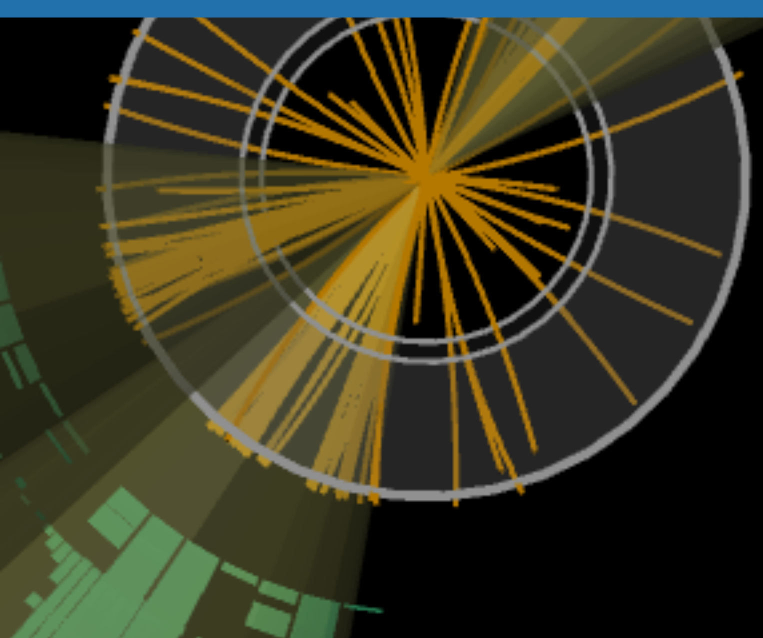
Perturbative techniques

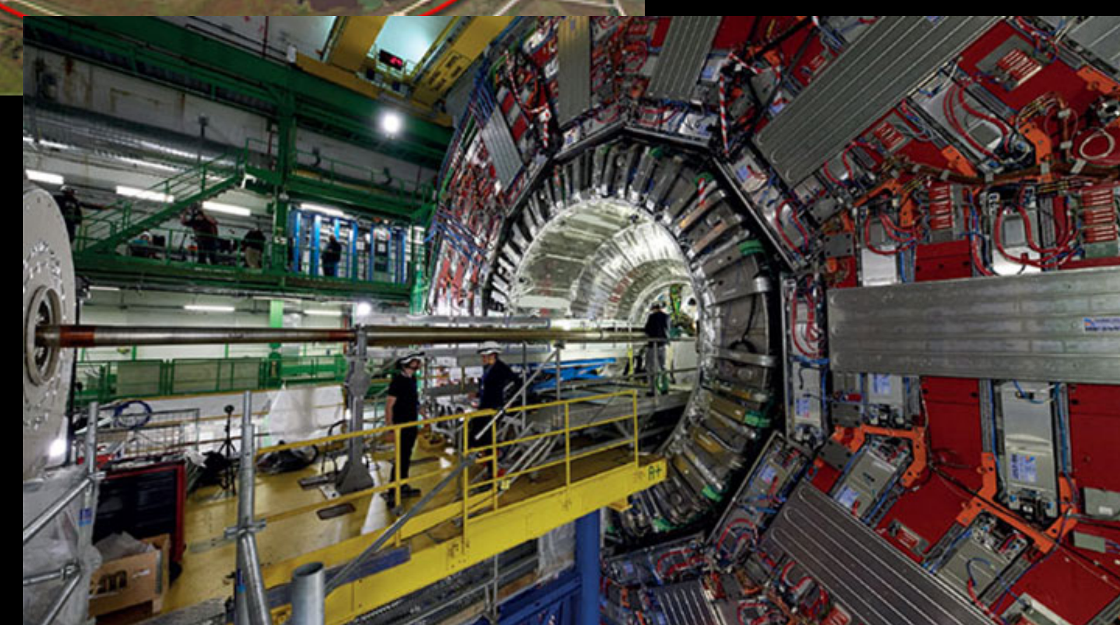
For precision collider physics and cosmology

50 Years of QCD



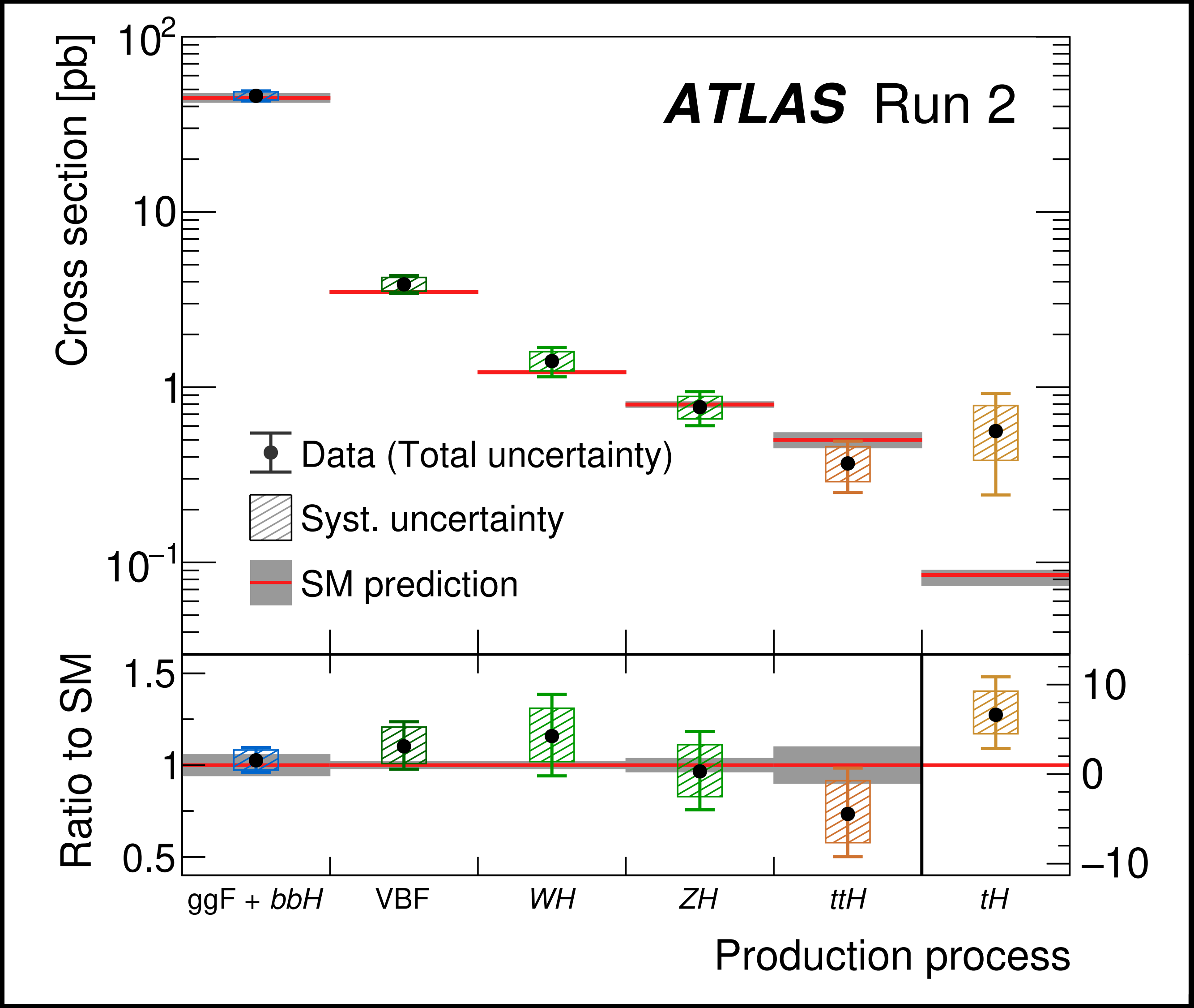
Babis Anastasiou, ETH Zurich

A celebration of QCD and exploring “Beyond QCD”

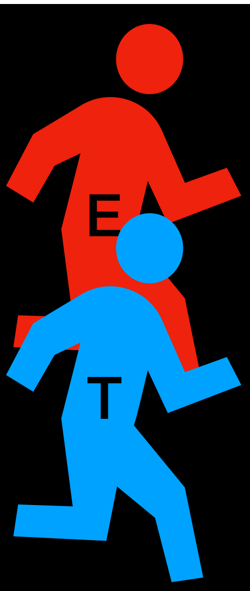


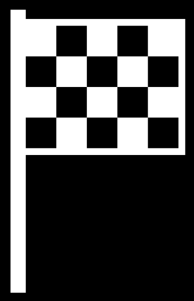
Testing the Higgs sector

LHC Measurements



- High precision measurements of a wide spectrum of observables.
- Precise comparisons with theory.
- Superb test of the Standard Model and powerful constraints on its extensions.
- A testament to the great understanding of Quantum Chromodynamics.

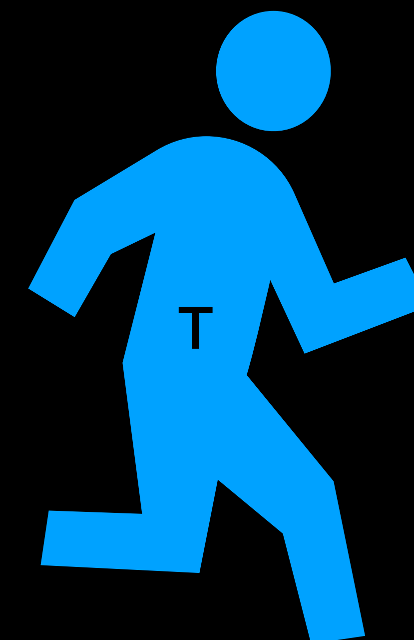




Why can we predict precisely?

