Running couplings in Higher derivatives theories

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D.B., J. Donoghue, G. Menezes, R. Percacci, *Physical running of couplings in quadratic gravity*, Phys.Rev.Lett. 133 (2024) 2, 021604, arXiv:2403.02397

D. B., J. F. Donoghue, G. Menezes and R. Percacci, *Renormalization and running in the 2D CP(1) model*, arXiv: 2408.13142

D. B., L. Parente and O. Zanusso, *Physical Running in Conformal Gravity and Higher Derivative Scalars*, arXiv: 2410.21475

Why higher derivative?

- Einstein general relativity as a QFT is perturbatively non renormalizable
- Higher derivative operators R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ contain a fourth derivative kinetic term for the metric

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The theory is now perturbatively renormalizable [Stelle, '77]

$$S = \int d^4x\sqrt{-g}\left[+2\Lambda - Z_N R + \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\right]$$

- Ostrogradsky instability, ghosts and breakdown of unitarity

Historical overview

RG equations of quadratic gravity:

- Julve & Tonin, ’78: perturbatively Asymptotically Free (AF) only with scalar tachyon

No Nakanishi-Lautrup ghost

- Fradkin & Tseytlin, ’82: AF without scalar tachyon

Numerical error

- Avramidi & Barvinski, ’85: again AF only with scalar tachyon

Problem:

Do these RG equations reproduce the momentum dependence of scattering amplitudes?

Running couplings from amplitudes (Gell-Mann & Low)

Consider a scattering amplitude renormalized with a momentum subtraction scheme at the mass scale

$$\mathrm{A}(g_i, m_j, p) \sim g_i + g_i^2 \log\left(\frac{p^2}{m_j^2}\right)$$

if $p \gg m_j$
Breakdown of perturbativity

Large logs are reabsorbed in $g_i(\bar{p}) = g_i + g_i^2 \log(\bar{p}^2/m_j^2)$ at a new energy scale $\bar{p} \sim p$

$$\mathrm{A}(g_i, m_j, p, \bar{p}) \sim g_i(\bar{p}) + g_i^2(\bar{p}) \log\left(\frac{p^2}{\bar{p}^2}\right)$$

The "physical'' running of g is defined by integrating an infinitesimal shift of $\bar{p}$:

$$\bar{p}\frac{d}{d\bar{p}} g(\bar{p}) = \beta^{\bar{p}}(g) \qquad g(\bar{p}') = \frac{g(\bar{p})}{1-\beta^{\bar{p}}(g)\log\left(\frac{\bar{p}'^2}{\bar{p}^2}\right)}$$

Extended perturbative regime

Running of coupling from regulators (Wilson)

Regularization and renormalization introduce an unphysical energy scale (Λ, k, μ,...)

A typical renormalized amplitude is

$$A(g_i, \mu, p) \sim g_i(\mu) + g_i^2(\mu)\, c \log\left(\frac{f(m_i, p)}{\mu^2}\right)$$

Physics independent of μ ⇒ Callan-Symanzik equations

$$\frac{d}{d\mu} A(g_i, \mu, p) = 0, \qquad \beta^{\mu}(g) \coloneqq \mu \frac{\partial}{\partial \mu} g(\mu)$$

In massless theories with only one regulator, only one dimensionless quantity p^2/μ^2

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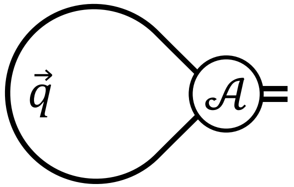
$$\beta^{\bar{p}}(g) = \beta^{\mu}_{m_i=0}(g)$$

Is this relation true in general?

