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Perturbative study of renormalization and mixing for asymmetric staple-shaped Wilson-line operators on the lattice

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Outline

A. Introduction: TMD quasi-PDFs on the lattice, Renormalization

B. Study of operator mixing through symmetries

C. Construction of regularization-independent (RI') renormalization prescriptions

D. One-loop perturbative study in dimensional regularization (DR)

E. One-loop perturbative study in lattice regularization (LR)

F. Conclusions and future prospects

Introduction

- **LaMET formulation of TMD PDFs:** [X. Ji et al., Rev. Mod. Phys. 93, 035005 (2021)], [X. Ji et al., PLB 811, 135946 (2020)]

$$f^{\mathrm{TMD}}(x,b,\mu,\zeta) = H\left(\frac{\zeta_z}{\mu^2}\right) e^{-\ln\left(\frac{\zeta_z}{\zeta}\right)K(b,\mu)} \tilde{f}(x,b,\mu,\zeta_z)\, S_r^{\frac{1}{2}}(b,\mu) + \ldots$$

- **TMD quasi-PDFs:**

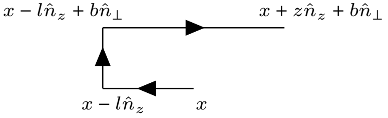
$$\tilde{f}(x,b,\mu,\zeta_z) = \lim_{l\to\infty} \int \frac{dz}{2\pi} e^{-iz\zeta_z} \frac{2P^z}{N_\Gamma} \frac{\tilde{B}_\Gamma(z,b,l,P^z,\mu)}{\sqrt{Z_E(2l+z,b,\mu)}},$$

- **Quasi-beam function:**

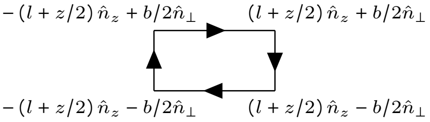
$$\tilde{B}_\Gamma(z,b,l,P^z) = \langle h(P^z)|\bar{\psi}(x+b\hat{n}_\perp + z\hat{n}_z)\Gamma \mathcal{W}_{\mathrm{staple}}(z,b,l)\psi(x)|h(P^z)\rangle,$$

- **Wilson loop:**

$$Z_E(2l+z,b) = \langle \mathcal{W}_{\mathrm{loop}}(2l+z,b)\rangle$$



$\mathcal{W}_{\mathrm{staple}}(z,b,l)$



$\mathcal{W}_{\mathrm{loop}}(2l+z,b)$

Introduction

- **Asymmetric staple-shaped Wilson-line quark bilinear operators:**

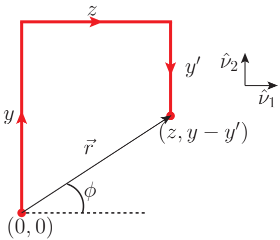
$$\mathcal{O}_\Gamma(x,z,y,y') \equiv \bar{\psi}(x)\ \Gamma\ \mathcal{W}(x,z,y,y')\ \psi(x + z\hat{\nu}_1 + (y-y')\hat{\nu}_2),$$

where

$$\mathcal{W}(x,z,y,y') \equiv \mathcal{P}\Big\{\mathcal{U}(x, y\ \hat{\nu}_2)\ \mathcal{U}(x + y\ \hat{\nu}_2, z\ \hat{\nu}_1)\ \mathcal{U}^\dagger(x + z\ \hat{\nu}_1 + (y-y')\ \hat{\nu}_2, y'\ \hat{\nu}_2)\Big\},$$

$$\mathcal{U}(r, \ell\ \hat{\mu}) \equiv \mathcal{P}\exp\Big[ig\int_0^\ell d\bar{\ell}\ A_\mu(r + \bar{\ell}\ \hat{\mu})\Big],$$

and $\Gamma = 1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}$.



- **Renormalization challenges:**
  1. Linear divergences: depend on the total length of the staple line.
  2. Cusp logarithmic divergences: depend on the number and angles of cusps.
  3. End-point and contact logarithmic divergences: depend on the number of end points and straight-line segments.
  4. Pinch-point singularity in the limit $y \to \infty$.
  5. Operator mixing: depend on the regularization (chiral vs non-chiral fermions)

Operator mixing

- **Different studies:**

1. Based on **one-loop perturbation theory**: [M. Constantinou et al., PRD 99, 074508 (2019)]

Mixing between $(\Gamma, \pm[\Gamma, \gamma_{\nu_2}]/2)$ for non-chiral fermions
Mixing sets: $(\gamma_5, \gamma_5\gamma_{\nu_2})$, $(\gamma_{\nu_1}, \sigma_{\nu_1\nu_2})$, $(\gamma_{\nu_3}, \sigma_{\nu_3\nu_2})$, $(\gamma_{\nu_4}, \sigma_{\nu_4\nu_2})$
Multiplicative renormalization: $1, \gamma_{\nu_2}, \gamma_5\gamma_{\nu_1}, \gamma_5\gamma_{\nu_3}, \gamma_5\gamma_{\nu_4}, \sigma_{\nu_3\nu_1}, \sigma_{\nu_4\nu_1}, \sigma_{\nu_4\nu_3}$

where $(\nu_1, \nu_2, \nu_3, \nu_4)$ are all different and $\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = +1$.