FOUNDATIONS

- Asymptotic freedom
- Infrared safety
- Factorization Theorems



Superb measurements

- Structure of hadrons
- Couplings and masses

Advances in mathematics and computation for perturbation theory

"Forced" to high orders

NLO DRELL-YAN

The corrections to both these cross sections coming from radiative corrections to the lowest-order annihilation diagram are found to be large at present values of Q_2 and S when the cross section is expressed in terms of parton densities derived from lepton production, for all Drell-Yan processes of practical interest.

Altarelli, Ellis, Martinelli [Nucl. Phys. B157 (1979) 461-497]

NLO HIGGS

We have computed the $\mathcal{O}\left(\alpha_s^3\right)$ contributions to Higgs boson production at hadron colliders in the infinite top-quark mass limit. These corrections typically increase the lowest-order prediction by about a factor of 1.5 to 2. However, the results are sensitive to the choice of renormalization scale and to the choice of structure functions. It does seem clear, though, that the radiative corrections *increase* the cross-section.

Dawson [Nucl. Phys. B359 (1991) 283-300]

NNLO DRELL-YAN

In the above we see a good agreement between the theoretical prediction and the experimental result. In particular we neeed the order α_s corrections to explain the UA2 result.

Van Neerven [J. Mod. Phys. A10 (1995) 2921-2940]

NNLO HIGGS

In conclusion, we have computed the full NNLO corrections to inclusive Higgs boson production at hadron colliders. We find reasonable perturbative convergence and reduced scale dependence.

Harlander and Kilgore [Phys. Rev. Lett. 88 (2002) 201801]

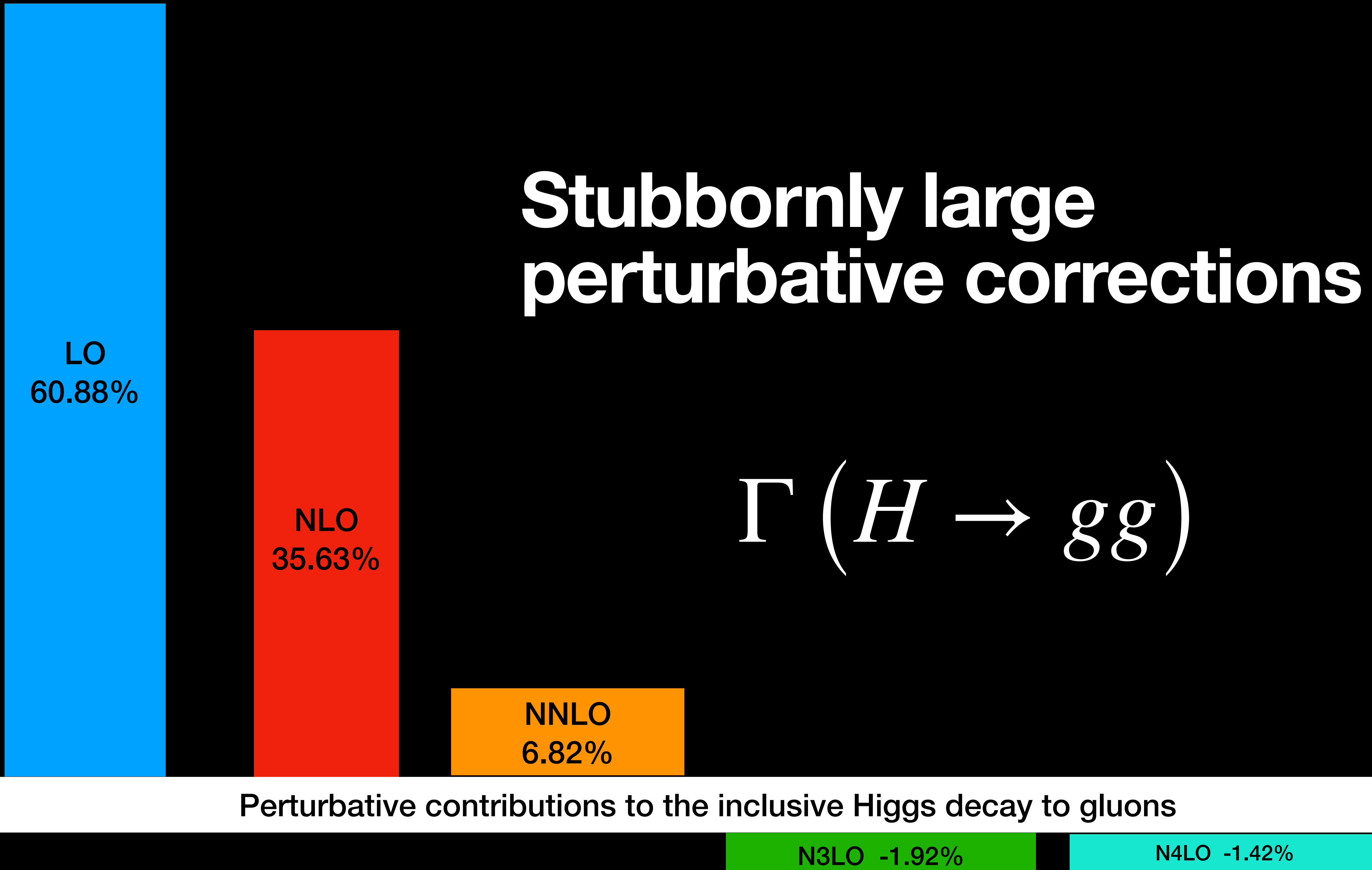
N3LO DRELL-YAN

In line with the case of Higgs production, we find that the hadronic cross section receives corrections at the percent level, and the residual dependence on the perturbative scales is reduced. However, unlike in the Higgs case, we observe that the uncertainty band derived from scale variation is no longer contained in the band of the previous order.

Duhr, Dulat, Mistlberger [Phys. Rev. Lett. 125 (2020) 172001]

Stubbornly large perturbative corrections

$$\Gamma\left(H \to gg\right)$$

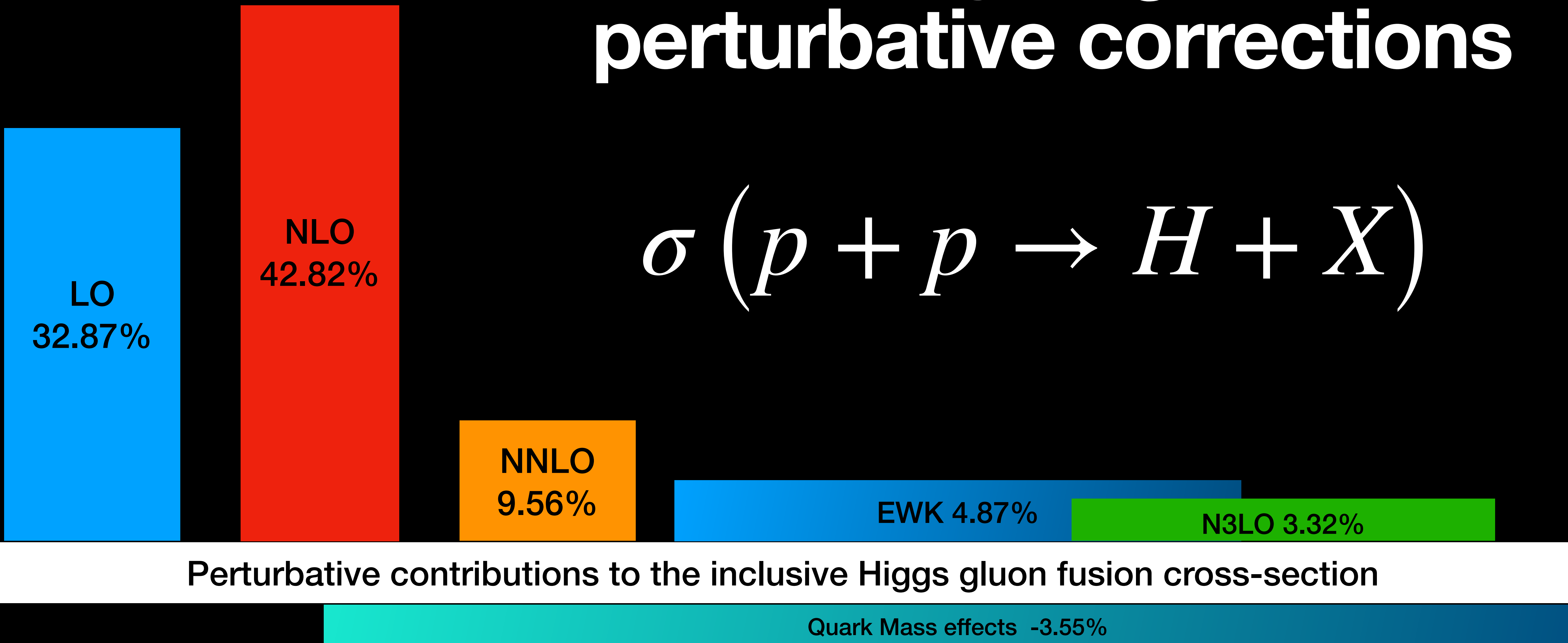


Perturbative contributions to the inclusive Higgs decay to gluons

Herzog, Ruijl, Ueda, Vermageren, Vogt [1707.01044]