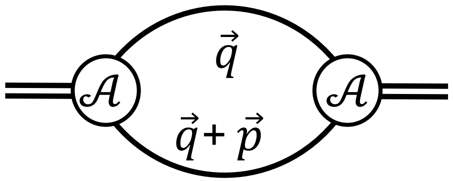
1-loop beta functions are universal in 2-derivative theories if $\bar{p}, \mu \gg m_i$

Higher derivatives theories



$$\mu^{2\epsilon}\int d^{4-2\epsilon}q\,\frac{\mathcal{A}}{m^2q^2+q^4}$$
$$\sim\frac{\mathcal{A}}{\epsilon}+\mathcal{A}\log\left(\frac{\mu^2}{m^2}\right)$$

No running with p

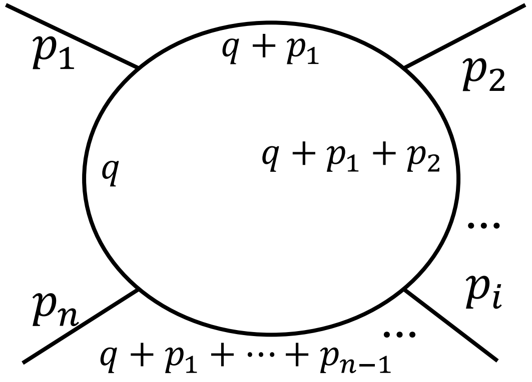


$$\mu^{2\epsilon}\int d^{4-2\epsilon}q\,\frac{\mathcal{A}^2}{(m^2q^2+q^4)(m^2(q+p)^2+(q+p)^4)}$$
$$\sim\frac{\mathcal{A}^2}{p^4}\log\left(\frac{m^2}{p^2}\right)+O\left(\frac{m}{p}\right)$$

p-running without $\frac{1}{\epsilon}$ poles if $\mathcal{A}\sim p^4$

General diagram

In higher derivatives theories there are off-shell IR divergences:



$$\sim \int d^4q \frac{N(p, q_i)}{q^4(q+p_1)^4 \times \cdots \times (q + p_1 + \cdots + p_{n-1})^4}$$

$q \ll p_i$ $\qquad$ $q \gg p_i$

$$\frac{1}{p_1^4 \times \cdots \times (p_1 + \cdots + p_{n-1})^4} \int d^4q \frac{N(p, q_i)}{q^4} + \cdots \qquad \int d^4q \frac{N(p, q_i)}{q^{4n}} + \cdots$$

If IR regulator $\neq$ UV regulator (ex. $m_i \neq 0$, IR cutoff λ)

$\beta^{\bar{p}}(g) \neq \beta^{\mu}(g)$ even when $\bar{p} \gg m_i$

[D.B., John Donoghue, Roberto Percacci, ’23]
[John Donoghue, Gabriel Menezes, ’23]

HD scalar in curved spacetime

$$S_{HDS} = \frac{1}{2}\int d^4x\sqrt{\bar{g}}\left[\phi\bar{\Box}^2\phi + \phi\bar{\nabla}_\mu\left((\xi_1\bar{R}^{\mu\nu} + \xi_2\bar{g}^{\mu\nu}\bar{R})\bar{\nabla}_\nu\phi\right) + U\phi\right]$$

$$U = \lambda_1\bar{C}^2 + \lambda_2\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \lambda_3\bar{R}^2 + \lambda_4\bar{\Box}\bar{R}$$

$$\Gamma_{1-loop}[\bar{g},\varphi] = \frac{1}{2}Tr\,\log\left(\frac{\delta^2 S}{\delta^2\phi}\right)$$

$$\Gamma_{1-loop}[\bar{g},0] = \int d^4x\sqrt{\bar{g}}\left[\bar{C}_{\mu\nu\rho\sigma}f_\lambda\left(\bar{\Box};\mu^2,m^2\right)\bar{C}^{\mu\nu\rho\sigma} + \bar{R}_{\mu\nu}f_\xi\left(\bar{\Box};\mu^2,m^2\right)\bar{R}^{\mu\nu} + O\left(R^3\right)\right]$$

when $\bar{\Box} \gg m^2$,

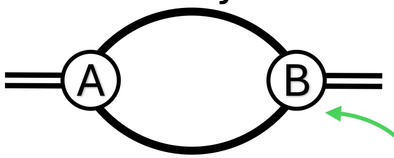
$$f_i(\Box,\mu,m) \sim b_i\log\left(\frac{\mu^2}{m^2}\right) + c_i\log\left(\frac{\Box}{m^2}\right)$$

$$\beta_i^\mu \propto -b_i, \qquad \beta_i^{\bar{p}} \propto c_i \qquad b_i = -c_i?$$

