

A Fuzzy Electroweak Model: Exorcism, UV/IR Mixing & Naturalness

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Based on [arXiv:2307.11741](https://arxiv.org/abs/2307.11741) & [arXiv:2311.08311](https://arxiv.org/abs/2311.08311)



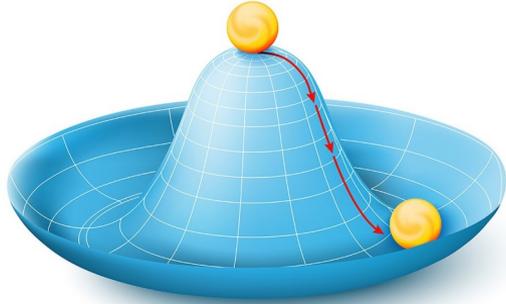
Summary:

- 1. Why Fuzziness?
- 2. Fuzzy Tachyon Condensation
- 3. Conclusion & Outlook

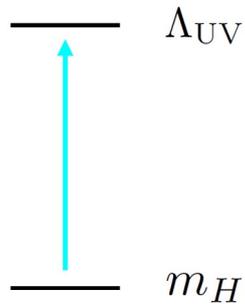
1. Why Fuzziness? → Beating Electroweak Naturalness

Higgs Mechanism

EWSB → Light Scalar:

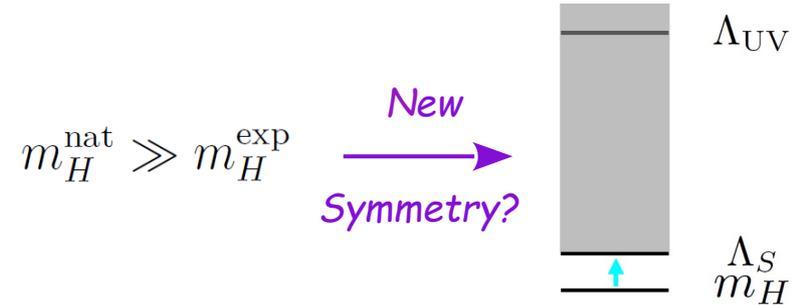


Wilsonian EFT → EW Naturalness:



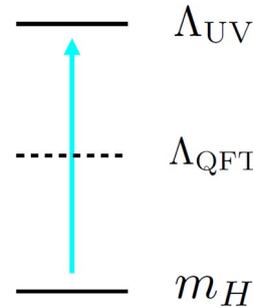
$$(m_H^{\text{nat}})^2 \sim \frac{y_t^2}{16\pi^2} \Lambda_{UV}^2$$

Gauge Hierarchy: $\Lambda_{EW} \sim 100 \text{ GeV} \ll \Lambda_P \sim 10^{18} \text{ GeV}$



→ Bunch of new particles @ $\Lambda_S \sim \text{TeV Scale}$ (LHC)

→ Little Hierarchy Problem!



Radical: Breakdown of (local) QFT
(e.g. Quantum Gravity, UV/IR mixing)

1. Why Fuzziness? → Exorcizing Higher-Derivative QFT's

∂^4 -Scalar QFT → Polynomial in ∂ → Local

$$S = \int d^4x \left[\frac{1}{2} \phi(\square - \alpha \square^2) \phi - \frac{1}{2} m^2 \phi^2 \right]$$

Propagator: UV $\sim 1/k^4$ → Better UV behavior!

$$m = 0 \rightarrow G(k^2) = -\frac{1}{k^2(1 + \alpha k^2)} = -\frac{1}{k^2} + \frac{\alpha}{1 + \alpha k^2}$$

Poles:

$$k^2 = 0$$

&

$$k^2 = -1/\alpha < 0$$



Particle ✓



Ghost ✗



Stability $(-i\epsilon)$
OR
Unitarity $(+i\epsilon)$

Woodard, [arXiv:1506.02210](https://arxiv.org/abs/1506.02210)

Platania, [arXiv:2206.04072](https://arxiv.org/abs/2206.04072)

Kubo, Kugo, [arXiv:2308.09006](https://arxiv.org/abs/2308.09006)

Prototype → ∂^∞ -Scalar QFT → Weakly Nonlocal!

$$S = \int d^4x \left[\frac{1}{2} \phi(x) \gamma(\square) \phi(x) - V(\phi) \right], \quad \gamma(\square) = \sum_{n=0}^{\infty} c_n \square^n$$

Ghost-free form factor → UV finiteness!

$$\gamma(\square) = (\square - m^2) e^{-l_*^2 \square} \rightarrow G(k^2) = -\frac{e^{-l_*^2 k^2}}{k^2 + m^2}$$

Scalar Naturalness: $\lambda \phi^4$

$$\delta m^2 = \frac{\lambda}{32\pi^2} \Lambda^2, \quad \Lambda = 1/l_*$$

Stabilization of EWSB?

TeV Fuzziness! → LHC?

Pole:

$$k^2 = -m^2$$



Particle ✓



Biswas, Okada, [arXiv:1407.3331](https://arxiv.org/abs/1407.3331)

Buoninfante, Lambiase, Mazumdar, [arXiv:1805.03559](https://arxiv.org/abs/1805.03559)

Summary:

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2. Fuzzy Tachyon Condensation → The Ghosts Strikes Back!

Tachyon Condensation $\phi^4 \rightarrow \mu^2 > 0$

→ Z_2 spontaneous symmetry breaking

$$\mathcal{L} = \underbrace{\frac{1}{2} \phi (\square + \mu^2) e^{-l_*^2 \square} \phi}_{\text{nonlocal mass}} - \underbrace{\frac{\lambda}{4!} \phi^4}_{\text{local mass}}$$