2. **Maximal prescription**: 16×16 mixing [P. Shanahan et al., PRD 101, 074505 (2020)]

one mixing set: $(1, \gamma_5, \gamma_{\nu_1}, \gamma_{\nu_2}, \gamma_{\nu_3}, \gamma_{\nu_4}, \gamma_5\gamma_{\nu_1}, \gamma_5\gamma_{\nu_2}, \gamma_5\gamma_{\nu_3}, \gamma_5\gamma_{\nu_4}, \sigma_{\nu_1\nu_2}, \sigma_{\nu_3\nu_1}, \sigma_{\nu_4\nu_1}, \sigma_{\nu_3\nu_2}, \sigma_{\nu_4\nu_2}, \sigma_{\nu_4\nu_3})$

3. Based on **symmetries**: [C. Alexandrou et al., arXiv:2305.11824], [Y. Ji et al., PRD 104, 094510 (2021)]

Mixing between $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2}, \pm\Gamma\gamma_{\nu_1}, \pm\Gamma\gamma_{\nu_2})$ for non-chiral fermions
Mixing sets: $(1, \sigma_{\nu_1\nu_2}, \gamma_{\nu_1}, \gamma_{\nu_2})$, $(\gamma_5, \sigma_{\nu_4\nu_3}, \gamma_5\gamma_{\nu_1}, \gamma_5\gamma_{\nu_2})$, $(\gamma_{\nu_3}, \gamma_5\gamma_{\nu_4}, \sigma_{\nu_3\nu_1}, \sigma_{\nu_3\nu_2})$, $(\gamma_{\nu_4}, \gamma_5\gamma_{\nu_3}, \sigma_{\nu_4\nu_1}, \sigma_{\nu_4\nu_2})$

Operator mixing

- **Discrete symmetries:**

1. Generalized time reversal:

$$\psi(\vec{x},x_\mu) \xrightarrow{\mathcal{T}_\mu} \gamma_\mu\gamma_5\psi(\vec{x},-x_\mu),$$
$$\bar\psi(\vec{x},x_\mu) \xrightarrow{\mathcal{T}_\mu} \bar\psi(\vec{x},-x_\mu)\gamma_5\gamma_\mu,$$
$$U_\mu(\vec{x},x_\mu) \xrightarrow{\mathcal{T}_\mu} U_\mu^\dagger(\vec{x},-x_\mu-\hat\mu),$$
$$U_{\nu\neq\mu}(\vec{x},x_\mu) \xrightarrow{\mathcal{T}_\mu} U_\nu(\vec{x},-x_\mu).$$

2. Generalized parity:

$$\psi(\vec{x},x_\mu) \xrightarrow{\mathcal{P}_\mu} \gamma_\mu\psi(-\vec{x},x_\mu),$$
$$\bar\psi(\vec{x},x_\mu) \xrightarrow{\mathcal{P}_\mu} \bar\psi(-\vec{x},x_\mu)\gamma_\mu,$$
$$U_\mu(\vec{x},x_\mu) \xrightarrow{\mathcal{P}_\mu} U_\mu(-\vec{x},x_\mu),$$
$$U_{\nu\neq\mu}(\vec{x},x_\mu) \xrightarrow{\mathcal{P}_\mu} U_\nu^\dagger(-\vec{x}-\hat\nu,x_\mu).$$

3. Charge Conjugation:

$$\psi(x) \xrightarrow{\mathcal{C}} C^{-1}\bar\psi^T(x),$$
$$\bar\psi(x) \xrightarrow{\mathcal{C}} -\psi^T(x)\ C,$$
$$U_\mu(x) \xrightarrow{\mathcal{C}} \left(U_\mu^\dagger(x)\right)^T.$$

4. Chiral transformations:

$$\psi(x) \xrightarrow{\mathcal{A}} e^{i\alpha\gamma_5}\psi(x),$$
$$\bar\psi(x) \xrightarrow{\mathcal{A}} \bar\psi(x)\ e^{i\alpha\gamma_5},$$
$$U_\mu(x) \xrightarrow{\mathcal{A}} U_\mu(x).$$

- **Action of symmetry transformations:**

$$\mathcal{O}_\Gamma(x,+z,+y,+y') \xrightarrow{\mathcal{T}_\mu} \mathcal{O}_{\gamma_5\gamma_\mu\Gamma\gamma_\mu\gamma_5}(x,(-1)^{\delta_{\mu\nu_1}}\ z,(-1)^{\delta_{\mu\nu_2}}\ y,(-1)^{\delta_{\mu\nu_2}}\ y'),$$
$$\mathcal{O}_\Gamma(x,+z,+y,+y') \xrightarrow{\mathcal{P}_\mu} \mathcal{O}_{\gamma_\mu\Gamma\gamma_\mu}(x,(-1)^{\delta_{\mu\nu_1}+1}\ z,(-1)^{\delta_{\mu\nu_2}+1}\ y,(-1)^{\delta_{\mu\nu_2}+1}\ y'),$$
$$\mathcal{O}_\Gamma(x,+z,+y,+y') \xrightarrow{\mathcal{C}} \mathcal{O}_{(C\Gamma C^{-1})^T}(x,-z,+y',+y),$$
$$\mathcal{O}_\Gamma(x,+z,+y,+y') \xrightarrow{\mathcal{A}} \mathcal{O}_\Gamma(x,+z,+y,+y'),\ \text{only for}\ \Gamma=\gamma_\mu,\gamma_5\gamma_\mu.$$

Operator mixing

- **Basis of operators:** We take 8 independent linear combinations of $(\mathcal{O}_\Gamma(x,+z,+y,+y^{'})$,$\mathcal{O}_\Gamma(x,-z,+y,+y^{'})$, $\mathcal{O}_\Gamma(x,+z,-y,-y^{'})$, $\mathcal{O}_\Gamma(x,-z,-y,-y^{'})$, $\mathcal{O}_\Gamma(x,+z,+y^{'},+y)$,$\mathcal{O}_\Gamma(x,-z,+y^{'},+y)$, $\mathcal{O}_\Gamma(x,+z,-y^{'},-y)$, $\mathcal{O}_\Gamma(x,-z,-y^{'},-y))$, which are odd/even under $\mathcal{T},\mathcal{P},\mathcal{C}$.