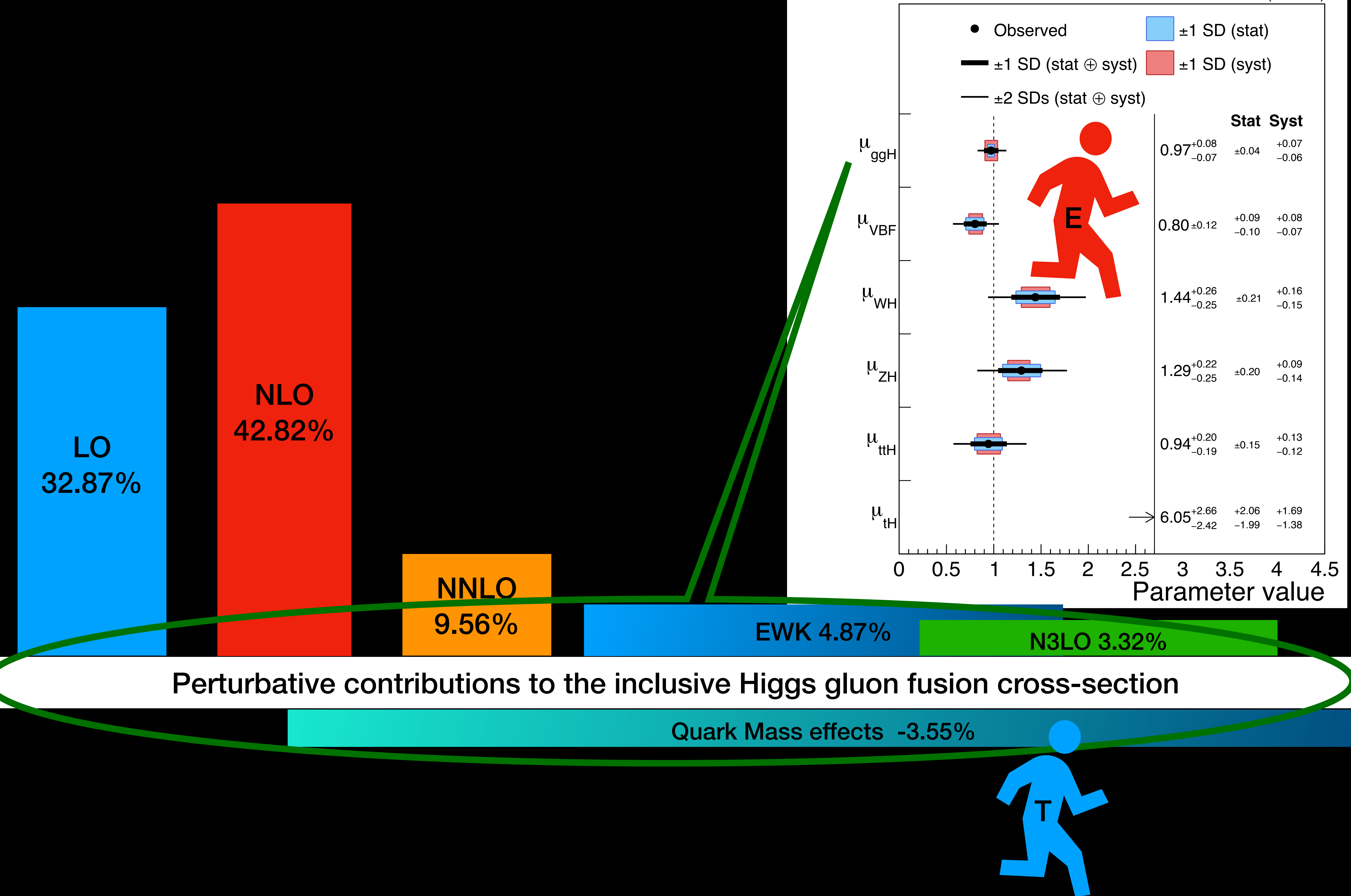
Stubbornly large perturbative corrections

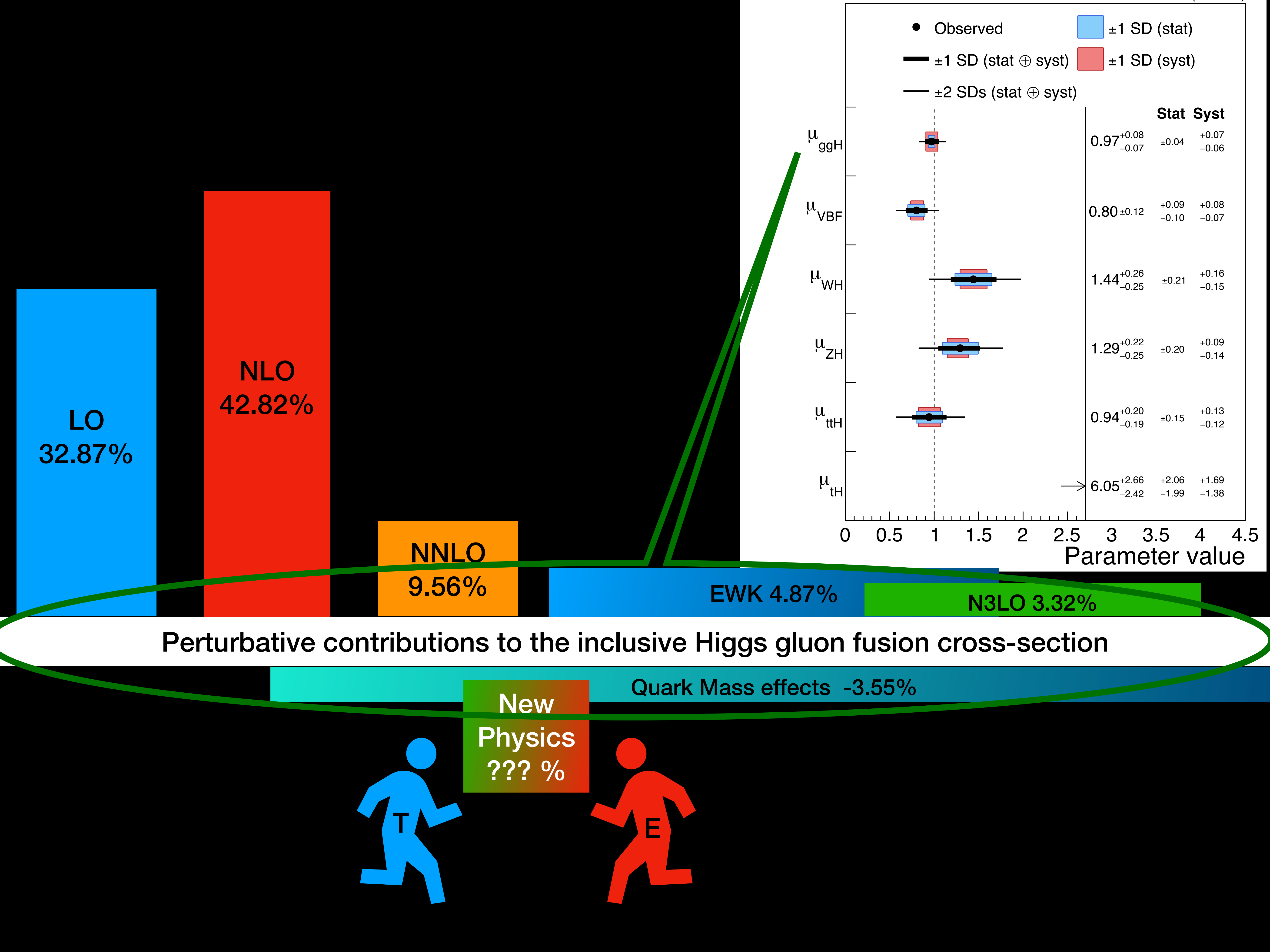
$$\sigma\left(p + p \to H + X\right)$$



Perturbative contributions to the inclusive Higgs gluon fusion cross-section

IHixs: Dulat, Lazopoulos, Mistlberger [1802.00827]


LO
32.87%
NLO
42.82%
NNLO
9.56%
EWK 4.87%
N3LO 3.32%
Perturbative contributions to the inclusive Higgs gluon fusion cross-section
Quark Mass effects -3.55%
E
T
Observed
±1 SD (stat ⊕ syst)
±2 SDs (stat ⊕ syst)
±1 SD (stat)
±1 SD (syst)
Stat Syst
μggH
μVBF
μWH
μZH
μttH
μtH
0.97 +0.08 −0.07 ±0.04 +0.07 −0.06
0.80 ±0.12 +0.09 −0.10 +0.08 −0.07
1.44 +0.26 −0.25 ±0.21 +0.16 −0.15
1.29 +0.22 −0.25 ±0.20 +0.09 −0.14
0.94 +0.20 −0.19 ±0.15 +0.13 −0.12
6.05 +2.66 −2.42 +2.06 −1.99 +1.69 −1.38
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5
Parameter value