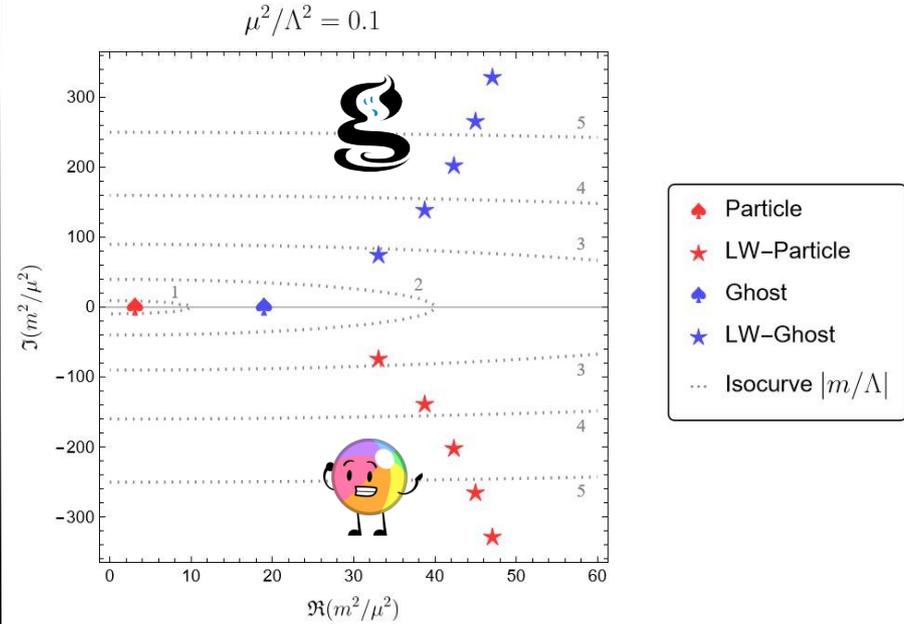
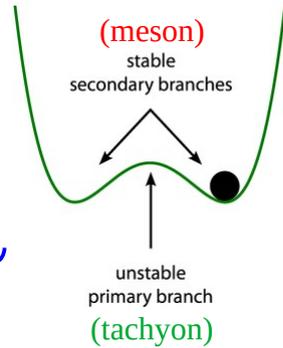
$$\phi(x) = v + \sigma(x)$$

nonlocal mass local mass

no ghost-free factorization!

$$G(k^2) = -\frac{e^{-l_*^2 k^2}}{k^2 + m^2} \quad \times$$

∞ tower
of ghost-like excitations!



→ Problem also for Higgs mechanism!
(gauge sector haunted by ghost-like tower)

Hashi, Isono, Noumi, Shiu, Soler, [arXiv:1805.02676](https://arxiv.org/abs/1805.02676)

Nortier, [arXiv:2307.11741](https://arxiv.org/abs/2307.11741)

2. Fuzzy Tachyon Condensation → Star-Product: A New Hope!

$\Phi(x)$ = field in a representation r of the gauge group

$\vartheta(z)$ = **entire function** on complex plane

$$\forall z \in \mathbb{C}, e^{\vartheta_r(z)} = \sum_{n=0}^{+\infty} c_r^{(n)} z^n, \quad c_r^{(n)} \in \mathbb{R}$$

Non-covariant star-product (asterisk) of fields:

$$\begin{aligned} \Phi^\dagger(x) \star_r \Phi(x) &= \Phi^\dagger(x) e^{\vartheta_r(\overleftarrow{\partial}_\mu \eta_r^{\mu\nu} \overrightarrow{\partial}_\nu)} \Phi(x), \quad \eta_r^{\mu\nu} = \ell_r^2 \eta^{\mu\nu}, \\ &= \Phi^\dagger(x) \cdot \Phi(x) + c_r^{(1)} \ell_r^2 \partial_\mu \Phi^\dagger(x) \cdot \partial^\mu \Phi(x) \\ &\quad + c_r^{(2)} \ell_r^4 \partial_\mu \partial_\nu \Phi^\dagger(x) \cdot \partial^\mu \partial^\nu \Phi(x) + \mathcal{O}(\ell_r^6) \end{aligned}$$

→ **Linear & commutative** but **non-associative** in general!

Important property: $v \star \Phi(x) = v \cdot \Phi(x)$

Nortier, [arXiv:2307.11741](https://arxiv.org/abs/2307.11741)

Chattopadhyay, Nortier, [arXiv:2311.08311](https://arxiv.org/abs/2311.08311)

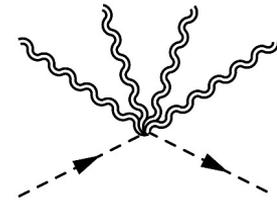
Covariant star-product of fields:

$$\begin{aligned} \Phi^\dagger(x) \star_r \Phi(x) &= \Phi^\dagger(x) e^{\vartheta_r(\overleftarrow{\mathcal{D}}_\mu \eta_r^{\mu\nu} \overrightarrow{\mathcal{D}}_\nu)} \Phi(x), \\ &= \Phi^\dagger(x) \cdot \Phi(x) + c_r^{(1)} \ell_r^2 [\mathcal{D}_\mu \Phi(x)]^\dagger \cdot \mathcal{D}^\mu \Phi(x) \\ &\quad + c_r^{(2)} \ell_r^4 [\mathcal{D}_\mu \mathcal{D}_\nu \Phi(x)]^\dagger \cdot \mathcal{D}^\mu \mathcal{D}^\nu \Phi(x) + \mathcal{O}(\ell_r^6), \\ &= \underbrace{\Phi^\dagger(x) \star_r \Phi(x)}_{\text{non-covariant star-product}} + \underbrace{\mathcal{O}(g \ell_r^2 |\Phi|^2 A)}_{\text{covariant dressing = gauge cloud}} \end{aligned}$$

non-covariant
star-product

covariant dressing
= gauge cloud

$$v \star v = v^2 + \mathcal{O}(A^2)$$



2. Fuzzy Tachyon Condensation → Higgs Mechanism

Goal → Propagators ~
→ **Ghost-free!**

$$G_h(k) = -\frac{e^{-\vartheta_h\left(\frac{k^2}{\lambda_h^2}\right)}}{k^2 + m_h^2}$$


$$G_A(k) = -\frac{e^{-\vartheta_0\left(\frac{k^2}{\lambda_0^2}\right)}}{k^2 + m_A^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_A^2} \right)$$

$$\mathcal{L} = -\frac{1}{4} \mathcal{A}_{\mu\nu} *_0 \mathcal{A}^{\mu\nu} - (\mathcal{D}_\mu H^*) \star_h (\mathcal{D}^\mu H) + \mu^2 |H|^2 - \lambda |H|^2 *_0 |H|^2$$

drop
gauge cloud:
★ ↦ *



unitary gauge:

$$H(x) = \frac{1}{\sqrt{2}} [v + h(x)]$$

$$\mathcal{L}_U \supset -\frac{1}{4} \mathcal{A}_{\mu\nu} *_0 \mathcal{A}^{\mu\nu} - \partial_\mu H *_h \partial^\mu H - g^2 (H \cdot A_\mu) *_h (H \cdot A^\mu) + \mu^2 |H|^2 - \lambda H^2 *_0 H^2$$