- **Mixing pattern for chiral fermions:** $(\Gamma,\pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2})$

  Mixing sets: $(1,\sigma_{\nu_1\nu_2})$, $(\gamma_{\nu_1},\gamma_{\nu_2})$, $(\gamma_5,\sigma_{\nu_4\nu_3})$, $(\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2})$, $(\gamma_{\nu_3},\gamma_5\gamma_{\nu_4})$, $(\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2})$, $(\gamma_{\nu_4},\gamma_5\gamma_{\nu_3})$, $(\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  In symmetric staples: $(\Gamma,\pm[\Gamma,\gamma_{\nu_1}\gamma_{\nu_2}]/2)$

  Mixing sets: $(\gamma_{\nu_1},\gamma_{\nu_2})$, $(\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2})$,$(\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2})$, $(\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  Multiplicative renormalization: $1,\gamma_5,\gamma_{\nu_3},\gamma_{\nu_4},\gamma_5\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_1\nu_2},\sigma_{\nu_4\nu_3}$

- **Mixing pattern for non-chiral fermions:** $(\Gamma,\pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\pm\Gamma\gamma_{\nu_1},\pm\Gamma\gamma_{\nu_2})$

  Mixing sets: $(1,\sigma_{\nu_1\nu_2},\gamma_{\nu_1},\gamma_{\nu_2})$, $(\gamma_5,\sigma_{\nu_4\nu_3},\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2})$, $(\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2})$, $(\gamma_{\nu_4},\gamma_5\gamma_{\nu_3},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  In symmetric staples: $(\Gamma,\pm[\Gamma,\gamma_{\nu_1}\gamma_{\nu_2}]/2,\pm[\Gamma,\gamma_{\nu_1}]/2,\pm[\Gamma,\gamma_{\nu_2}]/2)$

  Mixing sets: $(\sigma_{\nu_1\nu_2},\gamma_{\nu_1},\gamma_{\nu_2})$, $(\gamma_5,\gamma_5\gamma_{\nu_1},\gamma_5\gamma_{\nu_2})$, $(\gamma_{\nu_3},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2})$, $(\gamma_{\nu_4},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2})$

  Multiplicative renormalization: $1,\gamma_5\gamma_{\nu_3},\gamma_5\gamma_{\nu_4},\sigma_{\nu_4\nu_3}$

Renormalization prescriptions

$$\mathcal{O}^R_\Gamma(x,z,y,y') = Z^{R,X}_{\Gamma\Gamma'}\,\mathcal{O}_{\Gamma'}(x,z,y,y'), \qquad \psi^R(x) = (Z^{R,X}_\psi)^{1/2}\psi(x),$$

$$\Gamma \in \{S_1 \equiv (1, \sigma_{\nu_1\nu_2}, \gamma_{\nu_1}, \gamma_{\nu_2}), S_2 \equiv (\gamma_5, \sigma_{\nu_4\nu_3}, \gamma_5\gamma_{\nu_1}, \gamma_5\gamma_{\nu_2}),$$
$$S_3 \equiv (\gamma_{\nu_3}, \gamma_5\gamma_{\nu_4}, \sigma_{\nu_3\nu_1}, \sigma_{\nu_3\nu_2}), S_4 \equiv (\gamma_{\nu_4}, \gamma_5\gamma_{\nu_3}, \sigma_{\nu_4\nu_1}, \sigma_{\nu_4\nu_2})\}$$

- **Standard RI$'$ scheme:**

$$\boxed{\frac{1}{4N_c}(Z^{\mathrm{RI}',X}_\psi)^{-1} Z^{\mathrm{RI}',X}_{\Gamma\Gamma'}\,\mathrm{Tr}[\Lambda_{\Gamma'}(q,z,y,y')\mathcal{P}_{\Gamma''}]\Big|_{q=\bar{q}} = \delta_{\Gamma\Gamma''}},$$

where
$$\Lambda_\Gamma(q,z,y,y') = \textstyle\sum_x \langle\psi(q)|\mathcal{O}_\Gamma(x,z,y,y')|\bar{\psi}(q)\rangle_{\mathrm{amp.}},$$
$$Z^{\mathrm{RI}',X}_\psi = \frac{1}{4N_c}\,\mathrm{Tr}\Big[\langle\psi(q)\bar{\psi}(q)\rangle^{-1}\cdot\frac{i\not{q}}{q^2}\Big]\Big|_{q=\bar{q}},$$

$$\mathcal{P}^{[1]}_\Gamma = e^{-i\mathsf{q}\cdot\mathsf{r}}\,\Gamma^\dagger,$$

$$\mathcal{P}^{[2]}_\Gamma = \begin{cases} e^{-i\mathsf{q}\cdot\mathsf{r}}\left(1-\frac{\not{q}_T\ \not{q}_L}{q_T^2}\right)\Gamma^\dagger, & \Gamma\in S_1,S_2 \\ e^{-i\mathsf{q}\cdot\mathsf{r}}\left(1-\frac{(\not{q}_T-\not{q}_{\nu_3})(\not{q}_L+\not{q}_{\nu_3})}{q_T^2-q_{\nu_3}^2}\right)\Gamma^\dagger, & \Gamma\in\{\gamma_{\nu_3},\gamma_5\gamma_{\nu_3},\sigma_{\nu_3\nu_1},\sigma_{\nu_3\nu_2}\} \\ e^{-i\mathsf{q}\cdot\mathsf{r}}\left(1-\frac{(\not{q}_T-\not{q}_{\nu_4})(\not{q}_L+\not{q}_{\nu_4})}{q_T^2-q_{\nu_4}^2}\right)\Gamma^\dagger, & \Gamma\in\{\gamma_{\nu_4},\gamma_5\gamma_{\nu_4},\sigma_{\nu_4\nu_1},\sigma_{\nu_4\nu_2}\} \end{cases},$$

where $\vec{q}_L \equiv q_{\nu_1}\hat{\nu}_1 + q_{\nu_2}\hat{\nu}_2$ and $\vec{q}_T \equiv \vec{q} - \vec{q}_L = q_{\nu_3}\hat{\nu}_3 + q_{\nu_4}\hat{\nu}_4$.