LO
32.87%
NLO
42.82%
NNLO
9.56%
EWK 4.87%
N3LO 3.32%
Perturbative contributions to the inclusive Higgs gluon fusion cross-section
Quark Mass effects -3.55%
New
Physics
??? %
T
E
Observed
±1 SD (stat ⊕ syst)
±2 SDs (stat ⊕ syst)
±1 SD (stat)
±1 SD (syst)
Stat Syst
μ ggH
μ VBF
μ WH
μ ZH
μ ttH
μ tH
0.97 +0.08 −0.07 ±0.04 +0.07 −0.06
0.80 ±0.12 +0.09 −0.10 +0.08 −0.07
1.44 +0.26 −0.25 ±0.21 +0.16 −0.15
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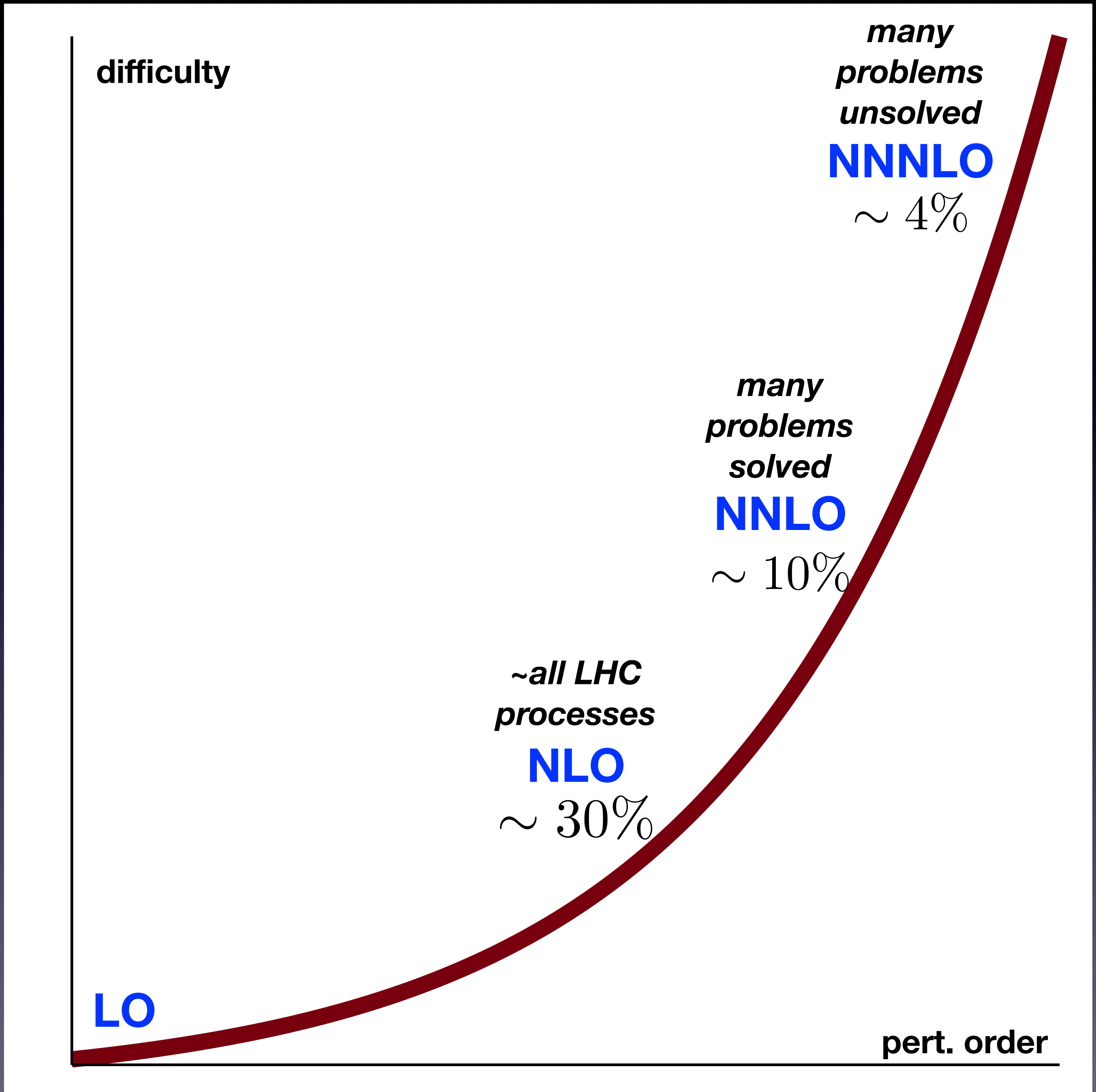
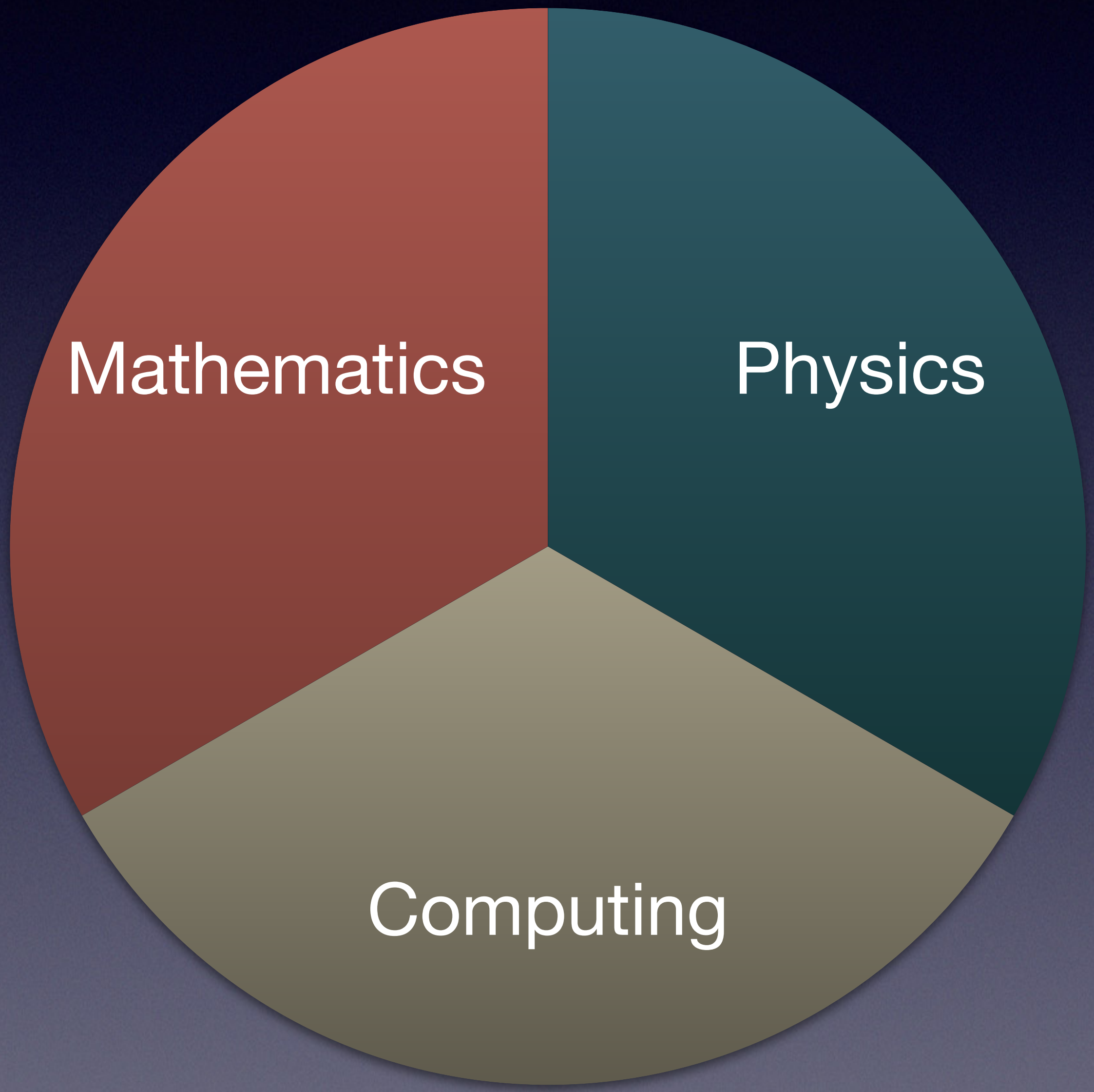
6.05 +2.66 −2.42 +2.06 −1.99 +1.69 −1.38
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5
Parameter value

What has been achieved for the LHC?