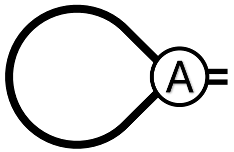
Heat kernel computation

$$\frac{\delta^2 S}{\delta^2\phi} = H = \Box^2 + V^{\rho\lambda}\nabla_\rho\nabla_\lambda + N^\mu\nabla_\mu + U$$

- Heat kernel (HK) technique permits to do manifestly covariant computations in perturbation theory. Divergencies are



$$-\frac{1}{\epsilon}\frac{1}{2(4\pi)^2}\int d^4x\, tr\left[\frac{R^2_{\mu\nu\rho\sigma}}{90} - \frac{R^2_{\mu\nu}}{90} + \frac{5R^2}{180} - U - \frac{1}{12}\left(2\,R_{\rho\lambda}V^{\rho\lambda} - R\,V^{\rho}{}_{\rho}\right) + \frac{1}{48}\left(V^{\rho}{}_{\rho}\,V^{\lambda}{}_{\lambda} + 2\,V_{\rho\lambda}\,V^{\rho\lambda}\right)\right]$$



HK introduces an unphysical IR regulator [Avramidi & Barvinski, ‘85]

Local HK expansion insensible to IR $\log(m^2/p^2)$

With Feynman diagrams

Expand around flat background ($\Lambda = 0$) [Julve & Tonin, '78]

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$$

Fourier space well defined ➡ I can use Feynman diagrams

Sensitive to IR $\log(\boldsymbol{p}^2)$!

$O(f^2)$ is enough to reconstruct R^2 terms in Γ_{1-loop}

$$H = \Delta + \mathcal{D}^{\rho\lambda\mu\nu}\partial_\rho\partial_\lambda\partial_\mu\partial_\nu + \mathcal{C}^{\rho\lambda\mu}\partial_\rho\partial_\lambda\partial_\mu + \mathcal{V}^{\rho\lambda}\partial_\rho\partial_\lambda + \mathcal{N}^\rho\partial_\rho + \mathcal{U}$$

Flat propagator

$$Tr\log(H) = Tr\log(\Delta + A)$$

$$\approx Tr\left[\log\Delta + A\frac{1}{\Delta} - \frac{1}{2}A\frac{1}{\Delta}A\frac{1}{\Delta} + \cdots\right]$$

$O(f^2)$ can be neglected in A

New IR divergent bubble diagrams

If $\Delta = \Box^2$

$$-\frac{1}{2} Tr\left[A\frac{1}{\Box^2}A\frac{1}{\Box^2}\right] = \int d^{4-2\epsilon}q\,\frac{Num}{q^4(q+p)^4}$$

vertices	numerator	$\log(p^2)$ term
$\mathcal{U}\mathcal{U}$	$-\frac{1}{2}\mathcal{U}_{AB}\mathcal{U}^{BA}$	$-\frac{\mathcal{U}^{AB}\mathcal{U}_{BA}}{16\pi^2 p^4}$
$\mathcal{U}\mathcal{N}$	$-\frac{i}{2}\mathcal{U}_{AB}\mathcal{N}^{\mu BA}q_\mu$	$i\frac{\mathcal{U}^{AB}\mathcal{N}^{\mu}_{BA}p_\mu}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{V}$	$\frac{1}{2}\mathcal{U}_{AB}\mathcal{V}^{\mu\nu BA}q_\mu q_\nu$	$\frac{\mathcal{U}^{AB}\mathcal{V}^{\mu\nu}_{BA}p_\mu p_\nu}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{C}$	$\frac{i}{2}\mathcal{U}_{AB}\mathcal{C}^{\mu\nu\rho BA}q_\mu q_\nu q_\rho$	$-i\frac{\mathcal{U}^{AB}\mathcal{C}^{\mu\nu\rho}_{BA}p_\mu p_\nu p_\rho}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{D}$	$-\frac{1}{2}\mathcal{U}_{AB}\mathcal{D}^{\mu\nu\rho\sigma BA}q_\mu q_\nu q_\rho q_\sigma$	$-\frac{1}{32\pi^2}\left(\frac{\mathcal{U}^{AB}\mathcal{D}^{\mu\nu\rho\sigma}_{BA}p_\mu p_\nu p_\rho p_\sigma}{p^4} - \frac{\mathcal{U}^{AB}\mathcal{D}^{\mu}{}_{\mu}{}^{\nu}{}_{\nu BA}}{8}\right)$

Apparently nonlocal contributions!

