langle J_i | \hexagon | j_i \rangle$ between physical states

= product of six $6j$ symbols

BM, X. Yao, arXiv: 2307.00045

Boundary conditions

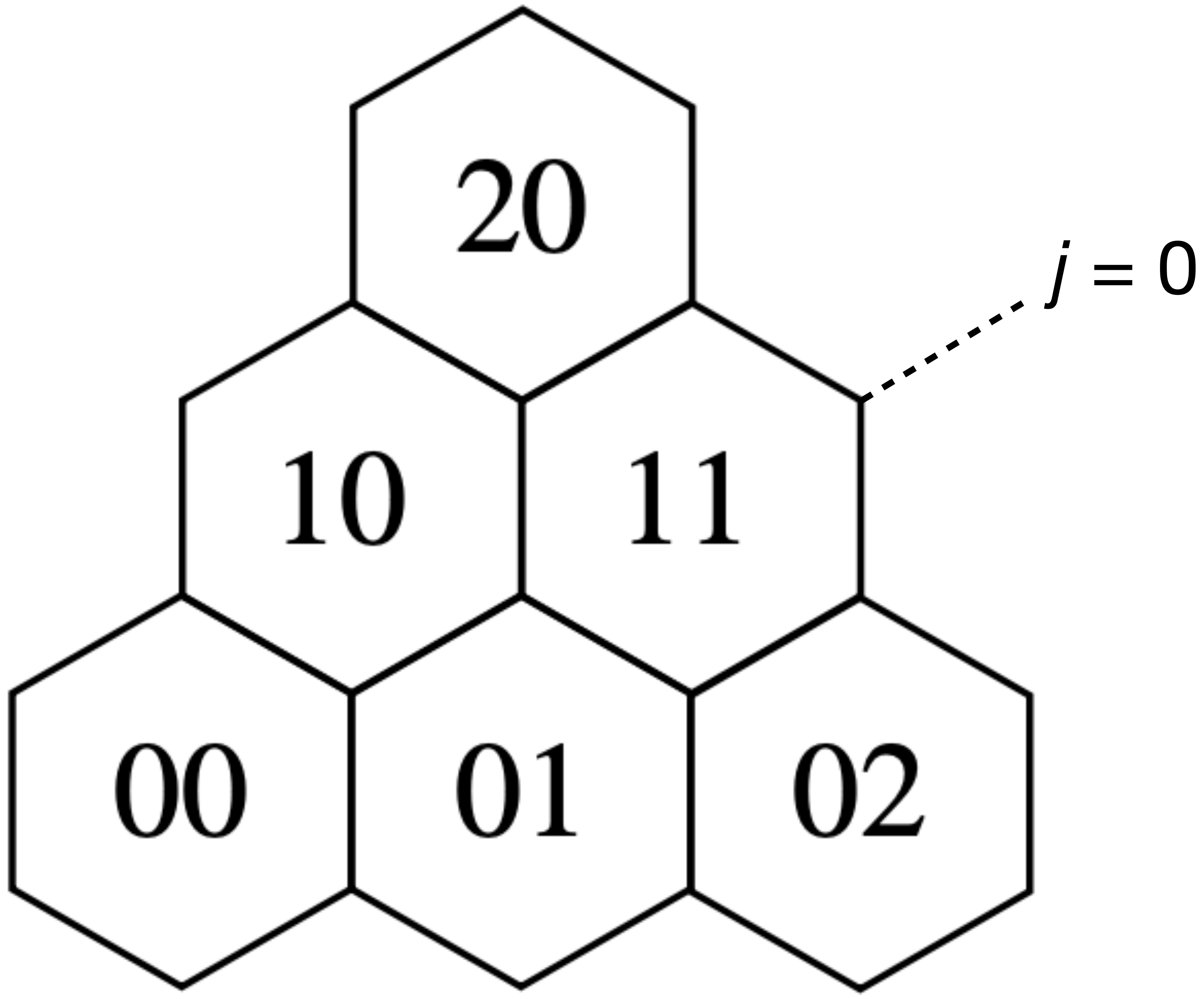
Periodic with periods N_x and N_y



(a) Parallelogram.

$N_x = 5, N_y = 4$ $k_x = k_y = 1$ sector

Closed (confining) with $\vec{n} \cdot \vec{E} = 0$



(b) Triangle.