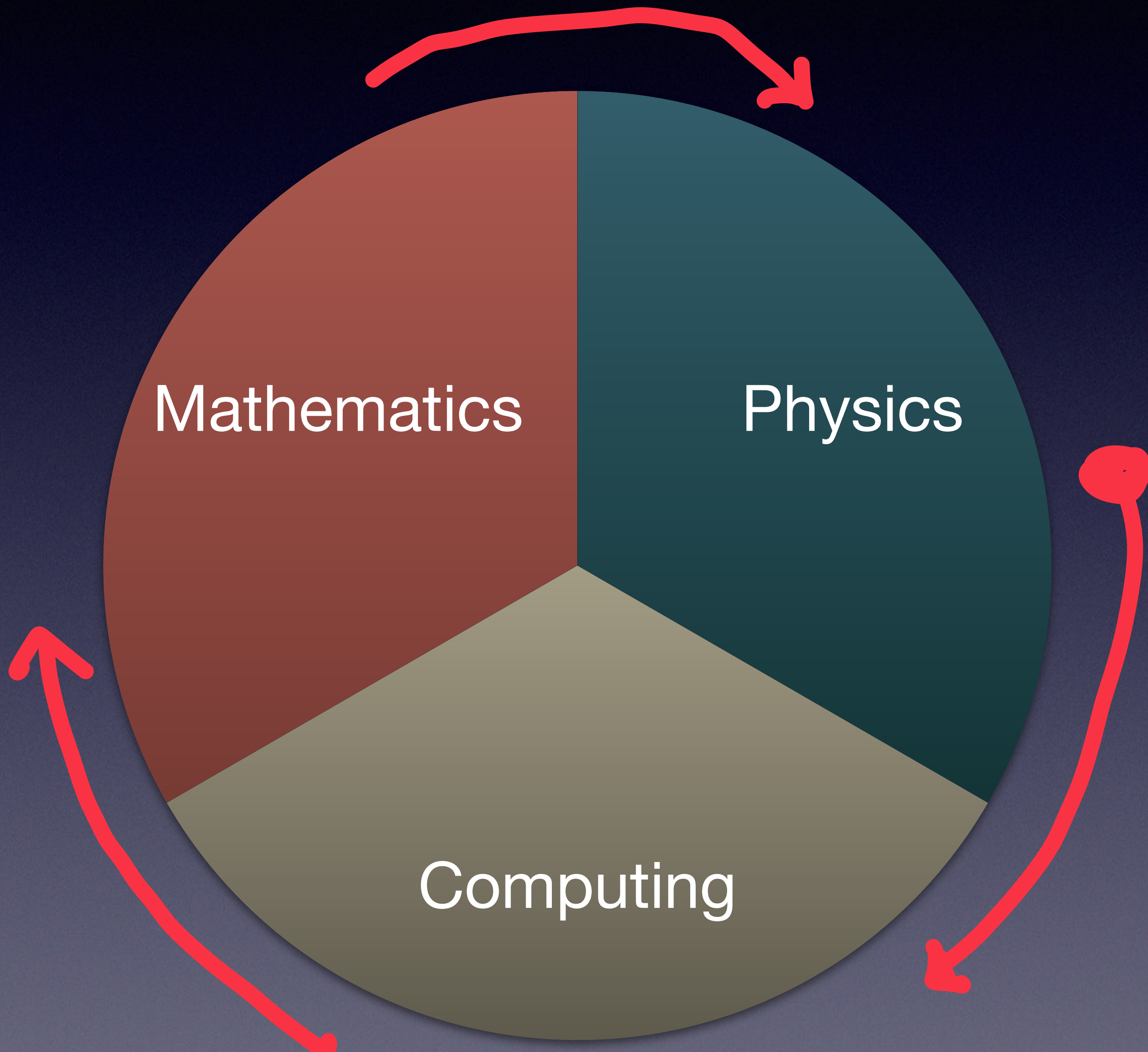
$$\sigma = \sigma_0 \alpha_s^n + \sigma_1 \alpha_s^{n+1} + \sigma_2 \alpha_s^{n+2} + \ldots$$

PERTURBATIVE COMPUTATIONS



Excellent ideas and methods!

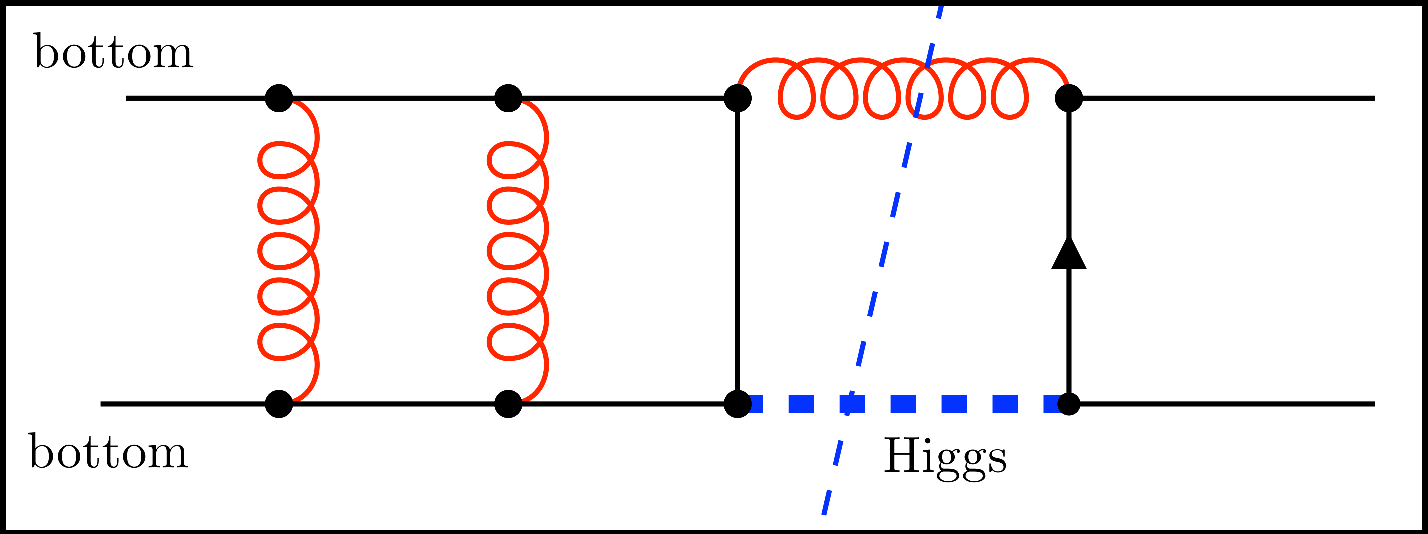
PERTURBATIVE COMPUTATIONS



Many and Challenging
Feynman integrals

- A diagram contributing to Higgs production in bottom fusion at NNLO.
- Gives rise to a rank-6 tensor integral.
- Which, in turn, gives rise to scalar $\mathcal{O}(500)$ integrals.
- At N3LO for the sinmplest type of processes, one needs to compute $\mathcal{O}(10^6)$ scalar integrals.

1 out of $\mathcal{O}(5000)$ diagrams



$$= Spin_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \int \delta\left(k_{10}^2 - M_h^2\right) \delta(k_9^2) \frac{k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4} k_5^{\mu_5} k_6^{\mu_6}}{k_1^2 k_2^2 k_3^2 k_4^2 k_5^2 k_6^2 k_7^2 k_8^2}$$

$$= Spin_{\mu_1 \dots} \int \delta\left(k_{10}^2 - M_h^2\right) \delta(k_9^2) \frac{k_1^{\alpha_1} \dots \mathcal{T}\left(u^{\alpha_1\mu_1\cdots}\right)}{k_1^2 k_2^2 k_3^2 k_4^2 k_5^2 k_6^2 k_7^2 k_8^2}$$

With $\mathcal{T}\left(u^{\alpha_1\mu_1} u^{\alpha_2\mu_2}\right) = u^{\alpha_1\mu_1} u^{\alpha_2\mu_2} + \eta_\perp^{\alpha_1\alpha_2} \eta_\perp^{\mu_1\mu_2} / D_\perp$

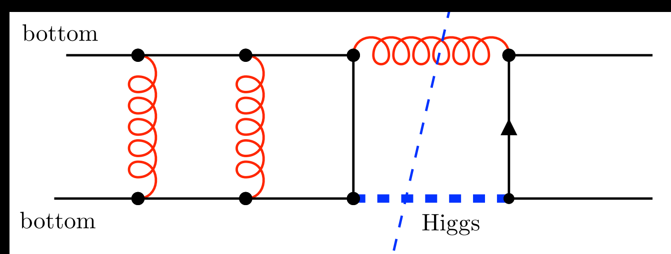
$$u^{\alpha\mu} = \frac{p_1^\alpha p_2^\mu + p_2^\alpha\ p_1^\mu}{p_1 \cdot p_2}$$

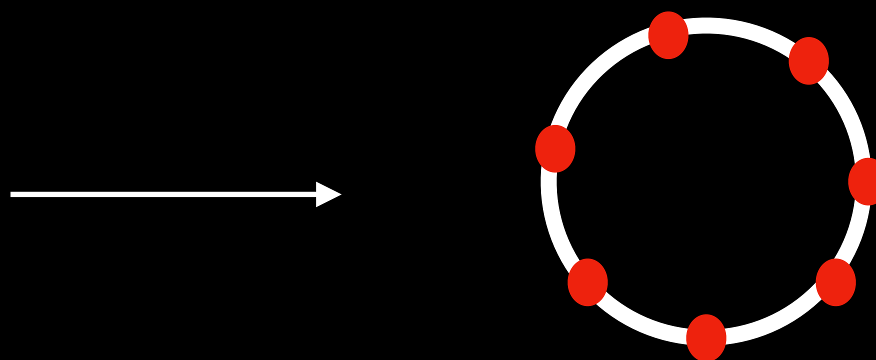
CA, Karlen, Vicini, [2308.1470]

Simplifying

Feynman integrals (toy example)