Ghost-free factorization

$$\Rightarrow *_0 \equiv *_h \Rightarrow \ell_0 \equiv \ell_h \ \& \ \vartheta_0(z) \equiv \vartheta_h(z)$$

$$\begin{aligned} \mathcal{L}_U \supset & -\frac{1}{4} \mathcal{A}_{\mu\nu} *_0 \mathcal{A}^{\mu\nu} - \frac{g^2 v^2}{2} A_\mu *_0 A^\mu \\ & - \frac{1}{2} \partial_\mu h *_0 \partial^\mu h - \lambda v^2 h *_0 h \\ & - \lambda v h *_0 h^2 - \frac{\lambda}{4} h^2 *_0 h^2 \end{aligned}$$

Nortier, [arXiv:2307.11741](https://arxiv.org/abs/2307.11741)

Chattopadhyay, Nortier, [arXiv:2311.08311](https://arxiv.org/abs/2311.08311)

Generalization to **Fuzzy Standard Model (FSM)**:

$$-\frac{1}{2} \text{tr} [\mathcal{G}_{\mu\nu} \star_c \mathcal{G}^{\mu\nu}] - \frac{1}{2} \text{tr} [\mathcal{W}_{\mu\nu} \star_w \mathcal{W}^{\mu\nu}] - \frac{1}{4} \mathcal{B}_{\mu\nu} \star_w \mathcal{B}^{\mu\nu}$$

Yukawa couplings (N generations):

$$-\lambda_d^{ij} (\bar{Q}_L^i \cdot H) \star_q d_R^j - \lambda_u^{ij} (\bar{Q}_L^i \cdot \bar{H}) \star_q u_R^j - \lambda_e^{ij} (\bar{L}_L^i \cdot H) \star_\ell e_R^j + \text{H.c.}$$

Linearity of star-product → **Same flavor structure as SM!**

2. Fuzzy Tachyon Condensation → UV/IR Mixing & Classification

Pure Higgs Sector Toy Model: $2H \rightarrow 2H$

Hard scattering limit: $\mathcal{U}(z) = -z$

$s \rightarrow +\infty, \quad t \rightarrow -\infty, \quad s/t \text{ fixed}$

$$A(s, t) \sim -\frac{16\lambda^2 v^2 e^{\ell_w^2 s}}{s}$$

→ **Unitarity violation?** $E \gg$ nonlocal scale

Field redefinition + Form factor expansion:

$$\alpha \ell_w^2 H^\dagger H \cdot \partial_\mu H^\dagger \partial^\mu H + \mathcal{O}(\ell_w^4)$$

→ **Classicalizing operator!** (Vainshtein screening)

→ Classicalons production:

$$\text{Vainshtein radius: } R_V \sim \sqrt{s} \ell_w^2$$

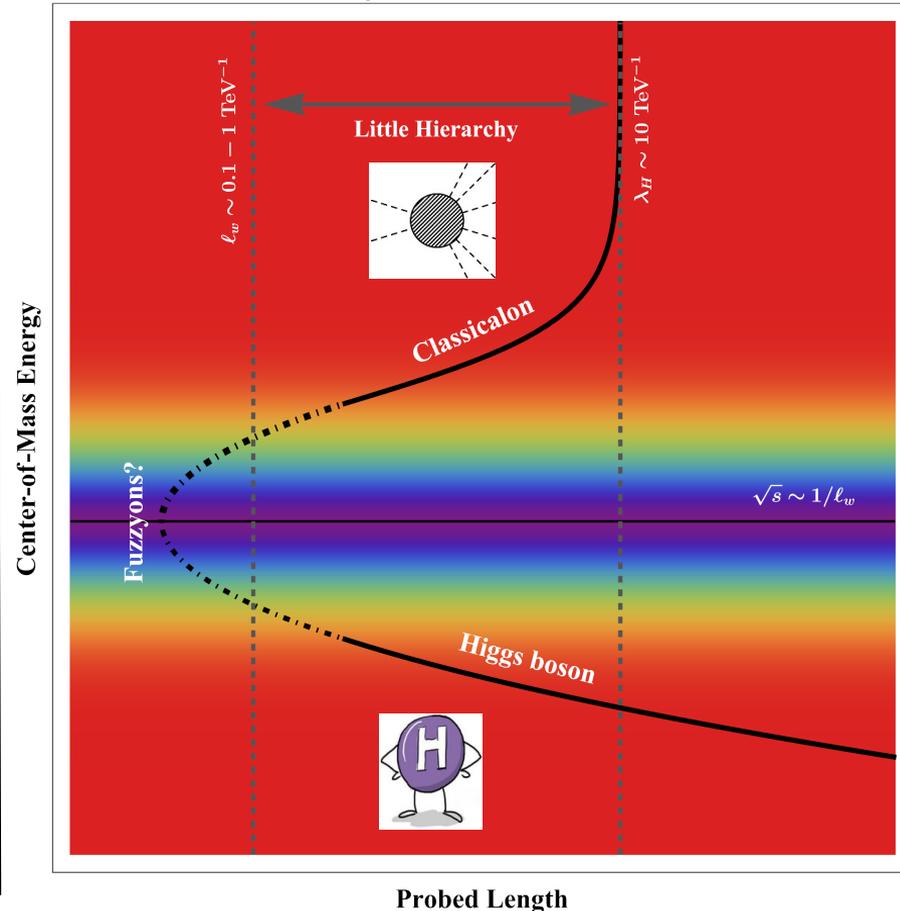
$$2H \rightarrow N \times H, \quad N \gg 1$$

Dvali, Giudice, Gomez, Kehagias, [arXiv:1010.1415](https://arxiv.org/abs/1010.1415)

Grojean, Gupta, [arXiv:1110.5317](https://arxiv.org/abs/1110.5317)

Chattopadhyay, Nortier, [arXiv:2311.08311](https://arxiv.org/abs/2311.08311)

UV/IR Mixing: Classification via Fuzziness



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3. Conclusion & Outlook

Recap:

Fuzzy Interactions → [Naturalness](#) & [Little Hierarchy!](#)

Ghost-Free Condition → Issues with Tachyon Condensation! → [New Star-Product](#) & [FSM](#)

Other attempts:

Hashi, Isono, Noumi, Shiu, Soler, [arXiv:1805.02676](#)

Modesto, [arXiv:2103.05536](#)

Outlook:

- Study of fuzzy gauge & Yukawa couplings
- Better understanding of “fuzzy regime”, quantum corrections, etc
- TeV scale phenomenology? → [Classicalons](#) & [Fuzzyons?](#)
- Include gravity
- ...