Renormalization prescriptions

- **Alternative RI$'$ scheme (RI$'$-bar):** [M. Ebert et al., JHEP 03 (2020) 099], [Y. Ji et al., PRD 104, 094510 (2021)]

$$\boxed{\frac{1}{4N_c}\left(Z_\psi^{\mathrm{RI}',X}\right)^{-1}\bar{Z}_{\Gamma\Gamma'}^{\mathrm{RI}',X}\mathrm{Tr}\left[\frac{\Lambda_{\Gamma'}(q,z,y,y')}{\sqrt{\langle L(z,y+y')\rangle}}\mathcal{P}_{\Gamma''}\right]\Bigg|_{\substack{q=\bar{q},\\ z=\bar{z},\\ y-y'=\delta\bar{y}}} = \delta_{\Gamma\Gamma''}},$$

where $L(z,y+y') \equiv$
$\frac{1}{N_c}\mathrm{Tr}[\mathcal{P}\{\mathcal{U}(x,(y+y')\hat{\nu}_2)\,\mathcal{U}(x+(y+y')\hat{\nu}_2,z\hat{\nu}_1)\,\mathcal{U}^\dagger(x+z\hat{\nu}_1,(y+y')\hat{\nu}_2)\,\mathcal{U}^\dagger(x,z\hat{\nu}_1)\}]$

$*$ **Main advantage:** Set small values of $\bar{z}$ and $\delta\bar{y}$ and use $\bar{Z}_{\Gamma\Gamma'}^{\mathrm{RI}',X}$ to renormalize correlators at all values of z and $y-y'$.
$\Rightarrow$ Avoid non-perturbative effects at large distances and residual linear divergences [LPC, PRL 129, 082002 (2022)].

- **Short-distance ratio scheme (SDR):** [LPC, PRL 129, 082002 (2022)]

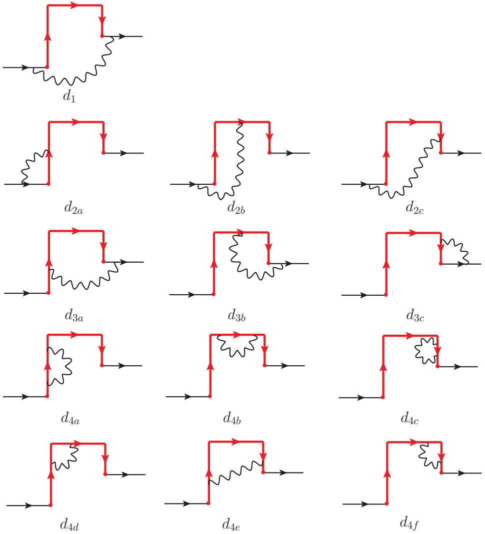
$$B_\Gamma^{\mathrm{SDR}}(z,\bar{z},\delta_y,\delta_{\bar{y}},P^z) = \frac{\lim_{y\to\infty} B_\Gamma(z,y,\delta_y,P^z,1/a)/\sqrt{\langle L(z,2y-\delta_y)\rangle}}{\lim_{y\to\infty} B_\Gamma(\bar{z},y,\delta_{\bar{y}},0,1/a)/\sqrt{\langle L(\bar{z},2y-\delta_{\bar{y}})\rangle}}$$

It is applicable when mixing does not occur or is negligible.

One-loop calculation in DR

$$\Lambda_\Gamma(q,z,y,y') = \sum_x \langle \psi(q)|\mathcal{O}_\Gamma(x,z,y,y')|\bar{\psi}(q)\rangle_{\text{amp.}}$$

- **Feynman diagrams:**



Divergent terms:

1. Endpoint divergences:

$\Lambda_\Gamma^{d2(a)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d3(c)}|_{\frac{1}{\varepsilon}} =$

$\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} \frac{1}{\varepsilon}(1-\beta),$

2. Contact divergences:

$\Lambda_\Gamma^{d4(a)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d4(b)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d4(c)}|_{\frac{1}{\varepsilon}} =$

$\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} \frac{1}{\varepsilon}(2+\beta),$

3. Cusp divergences:

$\Lambda_\Gamma^{d4(d)}|_{\frac{1}{\varepsilon}} = \Lambda_\Gamma^{d4(f)}|_{\frac{1}{\varepsilon}} =$

$\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} \frac{1}{\varepsilon}(-\beta),$

4. Pinch-point divergences:

$\Lambda_\Gamma^{d4(e)}|_y =$

$\Gamma e^{iq\cdot r} \frac{g^2 C_F}{16\pi^2} 4 \left[\frac{y}{z} \tan^{-1}\left(\frac{y}{z}\right) + \frac{y'}{z} \tan^{-1}\left(\frac{y'}{z}\right) \right]$