$N = 5$

Hamiltonian constrained to $j_{\max} = 1/2$

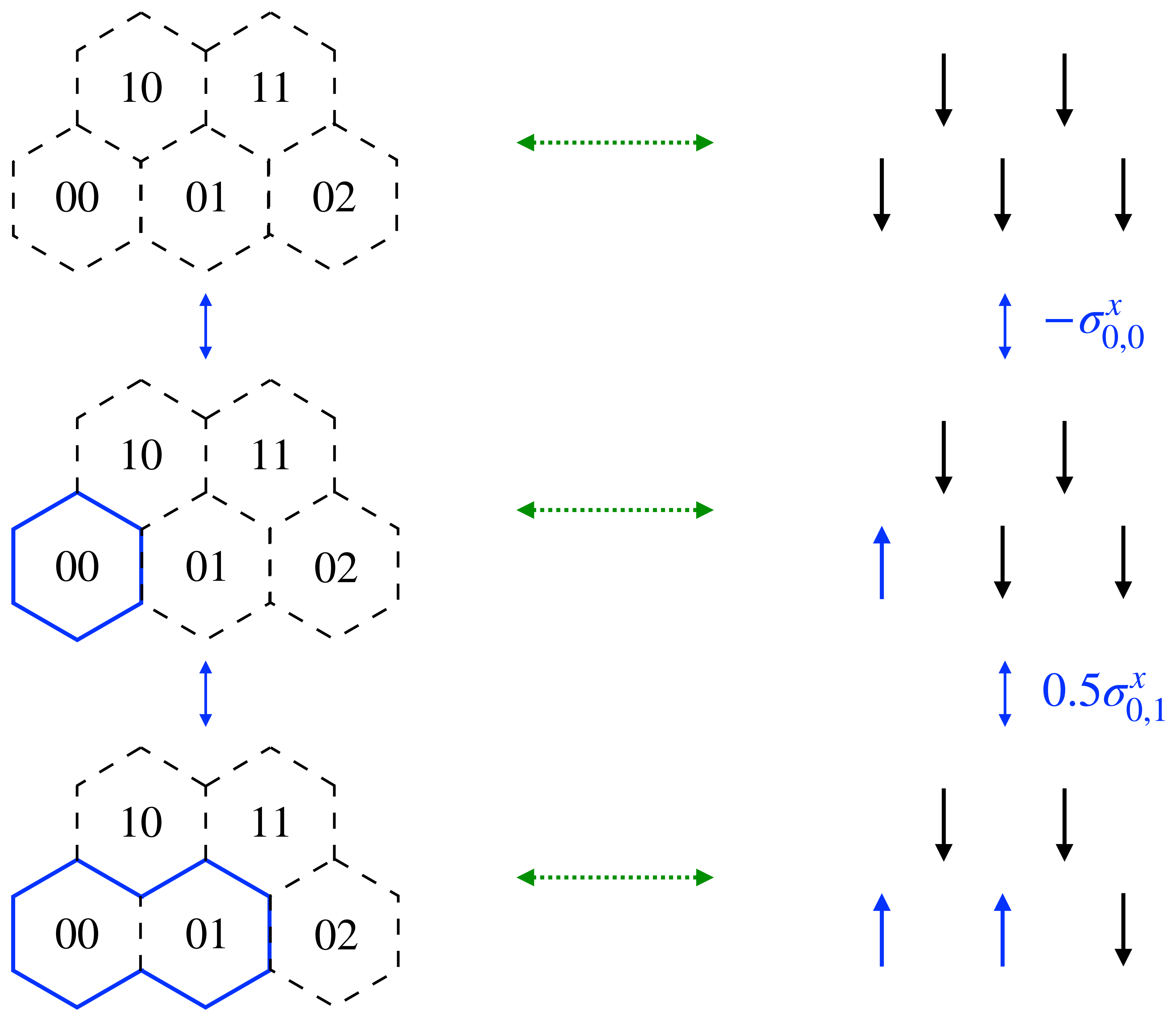
SU(2) with $j_{\max} = 1/2$ expressed at Ising-like model

BM, X. Yao, 2307.00045

$$aH = h_+ \sum_{(i,j)} \Pi^+_{i,j} - h_{++} \sum_{(i,j)} \Pi^+_{i,j} \left(\Pi^+_{i+1,j} + \Pi^+_{i,j+1} + \Pi^+_{i+1,j-1} \right) + h_x \sum_{(i,j)} (-0.5)^{c_{i,j}} \sigma^x_{i,j}$$