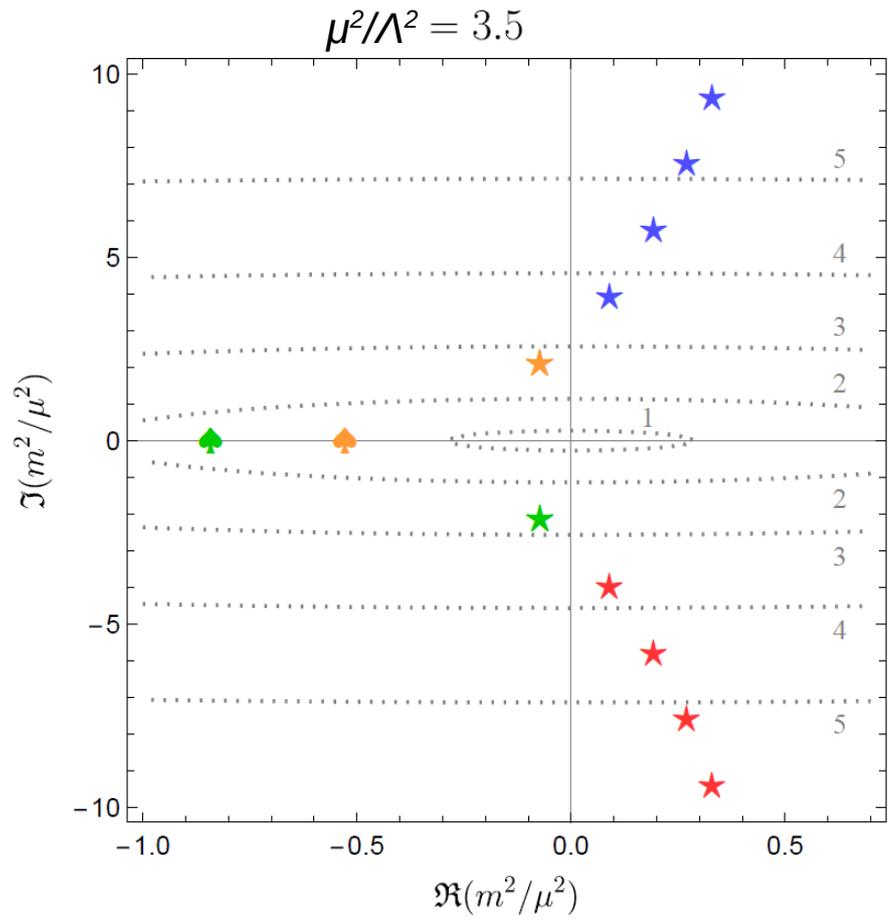
Fuzziness → Another “Flashy” BSM Scenario?



“Barbenheimer” Meme

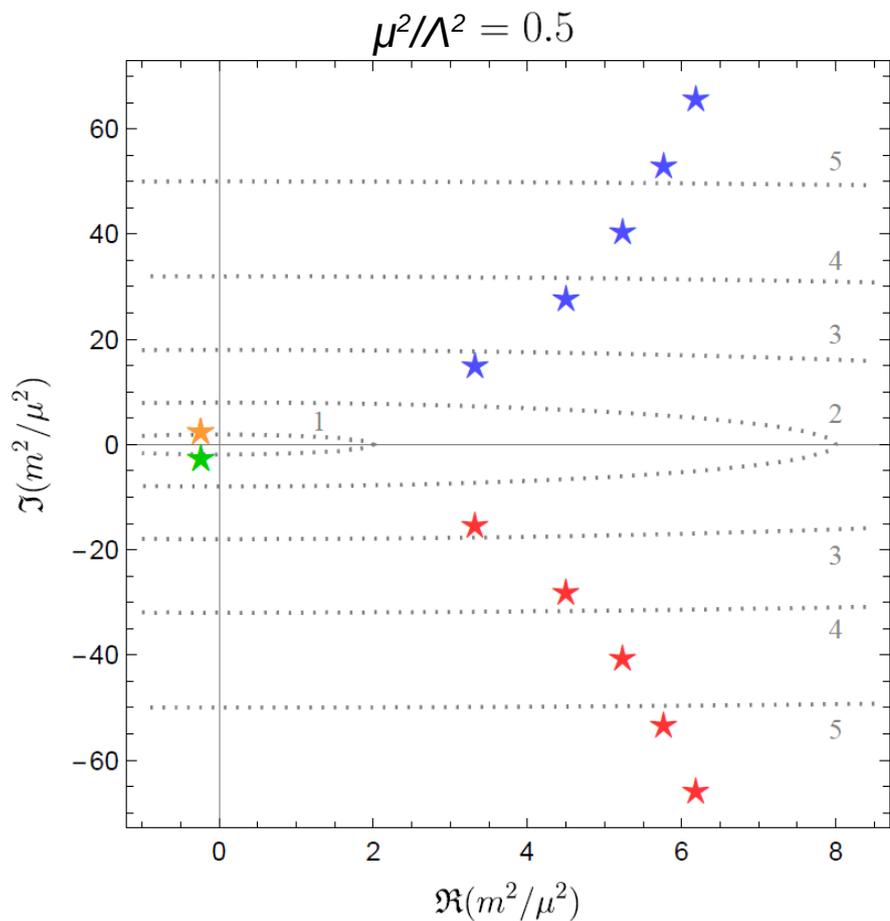
Thank You for your Attention!