One-loop calculation in DR

- **Dirac structures:**

$$\begin{aligned}
\Lambda_{1}(q,z,y,y') &= \Sigma_{1,1}\ \mathbb{1} + \Sigma_{1,2}\ \sigma_{\nu_1\nu_2} + \Sigma_{1,3}\ \gamma_{\nu_1}\not{q} + \Sigma_{1,4}\ \gamma_{\nu_2}\not{q},\\
\Lambda_{\gamma_{\nu_1}}(q,z,y,y') &= \Sigma_{2,1}\ \gamma_{\nu_1} + \Sigma_{2,2}\ \gamma_{\nu_2} + \Sigma_{2,3}\ \mathbb{1}\not{q} + \Sigma_{2,4}\ \sigma_{\nu_1\nu_2}\not{q},\\
\Lambda_{\gamma_{\nu_2}}(q,z,y,y') &= \Sigma_{3,1}\ \gamma_{\nu_2} + \Sigma_{3,2}\ \gamma_{\nu_1} + \Sigma_{3,3}\ \mathbb{1}\not{q} + \Sigma_{3,4}\ \sigma_{\nu_1\nu_2}\not{q},\\
\Lambda_{\gamma_{\mu}}(q,z,y,y') &= \Sigma_{4,1}\ \gamma_{\mu} + \Sigma_{4,2}\ \varepsilon_{\nu_1\nu_2\mu\rho}\gamma_5\gamma_\rho + \Sigma_{4,3}\ q_\mu\gamma_{\nu_1} + \Sigma_{4,4}\ q_\mu\gamma_{\nu_2}\\
&+ \Sigma_{4,5}\ \sigma_{\mu\nu_1}\not{q} + \Sigma_{4,6}\ \sigma_{\mu\nu_2}\not{q} + \Sigma_{4,7}\ q_\mu\mathbb{1}\not{q},\\
\Lambda_{\sigma_{\nu_1\nu_2}}(q,z,y,y') &= \Sigma_{5,1}\ \sigma_{\nu_1\nu_2} + \Sigma_{5,2}\ \mathbb{1} + \Sigma_{5,3}\ \gamma_{\nu_1}\not{q} + \Sigma_{5,4}\ \gamma_{\nu_2}\not{q},\\
\Lambda_{\sigma_{\mu\nu_1}}(q,z,y,y') &= \Sigma_{6,1}\ \sigma_{\mu\nu_1} + \Sigma_{6,2}\ \sigma_{\mu\nu_2} + \Sigma_{6,3}\ q_\mu\mathbb{1} + \Sigma_{6,4}\ q_\mu\sigma_{\nu_1\nu_2}\\
&+ \Sigma_{6,5}\ \gamma_\mu\not{q} + \Sigma_{6,6}\ \varepsilon_{\nu_1\nu_2\mu\rho}\gamma_5\gamma_\rho\not{q} + \Sigma_{6,7}\ q_\mu\gamma_{\nu_1}\not{q} + \Sigma_{6,8}\ q_\mu\gamma_{\nu_2}\not{q},\\
\Lambda_{\sigma_{\mu\nu_2}}(q,z,y,y') &= \Sigma_{7,1}\ \sigma_{\mu\nu_2} + \Sigma_{7,2}\ \sigma_{\mu\nu_1} + \Sigma_{7,3}\ q_\mu\mathbb{1} + \Sigma_{7,4}\ q_\mu\sigma_{\nu_1\nu_2}\\
&+ \Sigma_{7,5}\ \gamma_\mu\not{q} + \Sigma_{7,6}\ \varepsilon_{\nu_1\nu_2\mu\rho}\gamma_5\gamma_\rho\not{q} + \Sigma_{7,7}\ q_\mu\gamma_{\nu_1}\not{q} + \Sigma_{7,8}\ q_\mu\gamma_{\nu_2}\not{q},
\end{aligned}$$

$$\begin{aligned}
\Lambda_{\gamma_5}(q,z,y,y') &= \gamma_5\ \Lambda_{1}(q,z,y,y'), \qquad \Lambda_{\gamma_5\gamma_{\nu_1}}(q,z,y,y') = \gamma_5\ \Lambda_{\gamma_{\nu_1}}(q,z,y,y'),\\
\Lambda_{\gamma_5\gamma_{\nu_2}}(q,z,y,y') &= \gamma_5\ \Lambda_{\gamma_{\nu_2}}(q,z,y,y'), \qquad \Lambda_{\gamma_5\gamma_{\mu}}(q,z,y,y') = \gamma_5\ \Lambda_{\gamma_{\mu}}(q,z,y,y'),\\
\Lambda_{\sigma_{\nu_4\nu_3}}(q,z,y,y') &= \gamma_5\ \Lambda_{\sigma_{\nu_1\nu_2}}(q,z,y,y'),
\end{aligned}$$

where $\mu \neq \nu_1, \nu_2$ and $\Sigma_{i,j} \equiv \Sigma_{i,j}(\bar{\mu}^2, q_{\nu_1}, q_{\nu_2}, q^2, z, y, y')$ (See our forthcoming paper).

One-loop calculation in DR

- **Expectation value of Wilson loop:** Agree with [LPC, PRL 129, 082002 (2022)]

$$\langle L(z, y+y')\rangle = 1 + \frac{g^2}{16\pi^2} C_F\ 8\Big\{2 + \frac{1}{\varepsilon} + 2\gamma_E + \frac{y+y'}{z}\tan^{-1}(\frac{y+y'}{z}) + \frac{z}{y+y'}\tan^{-1}(\frac{z}{y+y'}) + \ln(\frac{\bar{\mu}^2 z^2}{4}) - \ln(1+\frac{z^2}{(y+y')^2})\Big\} + \mathcal{O}(g^4)$$

- **Renormalization functions in $\overline{\text{MS}}$:** Agree with [Brandt - Neri - Sato, PRD24 (1981) 879], [Korchemsky - Radyushkin, NPB 283 (1987) 342]

$$Z_{\Gamma\Gamma'}^{\overline{\text{MS}},\text{DR}} = \delta_{\Gamma\Gamma'}\left\{1 - \frac{g_{\overline{\text{MS}}}^2}{16\pi^2} C_F \frac{7}{\varepsilon} + \mathcal{O}\left(g_{\overline{\text{MS}}}^4\right)\right\}, \qquad \text{for } \mathcal{O}_\Gamma$$

$$\bar{Z}_{\Gamma\Gamma'}^{\overline{\text{MS}},\text{DR}} = \delta_{\Gamma\Gamma'}\left\{1 - \frac{g_{\overline{\text{MS}}}^2}{16\pi^2} C_F \frac{3}{\varepsilon} + \mathcal{O}\left(g_{\overline{\text{MS}}}^4\right)\right\}, \qquad \text{for } \mathcal{O}_\Gamma/\sqrt{\langle L\rangle}$$

∗ Diagonal

∗ Independent of the operator and the lengths of the staple segments

One-loop calculation in DR

- **Conversion matrices between RI$'$-type and $\overline{\text{MS}}$ schemes:** (See our forthcoming paper)

$$Z^{\overline{\text{MS}},X}_{\Gamma\Gamma'} = C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\Gamma''} Z^{\text{RI}',X}_{\Gamma''\Gamma'},$$

$$C^{\overline{\text{MS}},\text{RI}'} = \begin{pmatrix} C^{\overline{\text{MS}},\text{RI}'}_{\Gamma,\Gamma} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma,\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & 0 & 0 \\ C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & 0 & 0 \\ 0 & 0 & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_1}} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_2}} \\ 0 & 0 & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_1}} & C^{\overline{\text{MS}},\text{RI}'}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_2}} \end{pmatrix} + \mathcal{O}(g^4_{\overline{\text{MS}}}).$$