Localization

$$\mathcal{U} \sim \partial\partial\partial\partial f, \mathcal{N} \sim \partial\partial\partial f, \mathcal{V} \sim \partial\partial f, \mathcal{C} \sim \partial f, \mathcal{D} \sim f$$

$$\frac{\mathcal{U}\mathcal{A}p^{4-\dim[\mathcal{A}]}}{p^4} \sim \frac{p^4 f\, p^4 f}{p^4}$$

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“nonlocal logs” become local $+O(f^3)$ nonlocalities

$$\frac{\mathcal{U}\mathcal{A}p^{4-\dim[\mathcal{A}]}}{p^4} \sim p^2 f\, p^2 f$$

All f^2 terms can be rearranged in C^2 and R^2 up to topological terms

Comparing beta functions

$\boldsymbol{\mu}$-running

$$\beta_\lambda = -\frac{\lambda^2}{(4\pi)^2}\left[\frac{1}{30} + \frac{\xi_1}{24}(\xi_1 - 4) - 2\lambda_1 - \lambda_2\right]$$

$$\beta_\xi = -\frac{\xi^2}{(4\pi)^2}\left[\frac{1}{18} + \frac{\xi_1}{18} + \frac{5\xi_1^2}{72} + \frac{\xi_2}{3} + \frac{\xi_1\xi_2}{3} + \xi_2^2 - \frac{2\lambda_2}{3} - 2\lambda_3\right]$$

Physical running

$$\beta_\lambda = -\frac{\lambda^2}{(4\pi)^2}\left[\frac{1}{30} + \frac{\xi_1}{24}(\xi_1 - 4)\right]$$

$$\beta_\xi = -\frac{\xi^2}{(4\pi)^2}\left[\frac{1}{18} + \frac{\xi_1}{18} + \frac{5\xi_1^2}{72} + \frac{\xi_2}{3} + \frac{\xi_1\xi_2}{3} + \xi_2^2 - 2\lambda_4^2\right]$$

Discrepancy given by λ_i

Universality with $\lambda_i = 0$ $\Rightarrow$ $U = 0$ $\Rightarrow$ Shift symmetry

Conformal theory: $\lambda_i = 0, \xi_1 = 2, \xi_2 = \frac{3}{2}$

Conformal anomaly $< T > = -\frac{1}{15(4\pi)^2} C^2$ is universal!

Quadratic gravity

$$S_g = \int d^4x\sqrt{-g}\left[+2\Lambda - Z_N R + \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\right]$$

$$Z_N = \frac{m_p^2}{16\pi}$$

The Euler density $E = R_{\mu\nu\rho\sigma}^2 - 4\mathrm{R}_{\mu\nu}^2 + \mathrm{R}^2$ is topological in d=4

particle	spin	Mass^2
graviton	2	0
ghost	2	$\lambda\, m_p^2/2$
scalar	0	$-\xi\, m_p^2/12$

Background field method

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$S = S_g + S_{GF} + S_{FP}$$

$$S = \bar{S}_g + \int d^4x\sqrt{-\bar{g}}\frac{1}{2}h\frac{\delta^2 S_g}{\delta h^2}\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} h + \frac{1}{2a}\int d^4x\sqrt{-\bar{g}}F^{\mu}(h)Y_{\mu\nu}F^{\nu}(h) + \bar{c}^{\mu}\Delta_{GH}{}_{\mu\nu}c^{\nu} + O\left(h^3\right)$$

$$h\frac{\delta^2(S_g+S_{GF})}{\delta h^2}\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} h = hHh$$

One-loop effective action:

$$\Gamma_{1-loop} = \bar{S}_g + \frac{1}{2}Tr(\log H) - \frac{1}{2}Tr(\log Y) - Tr(\log \Delta_{GH})$$