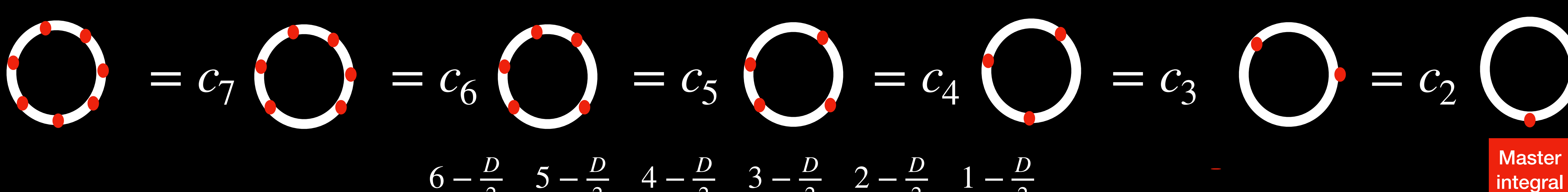
- Lets make a toy integral out of this diagram





$$\int \delta\left(k_{10}^2 - M_h^2\right)\delta(k_9^2)\frac{k_1^{\mu_1}k_2^{\mu_2}k_3^{\mu_3}k_4^{\mu_4}k_5^{\mu_5}k_6^{\mu_6}}{k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2k_7^2k_8^2} \to \int \frac{d^Dk}{i\pi^{\frac{D}{2}}}\frac{\left(k^2-M^2\right)^3}{\left[k^2-M^2\right]^{10}} = \frac{\Gamma\left(7-\frac{D}{2}\right)}{\Gamma(7)}\left(M^2\right)^{\frac{D}{2}-7}$$

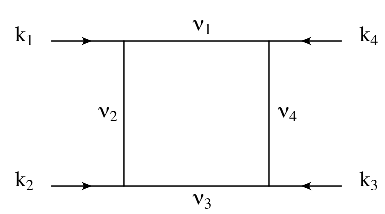
- Recursion: $\Gamma(x+1) = x\,\Gamma(x)$



$$c_2 = \frac{6-\frac{D}{2}}{6}\cdot\frac{5-\frac{D}{2}}{5}\cdot\frac{4-\frac{D}{2}}{4}\cdot\frac{3-\frac{D}{2}}{3}\cdot\frac{2-\frac{D}{2}}{3}\cdot\frac{1-\frac{D}{2}}{1}$$

Recursion and Reduction for general Feynman integrals

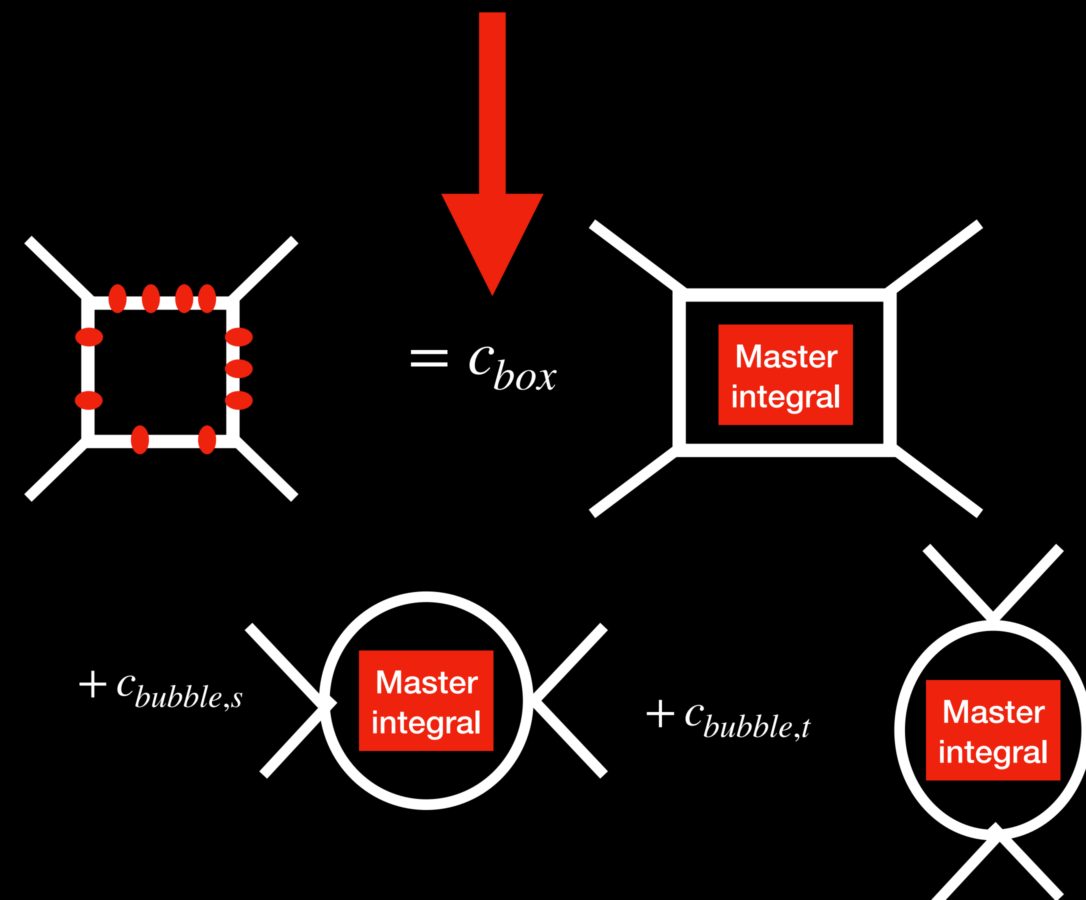
Feynman integrals are (generally uncharted) hypergeometric functions, i.e. infinite sums of products/ratios of factorials (Gamma functions).



$$
\begin{aligned}
&= (-1)^{\frac{D}{2}}\, t^{\frac{D}{2}-\sigma} \frac{\Gamma\left(\sigma-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\nu_{134}\right)\Gamma\left(\frac{D}{2}-\nu_{123}\right)}{\Gamma\left(\nu_2\right)\Gamma\left(\nu_4\right)\Gamma\left(D-\sigma\right)} \\
&\quad\times\ {}_3F_2\left(\nu_1,\nu_3,\sigma-\frac{D}{2},1+\nu_{134}-\frac{D}{2},1+\nu_{123}-\frac{D}{2},-\frac{s}{t}\right) \\
&+ (-1)^{\frac{D}{2}}\, s^{\frac{D}{2}-\nu_{123}} t^{-\nu_4} \frac{\Gamma\left(\nu_{123}-\frac{D}{2}\right)\Gamma\left(\nu_2-\nu_4\right)\Gamma\left(\frac{D}{2}-\nu_{23}\right)\Gamma\left(\frac{D}{2}-\nu_{12}\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)\Gamma\left(\nu_3\right)\Gamma\left(D-\sigma\right)} \\
&\quad\times\ {}_3F_2\left(\nu_4,\frac{D}{2}-\nu_{12},\frac{D}{2}-\nu_{23},1+\nu_4-\nu_2,1+\frac{D}{2}-\nu_{123},-\frac{s}{t}\right) \\
&+ (-1)^{\frac{D}{2}}\, s^{\frac{D}{2}-\nu_{134}} t^{-\nu_2} \frac{\Gamma\left(\nu_{134}-\frac{D}{2}\right)\Gamma\left(\nu_4-\nu_2\right)\Gamma\left(\frac{D}{2}-\nu_{14}\right)\Gamma\left(\frac{D}{2}-\nu_{34}\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_3\right)\Gamma\left(\nu_4\right)\Gamma\left(D-\sigma\right)} \\
&\quad\times\ {}_3F_2\left(\nu_2,\frac{D}{2}-\nu_{14},\frac{D}{2}-\nu_{34},1-\nu_4+\nu_2,1+\frac{D}{2}-\nu_{134},-\frac{s}{t}\right).
\end{aligned}
$$

$$
{}_2F_1(a,b;c;z) = \frac{c-2b+2+(b-a-1)z}{(b-1)(z-1)}\,{}_2F_1(a,b-1;c;z) + \frac{b-c-1}{(b-1)(z-1)}\,{}_2F_1(a,b-2;c;z)
$$

A Gauss recurrence identity for the common hypergeometric



A reduction to master integrals for a class of box integrals

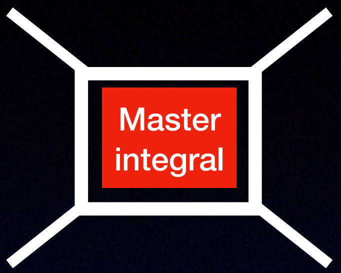
Recursion is intrinsic to hypergeometric functions, and Feynman integrals, in the form of difference equations.

A reduction of classes of integrals to fewer "master" integrals is always possible.

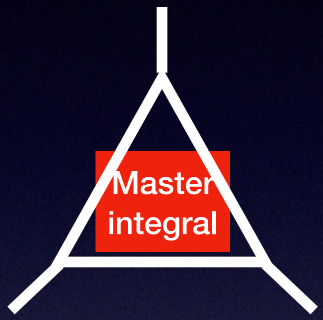
Physical reduction of amplitudes

"The NLO revolution"

$$\mathscr{A}_{1-loop} = c_{box}$$



$$+\, c_{triangle}$$



$$+\, c_{bubble}$$

$$+\, c_{tadp.}$$

- The Reduction of *one-loop amplitudes* to master integrals has a physical interpretation.
- Masters are integrals of a simple scalar field theory.
- Coefficients are Sums of Products of Tree Gauge Theory Amplitudes

$$C_{\text{Master}} = \sum \prod \mathscr{A}_{\text{tree}}$$

$$c_4 =$$

Britto, Cachazo, Feng [hep-th/0412103]

Britto, Feng, Mastrolia [hep-ph/0602178]

Del Aguila, Pittau [hep-ph/0404120]

Ossola, Papadopoulos, Pittau [hep-ph/0609007]

Forde [0704.1835]

Ellis, Giele, Kunszt [0708.2398]

Giele, Kunszt, Melnikov [0801.2237]

Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre [0803.4180]

- Generalisation at two-loops in amazing breakthroughs

Ita [1510.05626]

Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov [2009.11957]

When a complete physical solution is out of reach , compution comes to rescue.