Appendix → Inverted Hierarchy



- ♠ Canonical Particle
- ★ LW-Particle
- ♣ Tachyon
- ★ LW-Tachyon
- ♠ Ghost
- ★ LW-Ghost
- ♣ Tachyon-Ghost
- ★ LW-Tachyon-Ghost
- ... Isocurve $|m/\Lambda|$

Appendix → Ahierarchy



- ♠ Canonical Particle
- ★ LW-Particle
- ♣ Tachyon
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- ♠ Ghost
- ★ LW-Ghost
- ♠ Tachyon-Ghost
- ★ LW-Tachyon-Ghost
- ... Isocurve $|m/\Lambda|$

Appendix → No Ghosts in an EFT (Weinberg's Footnote, arXiv:0804.4291)

¹This is equivalent to what is generally done in deriving Feynman rules in effective flat-space quantum field theories. Consider for instance the very simple effective Lagrangian

$$\mathcal{L} = -\frac{1}{2}[\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2 + M^{-2}(\Box\varphi)^2] + J\varphi$$

where $M \gg m$ is some very large mass, and J is a c-number external current. We can easily find the connected part Γ of the vacuum persistence amplitude:

$$\Gamma = i \int d^4k \frac{|J(k)|^2}{k^2 + m^2 + k^4/M^2} .$$

If we took this result seriously, then we would conclude that in addition to the usual particle with mass $m + O(m^3/M^2)$, the theory contains an unphysical one particle state with mass $M + O(m^2/M)$. But if we regard \mathcal{L} as just the first two terms in a power series in $1/M^2$, then we must treat the term $M^{-2}(\Box\varphi)^2$ as a first-order perturbation, so that the vacuum persistence amplitude is

$$\Gamma = i \int d^4k |J(k)|^2 \left[\frac{1}{k^2 + m^2} - \frac{k^4}{M^2(k^2 + m^2)^2} + \dots \right] ,$$

and the only pole is at $k^2 = -m^2$. This is just the same result for Γ that we would find if we were to eliminate the second time derivatives in the $O(M^{-2})$ term in \mathcal{L} by using the field equation derived from the leading term in the Lagrangian

$$\Box\varphi = m^2\varphi - J .$$

In this case the effective Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}[\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2 + m^4M^{-2}\varphi^2] + (1 + m^2/M^2)J\varphi - J^2/2M^2 .$$

Taking into account all J -dependent terms, it is straightforward to see that with this Lagrangian we get the same vacuum persistence amplitude as found above for the the original Lagrangian, when $M^{-2}(\Box\varphi)^2$ is treated as a first-order perturbation.

Appendix → String Field Theory

String Field $\Psi \equiv$ Infinite Tower of Spins [Witten, NPB 268 (1986) 253-294]:

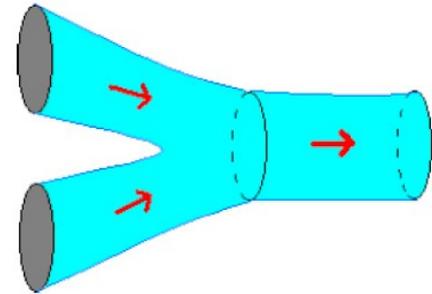
$$|\Psi\rangle = [\phi(x) + A_\mu(x)\alpha_{-1}^\mu + B_{\mu\nu}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu + \dots] c_1|0\rangle.$$

Open String Field Theory ($M_s = 1$), 0-level Truncated Action ($- + + \dots +$):

$$S = \frac{1}{g_s^2} \int d^d x \left[\frac{1}{2} \phi \square \phi - \frac{e^{3r_*}}{3} \tilde{\phi}^3 \right], \quad \tilde{\phi}(x) = e^{r_* \square} \phi(x), \quad r_* = \log \left(\frac{3^{3/2}}{4} \right) \simeq 0.2616.$$

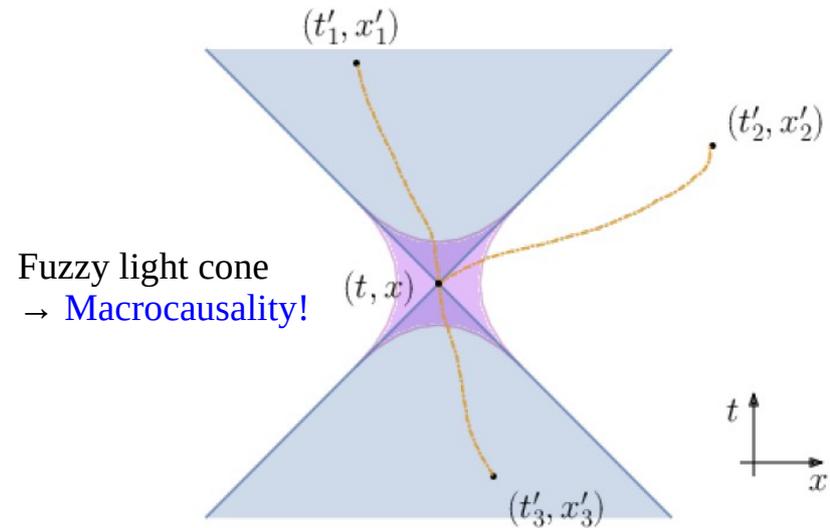
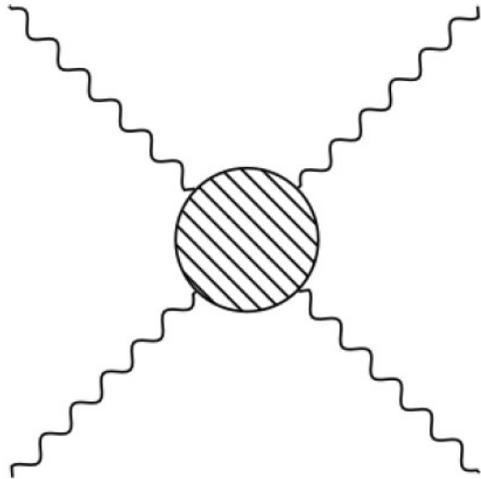
$\tilde{\phi}(x) \equiv$ Smeared Field via Nonlocality from ∞ -Derivative Operator (Nonlocal Length Scale η):

$$\begin{aligned} e^{\eta^2 \partial_x^2} \delta(x) &= \sqrt{\frac{1}{4\pi\eta^2}} e^{-\frac{x^2}{4\eta^2}}, \\ &= \delta(x) + \sum_{n=1}^{N-1} \frac{\eta^{2n}}{n!} \delta^{(n)}(x) + \mathcal{O}(\eta^{2N}). \end{aligned}$$