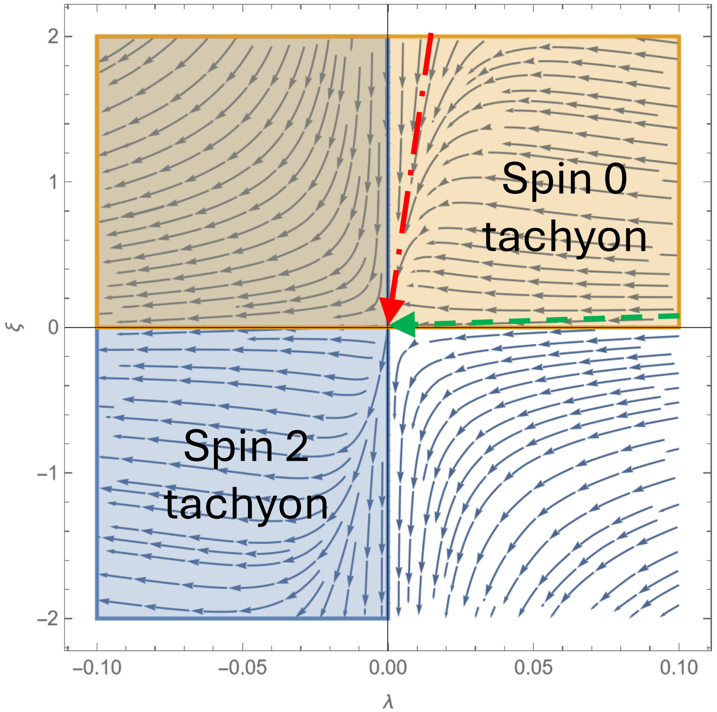
Beta functions of quadratic gravity

$$\omega = -\frac{3\lambda}{\xi}$$

μ-running

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\omega = -\frac{1}{(4\pi)^2}\frac{25 + 1098\omega + 200\omega^2}{60}\lambda$$



$$\lambda^* = 0$$

$$\omega_1^* \approx -5.5$$

$$\omega_2^* \approx -0.023$$

Spin 0 tachyon

Physical running

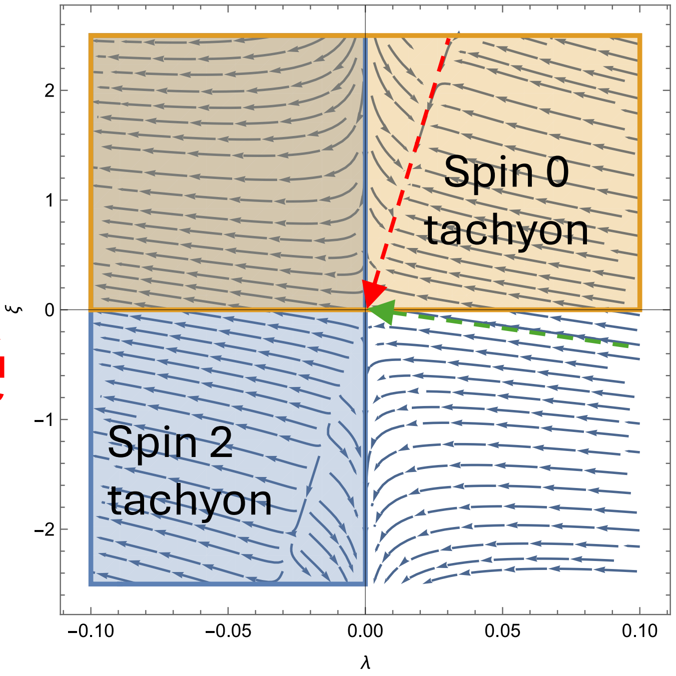
$$\beta_\lambda = -\frac{\lambda^2(539\omega + 20)}{240\pi^2}\omega$$

$$\beta_\omega = \lambda\frac{1400\omega^2 - 1138\omega - 45}{480\pi^2}$$

$$\lambda^* = 0$$

$$\omega_1^* \approx 0.85$$

$$\omega_2^* \approx -0.038$$



Conformal Gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\lambda}C^2 - \frac{1}{\rho}E\right]$$

Topological terms do not contribute to scattering amplitudes

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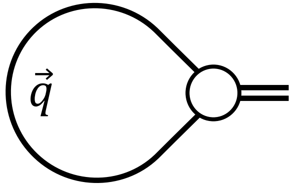
$\beta^{\bar{p}}(\rho)$ not defined

$$\beta_\lambda^{HK} = -\frac{1}{(4\pi)^2}\frac{199}{15}\,\lambda^2 \qquad \beta_\lambda^{\bar{p}} = -\frac{1}{(4\pi)^2}\frac{93}{5}\lambda^2$$

No qualitative differences

Conformal anomaly not well defined with dynamical metric

In $\boldsymbol{d} = 2$



$\sim \int d^2q \frac{1}{q^2}$ log divergent

Potentially the same problem!