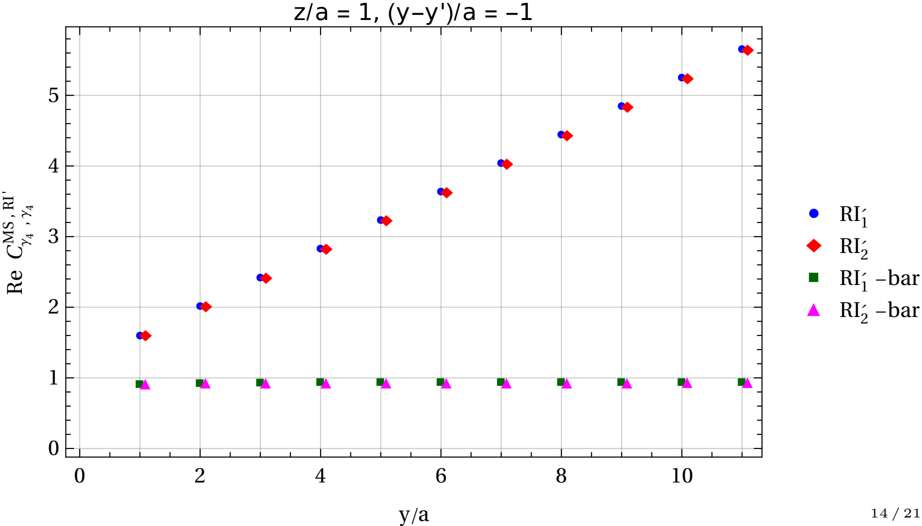
$*$ Block diagonal
$*$ Example: ($\mu \neq \nu_1, \nu_2$, F_i, G_i, $\bar{G}_i$, H_i, $\bar{H}_i$, I_i are integrals over Bessel functions)

$$C^{\overline{\text{MS}},\text{RI}'_1}_{\gamma_\mu,\gamma_\mu} = 1 + \frac{g_{\overline{\text{MS}}}^2}{16\pi^2} C_F \times$$
$$\Big\{15 - 4F_2 + \beta(-1 + 2F_1 - F_4) + s_0 + 2i(\bar{\mathsf{q}}\cdot\mathsf{r})\beta(F_1 - F_2)$$
$$-2iz\bar{q}_{\nu_1}(G_1 + \bar{G}_1 + G_2 + \bar{G}_2) + \bar{q}_\mu^2/\bar{q}^2(2\beta F_4 - 2(2F_1 - 4F_2 + F_4))$$
$$-(r^2\bar{q}_\mu^2)\beta F_3 + 2iy\bar{q}_{\nu_2}(-H_1 - H_2 + H_3 + H_4) - 4i(y - y')\bar{q}_{\nu_2}(I_1 + I_2)$$
$$+2iy'\bar{q}_{\nu_2}(\bar{H}_1 + \bar{H}_2 - \bar{H}_3 - \bar{H}_4) + 4i(\bar{\mathsf{q}}\cdot\mathsf{r})\bar{q}_\mu^2/\bar{q}^2 F_3\Big\} + \mathcal{O}(g^4_{\overline{\text{MS}}}),$$

$$s_0 = 2(6+\beta)\gamma_E + 4\left[y/z\tan^{-1}(y/z) + y'/z\tan^{-1}\left(y'/z\right) - (y-y')/z\tan^{-1}((y-y')/z)\right] + (1-\beta)\ln\left(\bar{\mu}^2/q^2\right)$$
$$+(2+\beta)\ln\left(\bar{\mu}^2r^2/4\right) + 4\ln\left(\bar{\mu}^2z^2/4\right) + 2\left[\ln\left(1+z^2/y^2\right) + \ln\left(1+z^2/y'^2\right)\right].$$

One-loop calculation in DR

- **Example: Unpolarized TMD PDF** $\Gamma = \gamma_4$, $(\nu_1, \nu_2, \nu_3, \nu_4) = (2, 3, 1, 4)$
  ETMC ensembles: $\beta = 1$, $\bar{\mu} = 2$ GeV, $a \simeq 0.09$ fm, $L/a = 24$, $T/a = 48$,
  $(a\bar{q}) = 2\pi a(n_1/L, n_2/L, n_3/L, (n_4 + 0.5)/T)$, $(n_1, n_2, n_3, n_4) = (3, 3, 3, 5)$



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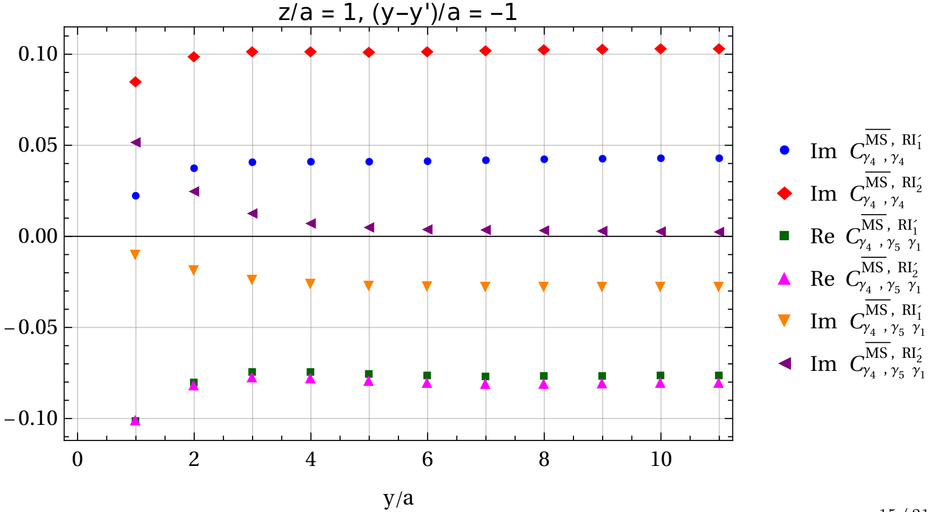