Appendix → Macrocausality

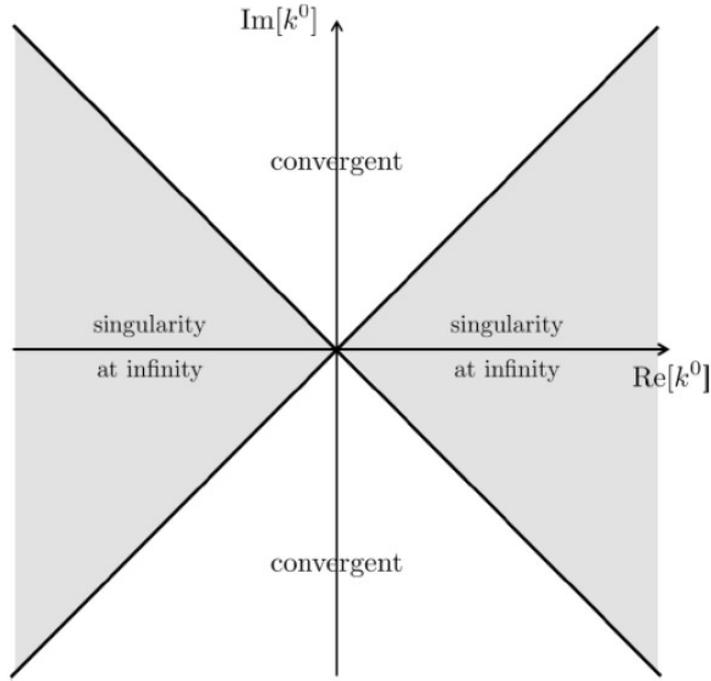
- Initial Value Problem: ∞ -Derivatives \nRightarrow ∞ Number of Initial Data N_0 .
 $\Rightarrow N_0 = 2 \times$ Number of Poles [Barnaby, Kamran, JHEP 02 (2008) 008].
- No Superluminal Propagation [Erbin, Firat, Zwiebach, JHEP 01 (2022) 167].
- Weak Nonlocality \Rightarrow Smeared Vertices \Rightarrow Minimal Uncertainty in Time Resolution!
[Carone, PRD 95 (2017) 4, 045009] & [Giaccari, Modesto (2018), arXiv:1803.08748]:



Appendix → Analyticity & Unitarity (1/2)

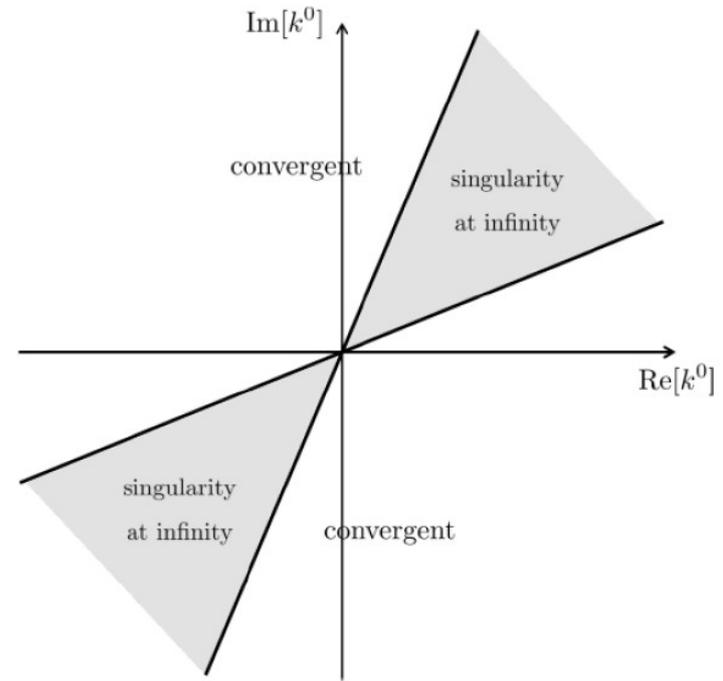
Some Form Factors Blow Up for $E \gg \Lambda_\phi \Rightarrow$ Perturbative Unitarity Lost!

[Koshelev, Tokareva, PRD 104 (2021) 2, 025016]



(a)

(a) Stringy Form Factor $e^{-\square/\Lambda_\phi^2}$.

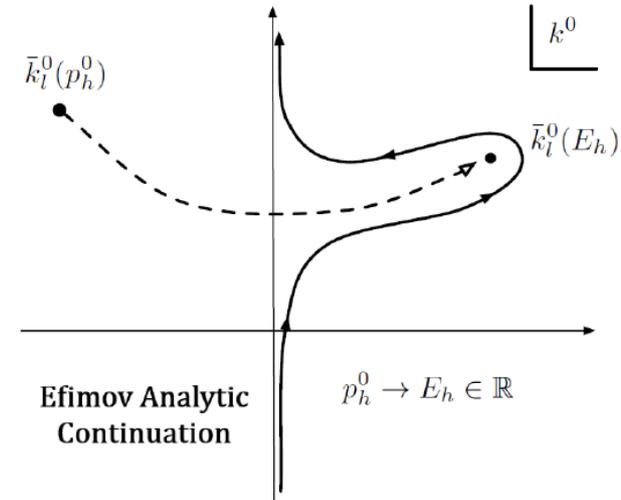
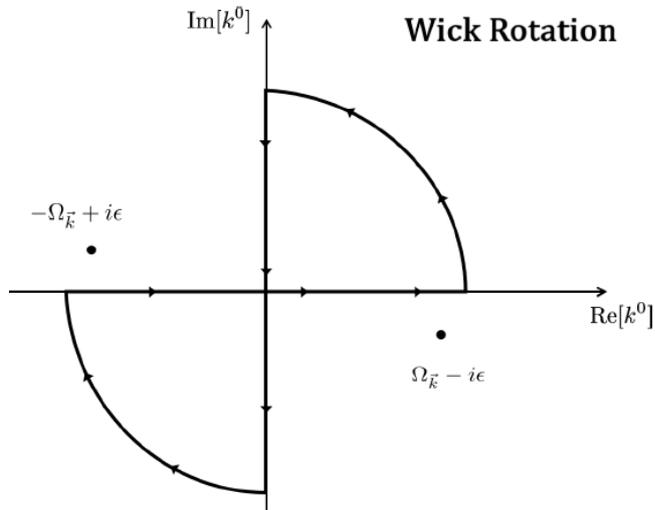


(b)

(b) Form Factor $e^{\square^2/\Lambda_\phi^4}$.