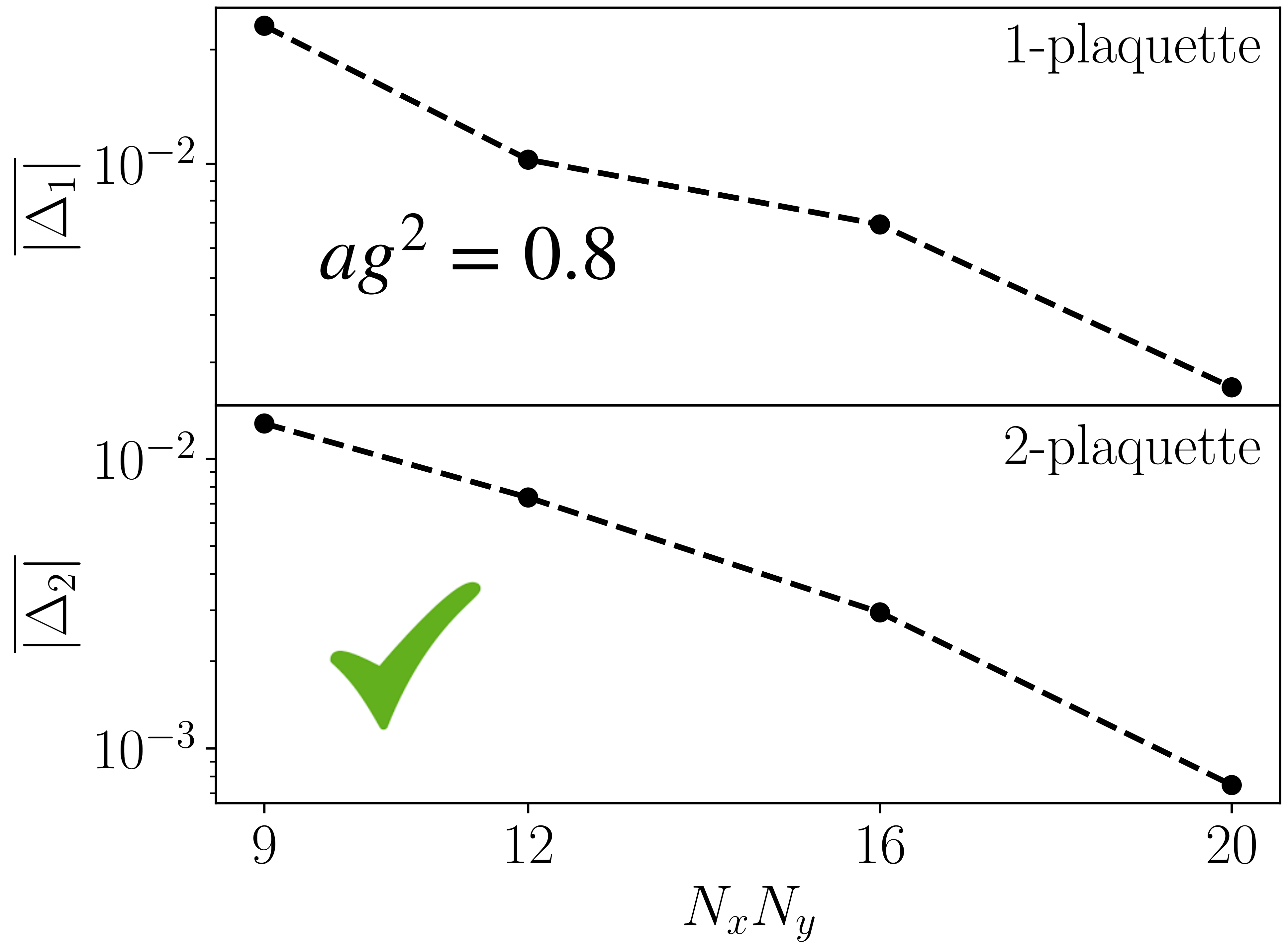
$$\Pi^+_{i,j} = (1 + \sigma^z_{i,j})/2$$

$$h_+ = \frac{27\sqrt{3}}{8} a g^2 \,, \; h_{++} = \frac{9\sqrt{3}}{8} a g^2 \,, \; h_x = \frac{4\sqrt{3}}{9 a g^2}$$