$\boldsymbol{CP}(1)$ (or $\boldsymbol{O}(3)$) NLSM

$$S = \frac{1}{2g}\int d^2x \frac{\partial_\mu \phi^a \partial^\mu \phi^a}{\left[1 + \frac{1}{4}\phi^a\phi^a\right]^2} \quad a = 1,2$$

Usually β_g computed from tadpoles [Shifman ‘12,...]

unphysical?

$2 \to 2$ amplitude

The theory is integrable: all amplitudes writable in terms of $2 \to 2$ processes

$$\phi_1 + \phi_1 \to \phi_2 + \phi_2$$

is IR safe at one-loop

$$\mathcal{M} = g^2(\mu)s - \frac{g^4 s}{8\pi}\left[\log\left(-\frac{t}{\mu^2}\right) + \log\left(-\frac{u}{\mu^2}\right)\right] - \frac{g^4}{8\pi}(t-u)\log\left(\frac{t}{u}\right)$$

Logs of the artificial IR cutoff m from tadpole diagrams are cancelled by IR divergent bubble diagrams and replaced with $\log(p)$

$$\beta^{\bar{p}}(g) = \beta^{\mu}(g) = -\frac{g^3}{4\pi}$$

Conclusions

- Renormalization and running are not equivalent concepts
- In higher derivatives theories UV-IR mixing in beta fuctions
- The heat kernel misses IR running
- Taking in account also IR contributions, there exists a unique AF trajectory without tachyons in quadratic gravity
- In $d = 2$ $CP(1)$ NLSM universality recovered thanks to unitarity (Kinoshita–Lee–Nauenberg theorem)

Future perspectives

- **Beta functions of nonlocal partner of R and Λ**

Beta functions of $m_p^2 R \frac{1}{\Box} R$ and $m_p^4 R \frac{1}{\Box^2} R$

We perturbed around flat spacetime, extension to $\Lambda \neq 0$?

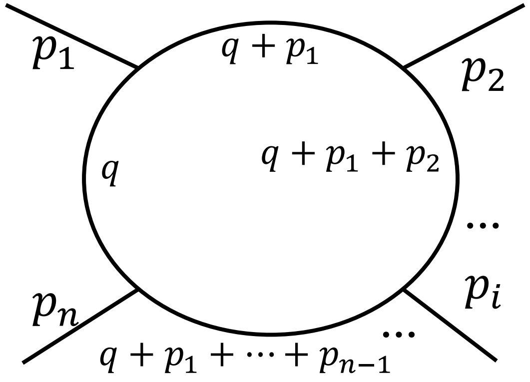
- **The same result can be reproduced with non-local HK expansion?**

- **The effective action is off-shell and gauge-fixing dependent: compare with running couplings extracted from scattering amplitudes in quadratic gravity**

Thank you!

2 derivatives theories

In general $\log(p^2/\mu^2)$ can arrive only from the UV region of loop integrals



$$\sim \int d^4q \, \frac{N(p, q_i)}{q^2(q+p_1)^2 \times \cdots \times (q+p_1+\cdots+p_{n-1})^2}$$

$q \ll p_i$

$$\frac{1}{p_1^2 \times \cdots \times (p_1+\cdots+p_{n-1})^2} \int d^4q \, \frac{N(p, q_i)}{q^2} + \cdots$$

$q \gg p_i$

$$\int d^4q \, \frac{N(p, q_i)}{q^{2n}} + \cdots$$

The UV regions are equal in massive and massless theories, up to $O(m_i/p)$

$$\beta^{\bar{p}}(g) = \beta^{\mu}_{m_i=0}(g) = \beta^{\mu}_{m_i \neq 0}(g) \qquad \text{when } \bar{p} \gg m_i$$