Appendix → Analyticity & Unitarity (2/2)

- Essential Singularity at Complex Infinity \Rightarrow Wick Rotation is Forbidden!
- Kallen–Lehmann Spectral Representation \nRightarrow Unitarity Violation!
[Calcagni, Rachwal (2022), arXiv:2210.04914]
- Euclidean \rightarrow Minkowskian Signature by Efimov Analytic Continuation \Rightarrow Cutkosky Rules:
 - Contour Prescription: [Efimov, Sov.J.Nucl.Phys. 4 (1967) 2, 309-315].
 - SFT: [Pius, Sen, JHEP 10 (2016) 024] & [de Lacroix, Erbin, Sen, JHEP 05 (2019) 139].
 - Review Nonlocal Scalars: [Buoninfante, PRD 106 (2022) 12, 126028].



Appendix → Kuz'min-Tomboulis Form Factors (1/2)

Fuzziness + Gauge invariance (local) → **Competition**: propagators vs vertices

$$\mathcal{L}_{YM} = -\frac{1}{2} \text{tr} \left[F_{\mu\nu} e^{H(-\ell_*^2 \mathcal{D}^2)} F^{\mu\nu} \right] \quad (\text{Yang-Mills})$$

$$\mathcal{L}_{GR} = -\frac{2}{\kappa_D^2} \sqrt{-g} \left[R - G_{\mu\nu} \frac{e^{H(-\ell_*^2 \square)} - 1}{\square} R^{\mu\nu} \right] \quad (\text{Gravity})$$

Power Counting Theorem → **Asymptotically Polynomial**: $p(z)$

$$e^{H(z)} = e^{\frac{1}{2} [\Gamma(0, p(z)^2) + \gamma_E + \log(p(z)^2)]}$$

$$= \underbrace{e^{\frac{\gamma_E}{2}} \sqrt{p(z)^2}}_{\text{UV Locality}} \left\{ 1 + \underbrace{\left[\frac{e^{-p(z)^2}}{2p(z)^2} \left(1 + O\left(\frac{1}{p(z)^2}\right) \right) + O\left(e^{-2p(z)^2}\right) \right]}_{\text{IR Nonlocal Dressing}} \right\}$$

→ Interpolates btw “Normal QFT’s” & “Higher-Derivative QFT’s”

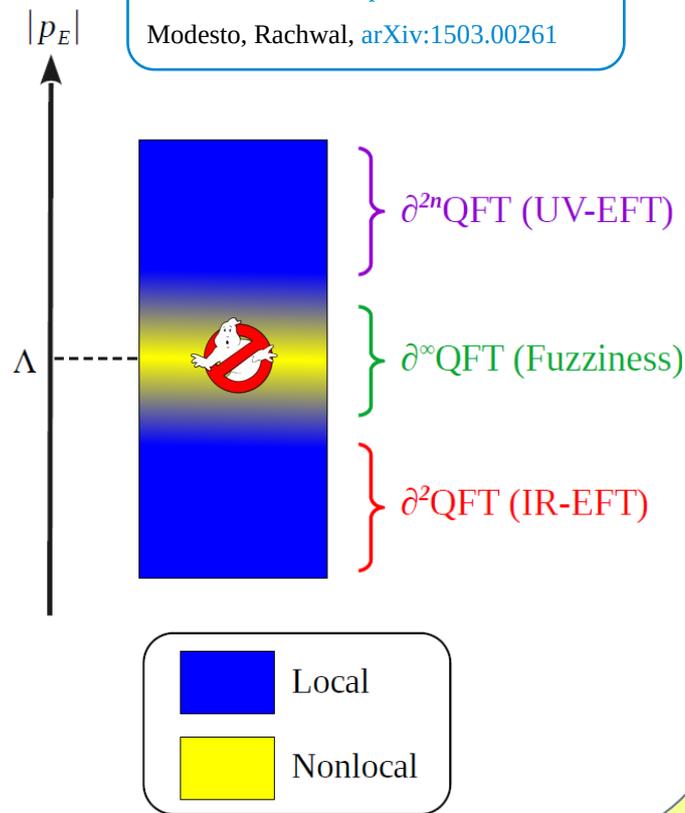
Even D → **Super-renormalizability!** (at least 1 loop UV-divergences)

Odd D → **UV finiteness!**

Kuz'min, *Sov.J.Nucl.Phys.* 50 (1989) 1011

Tomboulis, [arXiv:hep-th/9702146](https://arxiv.org/abs/hep-th/9702146)

Modesto, Rachwal, [arXiv:1503.00261](https://arxiv.org/abs/1503.00261)



Appendix → Kuz'min-Tomboulis Form Factors (2/2)

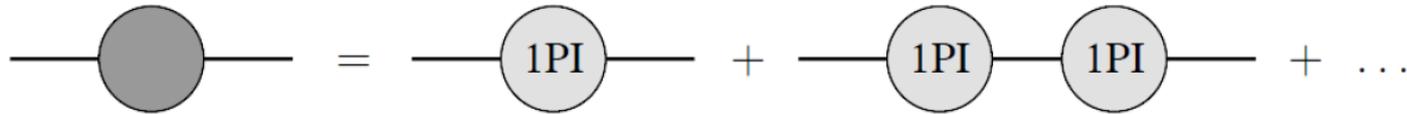
- Renormalized ∞ -Derivative Yang-Mills Lagrangian [Tomboulis (1997), arXiv:hep-th/9702146]:

$$\mathcal{L}_{YM} = - \underbrace{\frac{1}{2g_{YM}^2(\eta)} \text{Tr} [\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}]}_{\text{Local YM (Renormalized)}} - \frac{\alpha}{2} \text{Tr} \left[\mathcal{F}_{\mu\nu} \left(e^{\vartheta(\eta^2 D_\mu^2)} - 1 \right) \mathcal{F}^{\mu\nu} \right] + \text{G.F. terms} + \text{counterterms}$$

Higher-Deriv. (Not Renormalized)

Ghost-Free Renormalization Scheme: $\alpha g_{YM}^2(\eta) = 1$ at Scale $Q_0 \sim 1/\eta$.

- No Ghosts at Perturbative Level as in SFT!
[Pius, Sen, JHEP 10 (2016) 024] & [de Lacroix, Erbin, Sen, JHEP 05 (2019) 139].
- Dressed Propagator \Rightarrow Nonperturbative Resummation of 1PI Diagrams:
 - Shapiro: Infinite Tower of Complex Conjugate Poles [Shapiro, PLB 744 (2015) 67-73].
 - Modesto: Shapiro Ghosts Outside Radius of Convergence of Dressed Propagator \Rightarrow Not an Issue! \Rightarrow No Proof of Ghost-Freedom at Nonperturbative Level (Open Issue).