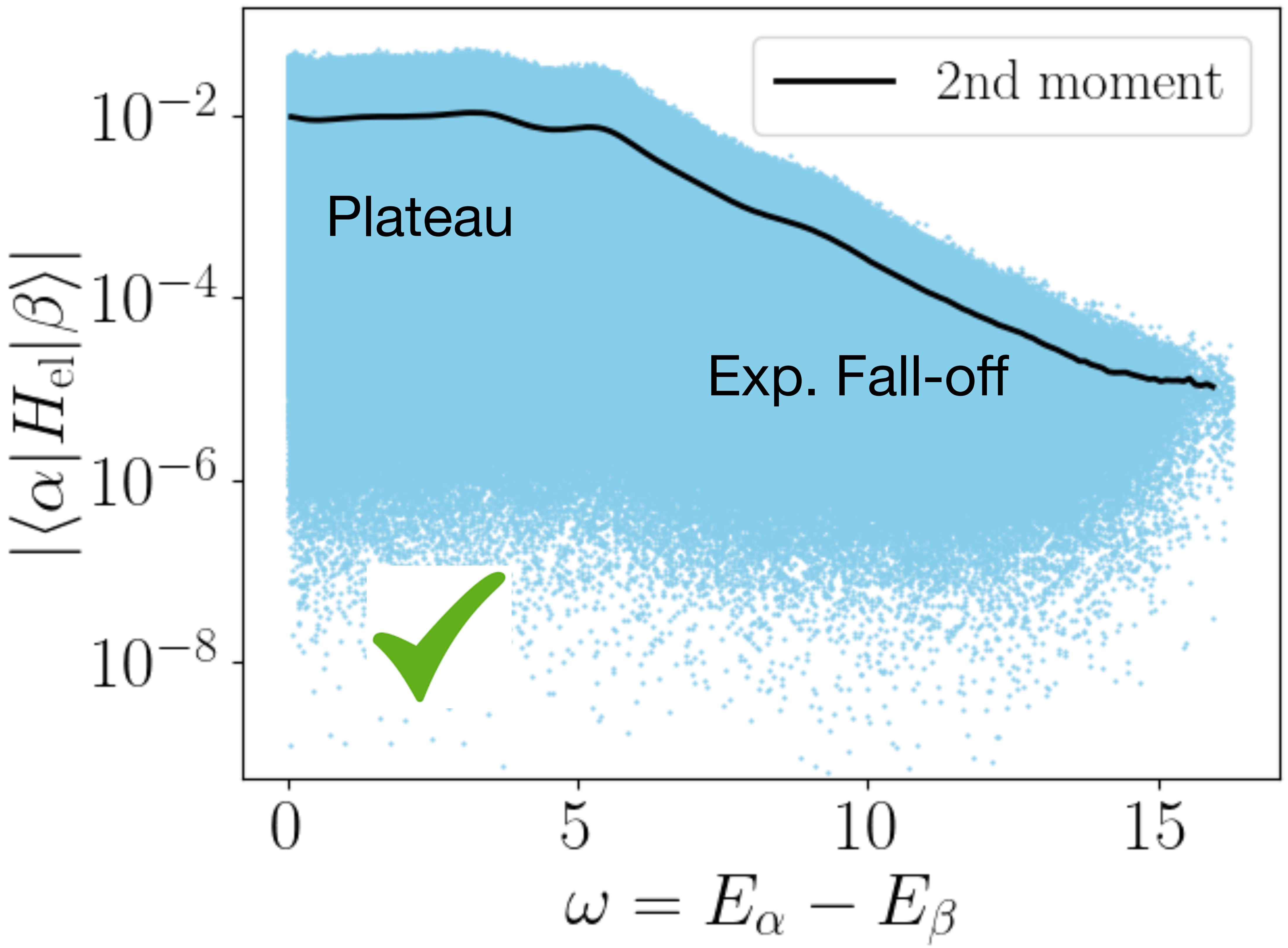


ETH Tests for Honeycomb Lattice with $j_{\max} = 1/2$

Diagonal matrix element test for local operators (1 and 2 plaquettes)



Off-diagonal matrix elements of H_{el}



Summary Plans

We obtained clear and eextensive numerical evidence for ETH in (2+1)-D SU(2) lattice gauge theory.

We studied three cases by direct dialgonalization of the KS Hamiltonian:

(1) long chain with $j_{\max} = 1/2$ ✓

(2) short chain with $j_{\max} = 7/2$ and fully converged spectrum ✓

(3) 2D honeycomb with $j_{\max} = 1/2$ ✓

We found:

- Wigner-Dyson level spacing statistics
- Clustering of diagonal matrix elements around micro canonical average
- Random matrix behavior of off-diagonal matrix elements
- Transport peak in spectral function at small |ω|

Future Plans

There are many possible directions for future research, e.g.:

(1) (2+1)-D honeycomb with higher $j_{\max}$

(2) (3+1)-D SU(2)

(3) SU(3) and include fermions

(4) Implementation on a quantum computer

Extent of further investigations will depend on availability of computing resources.

More efficient algorithms than full diagonalization of H_{KS} must also be explored.

Andreas at work

Danke für die lange, rege und fruchtbare Zusammenarbeit - There's surely more to come!