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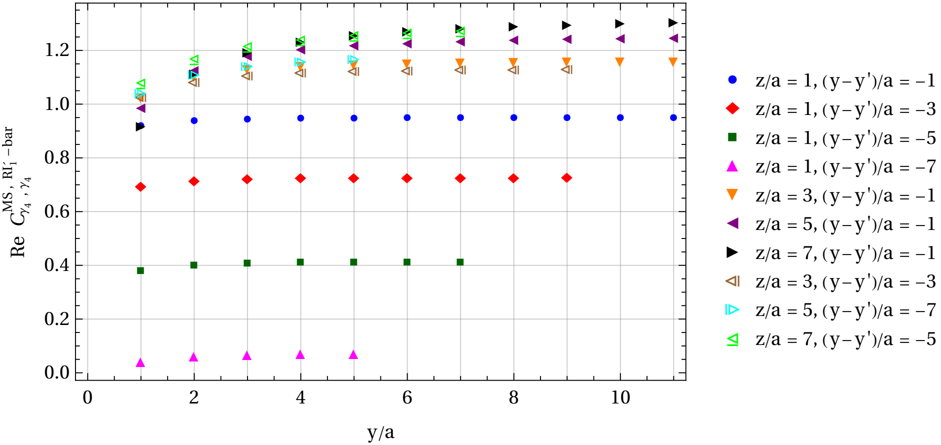

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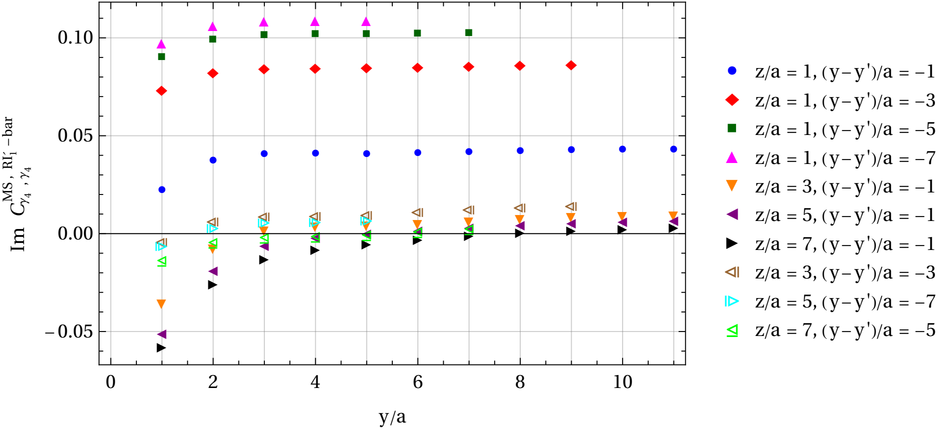

One-loop calculation on the lattice

- **Green's function of staple-shaped operator with n cusps and total length ℓ using Wilson/clover fermions and Symanzik-improved gluons:**

$$\Lambda_\Gamma^{\rm LR}(q,\ell_1,\ell_2,\dots\ell_{n+1}) = \Lambda_\Gamma^{\overline{\rm MS}}(q,\ell_1,\ell_2,\dots\ell_{n+1},\bar\mu) - \frac{g^2\,C_F}{16\,\pi^2}\,e^{i\,\mathsf{q}\cdot\mathsf{r}}\,\times$$

$$\Big\{2\,\Gamma\Big[\boldsymbol{\alpha_1} + 16\pi^2\boldsymbol{P_2}\,\beta + (1-\beta)\ln(a^2\bar\mu^2)\Big] + (\Gamma\hat{\not{\nu}}_i + \hat{\not{\nu}}_f\Gamma)(\boldsymbol{\alpha_2}\,r + \boldsymbol{\alpha_3}\,c_{SW})$$

$$+(n+1)\Gamma\Big[\boldsymbol{\alpha_4} - 16\pi^2\boldsymbol{P_2}\,\beta + (2+\beta)\ln(a^2\bar\mu^2)\Big] + \Gamma\boldsymbol{\alpha_5}\,\frac{\ell}{a},$$

$$+n\Gamma\Big[\boldsymbol{\alpha_6} + 16\pi^2\boldsymbol{P_2}\,\beta - \beta\ln(a^2\bar\mu^2)\Big]\Big\} + \mathcal{O}\left(g^4\right),$$

where $P_2 = 0.024013181119464 89(1)$, $r(c_{\rm SW})$ is the Wilson (clover) parameter, ℓ_i is the length of the $i^{\rm th}$ segment ($\sum_{i=1}^{n+1}\ell_i = \ell$), r is the vector connecting the two end points, and $\hat{\nu}_i$ ($\hat{\nu}_f$) is the direction of the Wilson line in the initial (final) end point.

* end-point divergences, mixing, contact divergences, linear divergences, cusp divergences

Gluon action	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\alpha_3}$	$\boldsymbol{\alpha_4}$
Wilson	-4.464066(5)	7.224955(7)	-4.142333(4)	-4.52575(1)
Tree-Level Symanzik	-4.341269(5)	6.377911(6)	-3.836778(4)	-3.93028(1)
Iwasaki	-4.163735(5)	4.968266(5)	-3.263819(3)	-1.90532(1)

One-loop calculation on the lattice

Gluon action	α_5	α_6	b_1	b_2
Wilson	19.95484(2)	0	-18.10303(1)	39.90968(4)
Tree-Level Symanzik	17.29374(2)	-0.809890(1)	-18.96069(3)	34.58748(3)
Iwasaki	12.97809(1)	-2.101083(2)	-16.02564(3)	25.95618(3)

- **Green's function of rectangular Wilson loop with lengths ℓ_1, ℓ_2 ($\ell_1 + \ell_2 = \ell$) using Symanzik-improved gluons:**

$$\langle L(\ell_1,\ell_2)\rangle^{\text{LR}} = \langle L(\ell_1,\ell_2,\bar{\mu})\rangle^{\overline{\text{MS}}} - \frac{g^2}{16\pi^2} C_F \left\{ \boldsymbol{b_1} + [4(2+\beta) - 4\beta] \ln(a^2\bar{\mu}^2) + \boldsymbol{b_2}\, \frac{\ell}{a} \right\} + \mathcal{O}\left(g^4\right).$$

  ∗ contact divergences, cusp divergences, linear divergences.