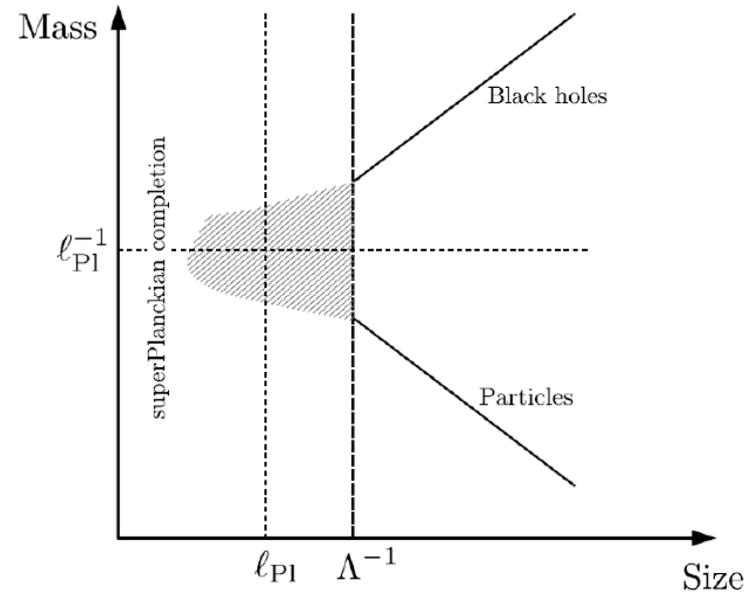
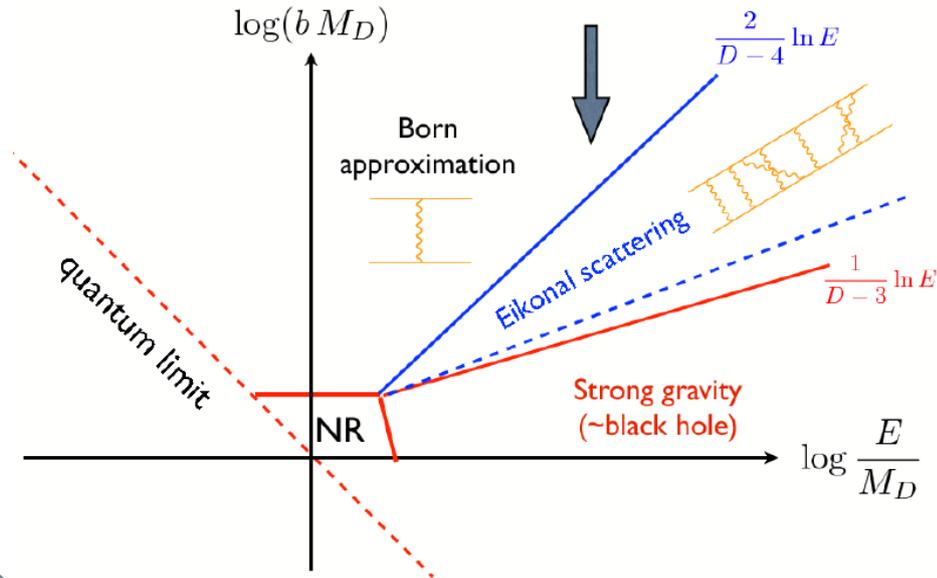


Appendix → Black Holes vs Fuzzstar (1/2)

- $\sqrt{s} \gg \Lambda_P$ & $b \lesssim r_{sch} = 2G\sqrt{s} \Rightarrow$ Black Hole Production (Mass $m = \sqrt{s}$):
[Dvali, Gomez, Isermann, Lust, Stieberger, NPB 893 (2015)]

$$G + G \longrightarrow BH \longrightarrow N \gg 1.$$

- Multiparticle Scattering \Rightarrow Collective Transmutation of $\Lambda_{nloc} \Rightarrow \eta_{eff} = N^\beta \eta$ with $\beta > 0$.
[Buoninfante, Ghoshal, Lambiase, Mazumdar, PRD 99 (2019) 4, 044032].



Appendix → Black Holes vs Fuzzstar (2/2)

- String-Inspired Nonlocal Gravity [Biswas, Gerwick, Koivisto, Mazumdar, PRL 108 (2012) 031101]:

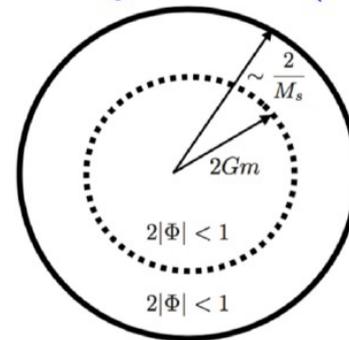
$$S_{GR} = \frac{1}{2k^2} \int d^4x \sqrt{g} \left[\mathcal{R} + \mathcal{G}_{\mu\nu} \frac{e^{-\square/M_s^2} - 1}{\square/M_s^2} \mathcal{R}^{\mu\nu} \right], \quad M_s \mapsto \frac{M_s}{\sqrt{N}}.$$

- BH \mapsto Fuzzstar (No Singularity & Horizon) \sim Fuzzball in String Theory:
 - [Koshelev, Mazumdar, PRD 96 (2017) 8, 084069];
 - [Buoninfante, Koshelev, Lambiase, Mazumdar, JCAP 09 (2018) 034];
 - [Buoninfante, Koshelev, Lambiase, Marto, Mazumdar, JCAP 06 (2018) 014];
 - Echoes in Gravitational Waves? [Buoninfante, Mazumdar, Peng, PRD 100 (2019) 10, 104059].
- Regular Gravitational Potential:

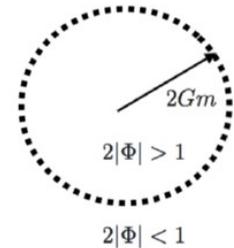
$$\Phi(r) = -\frac{Gm}{r} \operatorname{erf}\left(\frac{M_s r}{2}\right),$$

$$\xrightarrow{r \gg 2/M_s} -\frac{Gm}{r},$$

$$\xrightarrow{r \ll 2/M_s} \frac{Gm M_s}{\sqrt{\pi}}.$$



(a) BGKM



(b) Einstein's GR