Reduction identies are
obtained simply, with integration by parts (IBP).

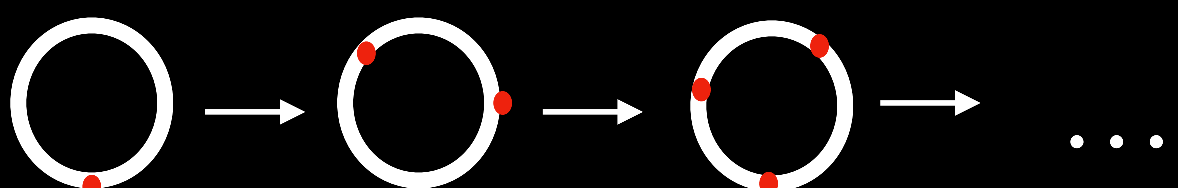
$$0 = \int d^D k \, \partial_\mu \frac{k^\mu}{\left(k^2 - M^2\right)^6} \rightsquigarrow \int d^D k \, \frac{1}{\left(k^2 - M^2\right)^7} = \frac{\frac{D}{2} - 6}{6} \int d^D k \, \frac{1}{\left(k^2 - M^2\right)^6}$$

Chetyrkin, Tkachov [Nucl. Phys. B192 (1981) 159-204]

Tkachov [Phys. Lett. B100 (1981) 65-68]

IBP identities can be "diagonalised" automatically with the "Laporta Algorithm", which is a optimised Gauss elimination method.

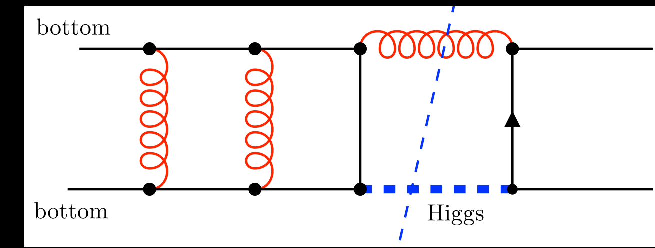
Laporta [hep-ph/0102033]



Master integral ⟶ **complicated integrals**

$$1 \rightarrow 2! \rightarrow 3! \rightarrow \ldots \left(n_{loops} + n_{legs}\right)!$$

1 out of $\mathcal{O}(500)$ scalar integrals in



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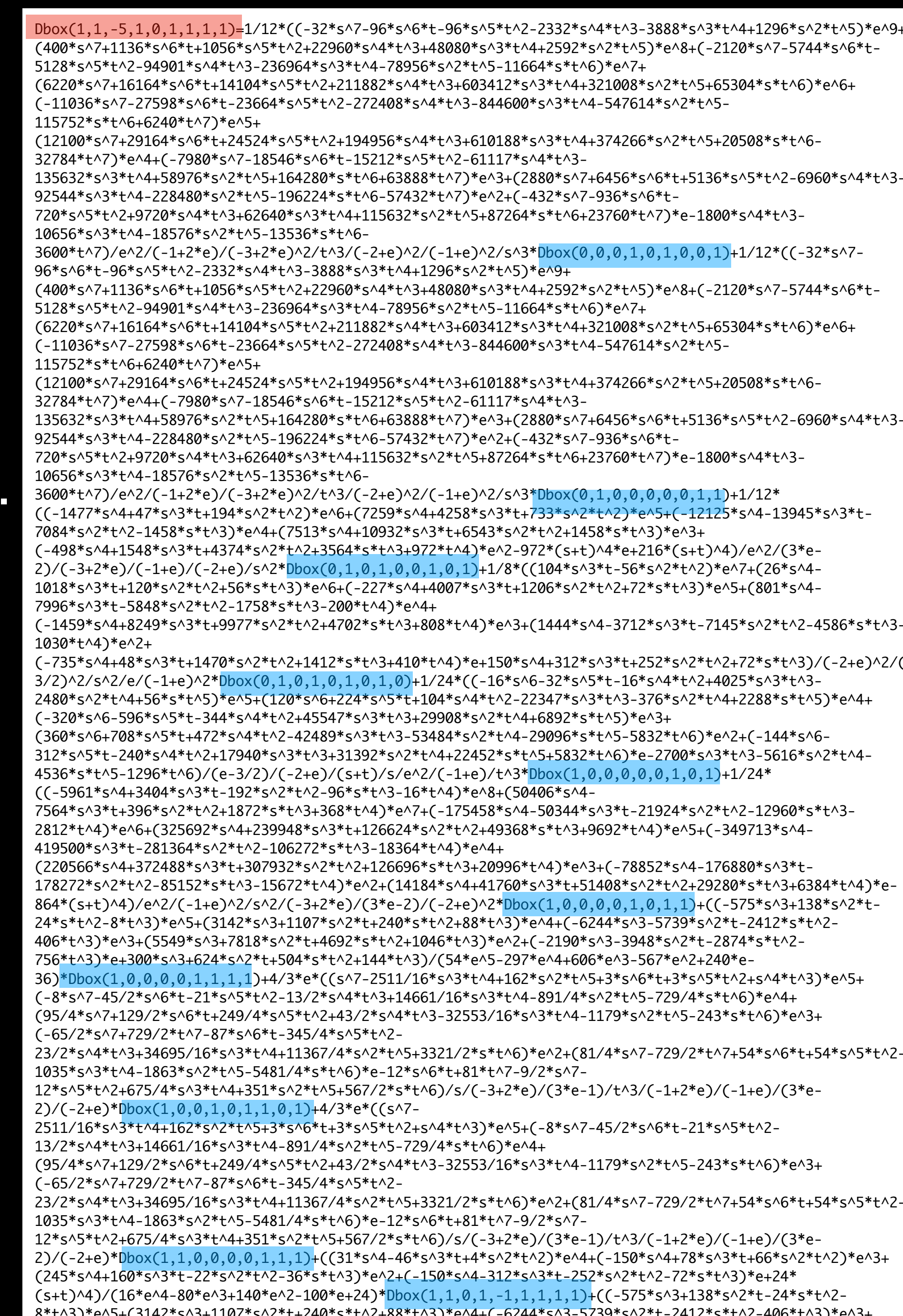
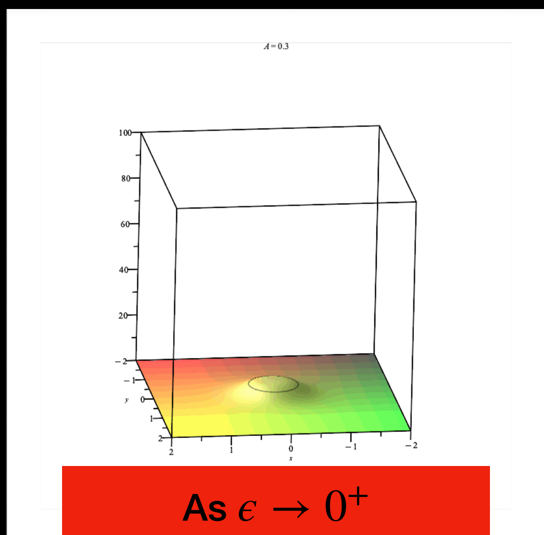


Simple and Powerful but Costly

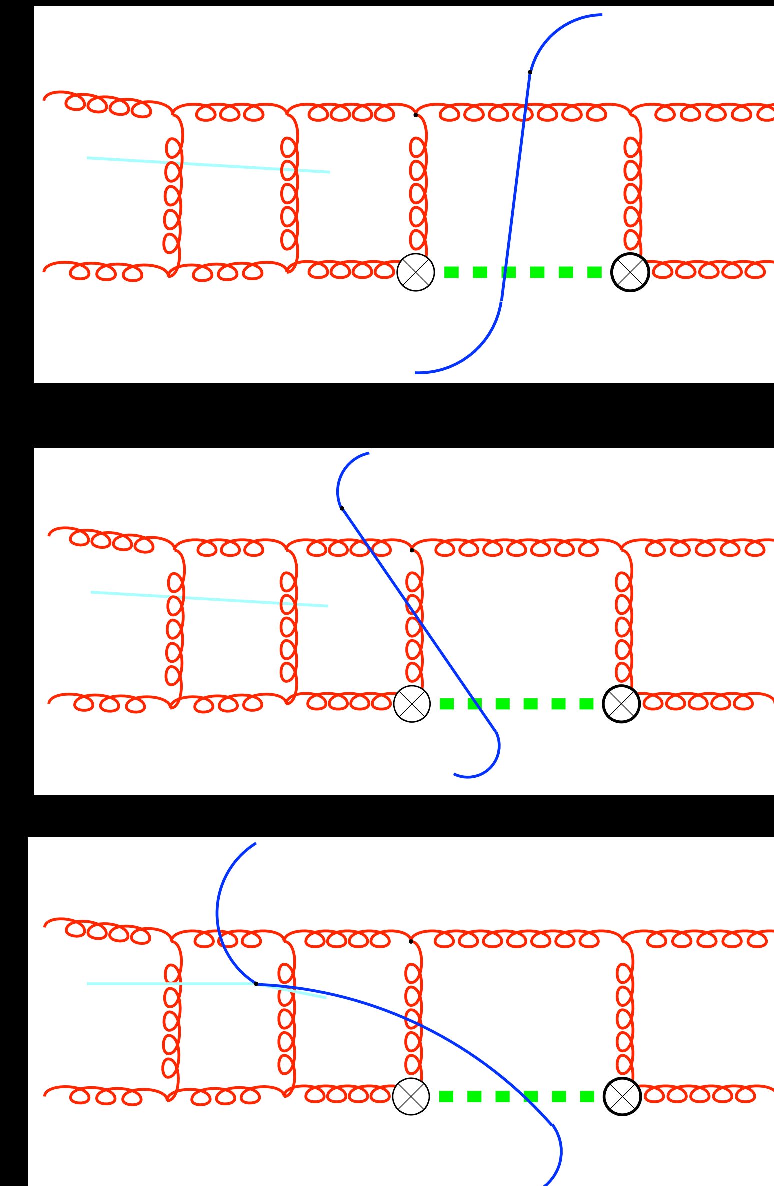
Reduction to master integrals

- "Inclusive" Phase-space integrations can also be simplified with integration by parts.