  ∗ $b_1 = 4(\alpha_4 + \alpha_6)$, $b_2 = 2\alpha_5$

- **Renormalization functions in $\overline{\text{MS}}$:**

$$Z^{\overline{\text{MS}},\text{LR}} = \begin{pmatrix} Z^{\overline{\text{MS}},\text{LR}}_{\Gamma,\Gamma} & 0 & 0 & Z^{\overline{\text{MS}},\text{LR}}_{\Gamma,\Gamma\gamma_{\nu_2}} \\ 0 & Z^{\overline{\text{MS}},\text{LR}}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & Z^{\overline{\text{MS}},\text{LR}}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma\gamma_{\nu_1}} & 0 \\ 0 & Z^{\overline{\text{MS}},\text{LR}}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & Z^{\overline{\text{MS}},\text{LR}}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_1}} & 0 \\ Z^{\overline{\text{MS}},\text{LR}}_{\Gamma\gamma_{\nu_2},\Gamma} & 0 & 0 & Z^{\overline{\text{MS}},\text{LR}}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_2}} \end{pmatrix} + \mathcal{O}(g^4_{\overline{\text{MS}}}).$$

One-loop calculation on the lattice

- **Renormalization functions in $\overline{\text{MS}}$:**

$$Z^{\overline{\text{MS}},\text{LR}}_{\Gamma,\Gamma} = 1 - \frac{g^2_{\overline{\text{MS}}} C_F}{16\pi^2}\Big[\boldsymbol{e^{\Gamma}(n)} - \boldsymbol{\alpha_5}\frac{\ell}{a} + \boldsymbol{e_2^{\psi}}\; c_{SW} + \boldsymbol{e_3^{\psi}}\; c_{SW}^2 - (2n+3)\log\left(a^2\bar{\mu}^2\right)\Big] + \mathcal{O}(g^4_{\overline{\text{MS}}}),$$

$$Z^{\overline{\text{MS}},\text{LR}}_{\Gamma,(\Gamma\hat{\psi}_i+\hat{\psi}_f\Gamma)} = \frac{g^2_{\overline{\text{MS}}} C_F}{16\pi^2}\Big[\boldsymbol{\alpha_2} r + \boldsymbol{\alpha_3} c_{SW}\Big] + \mathcal{O}(g^4_{\overline{\text{MS}}}),$$

$$\bar{Z}^{\overline{\text{MS}},\text{LR}}_{\Gamma,\Gamma} = 1 - \frac{g^2_{\overline{\text{MS}}} C_F}{16\pi^2}\Big[\boldsymbol{e^{\Gamma}(0)} + \boldsymbol{e_2^{\psi}}\; c_{SW} + \boldsymbol{e_3^{\psi}}\; c_{SW}^2 - 3\log\left(a^2\bar{\mu}^2\right)\Big] + \mathcal{O}(g^4_{\overline{\text{MS}}}), \qquad (n=2),$$

$$\bar{Z}^{\overline{\text{MS}},\text{LR}}_{\Gamma,(\Gamma\hat{\psi}_i+\hat{\psi}_f\Gamma)} = Z^{\overline{\text{MS}},\text{LR}}_{\Gamma,(\Gamma\hat{\psi}_i+\hat{\psi}_f\Gamma)} + \mathcal{O}(g^4_{\overline{\text{MS}}}), \qquad (n=2),$$

where $\boldsymbol{e^{\Gamma}(n)} = \Big[\boldsymbol{e_1^{\psi}} + 1 - 2\boldsymbol{\alpha_1} - (n+1)\boldsymbol{\alpha_4} - n\boldsymbol{\alpha_6}\Big]$.

Gluon action	$\boldsymbol{e_1^{\psi}}$	$\boldsymbol{e_2^{\psi}}$	$\boldsymbol{e_3^{\psi}}$
Wilson	11.8524	-2.2489	-1.3973
TL Symanzik	8.2313	-2.0154	-1.2422
Iwasaki	3.3246	-1.6010	-0.9732

- **Renormalization functions in RI$'$/RI$'$-bar:** (See our forthcoming paper)

$$Z^{\text{RI}',\text{LR}} = \begin{pmatrix} Z^{\text{RI}',\text{LR}}_{\Gamma,\Gamma} & Z^{\text{RI}',\text{LR}}_{\Gamma,\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & 0 & Z^{\text{RI}',\text{LR}}_{\Gamma,\Gamma\gamma_{\nu_2}} \\ Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma} & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_1}\gamma_{\nu_2},\Gamma\gamma_{\nu_1}} & 0 \\ 0 & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_1}\gamma_{\nu_2}} & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_1}} & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_1},\Gamma\gamma_{\nu_2}} \\ Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_2},\Gamma} & 0 & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_1}} & Z^{\text{RI}',\text{LR}}_{\Gamma\gamma_{\nu_2},\Gamma\gamma_{\nu_2}} \end{pmatrix} + \mathcal{O}(g^4_{\overline{\text{MS}}}).$$

Conclusions and future prospects

- One-loop perturbative study of asymmetric staple-shaped operators in both DR and LR.
- Provide different RI$'$-type prescriptions for renormalization (Different projectors, standard RI$'$ vs RI$'$-bar).
- Identify mixing sets through symmetries:
  chiral fermions: $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2})$
  non-chiral fermions: $(\Gamma, \pm\Gamma\gamma_{\nu_1}\gamma_{\nu_2}, \pm\Gamma\gamma_{\nu_1}, \pm\Gamma\gamma_{\nu_2})$
- Confirm (in one-loop perturbation theory) that RI$'$-bar addresses all divergences: linear, cusp, end-point, contact and pinch-point singularities
- Extract one-loop conversion functions for all 16 quark bilinear operators w.r.t. the operator mixing based on symmetries
- **Future plans:**
  - Extend our study to 2 loops (Conversion functions)
  - Calculate one-loop lattice artifacts to all orders in lattice spacing
  - Employ stout smearing in our perturbative calculations
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THANK YOU!