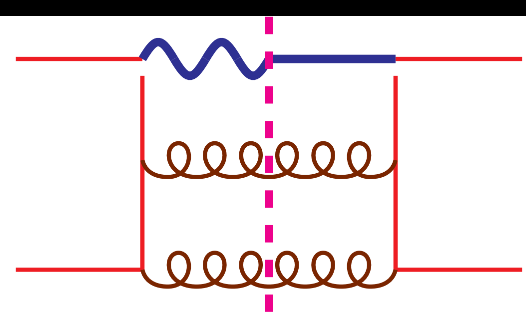
CA, Melnikov [1810.09462]



$$\lim_{\epsilon \to 0} \left(\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = 2\,\pi\, i\, \delta(x)$$



- Kinematic constraints can also be included.

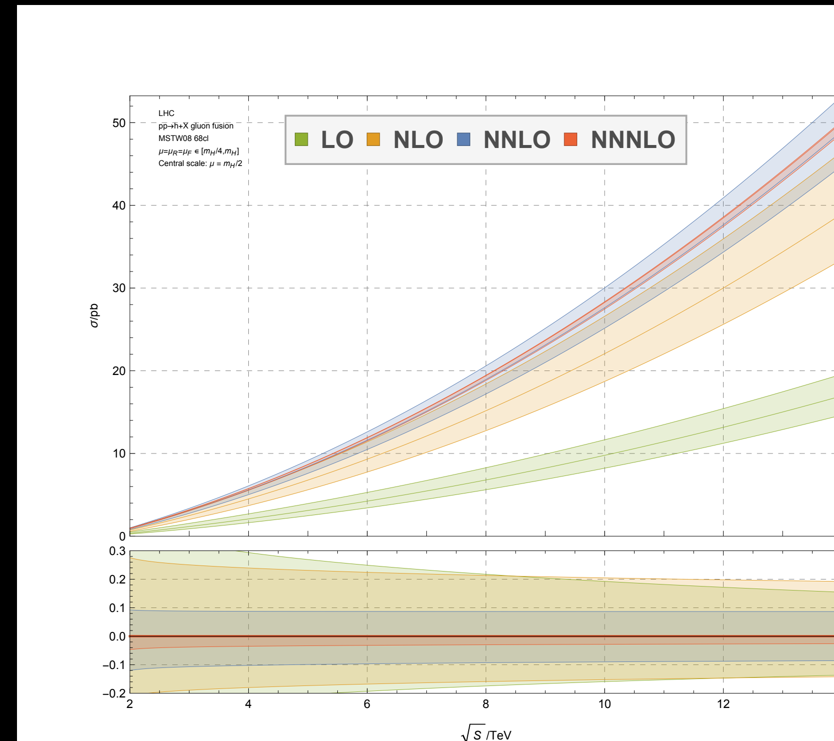


$$\delta\left(\frac{p_V \cdot p_1}{p_V \cdot p_2} - u\right) \;\to\; \frac{p_V \cdot p_2}{p_V \cdot (p_1 - u p_2) - i0} - (c.c.)$$

CA, Dixon, Melnikov, Petriello [1503.06056]

- Commutes with asymptotic expansions around simplifying limits (such as threshold production)

CA, Duhr, Dulat, Herzog, Mistlberger [1503.06056]



Analytic structure of master integrals

One-loop analytic structure at one-loop is simpler (CLOSED)

$$\mathscr{A}_{1-loop} = c_0 + c_1 \log + c_2 \log^2 + c_3 \,\mathrm{Li}_2\,.$$

$$Li_2(x) = -\int_0^x \log(1-t)\, d\log(t)$$

- Analytic structure is richer at two loops and beyond.
- Number and classes of special functions grows. Not known fully.
- Even a partial understanding has triggered an excellent progress in computing amplitudes.
- But it Is hard to go further. In need of further ideas/ alternatives.

$$\mathscr{A}_{1-loop} \ni \mathrm{Li}_2 + \mathrm{Li}_3 + \mathrm{Li}_4 + \mathrm{S}_{22}$$

NNLO Inclusive jet

$+\ldots$ harmonic polylogaritthms

NNLO Higgs+jet, Z+jet,

$+\ldots$ multiple polylogaritthms

NNLO WW, ZZ, …

$+\ldots$ elliptic polylogaritthms

NNLO top, N3LO Higgs inclusive, …

$+\ldots$???

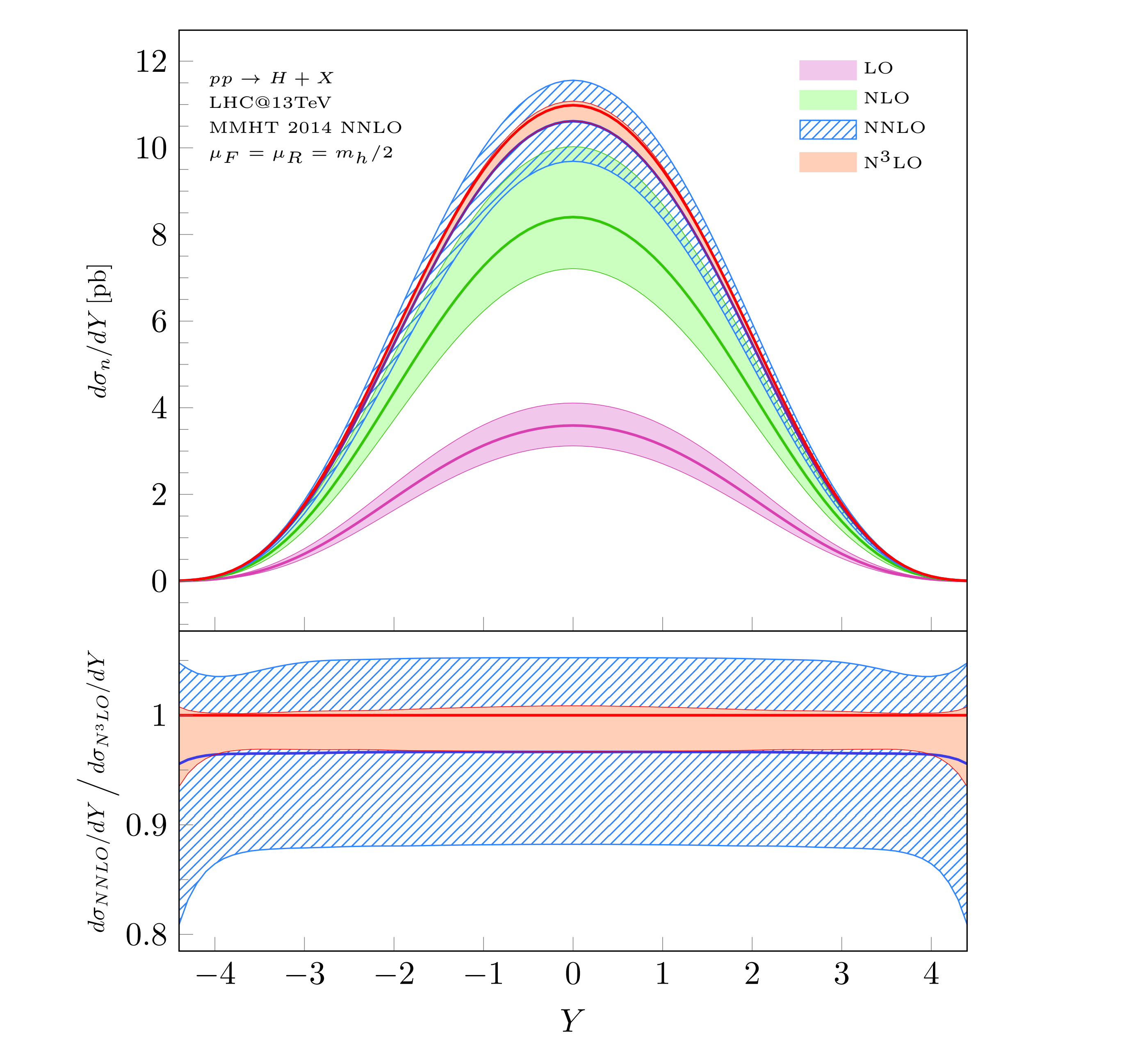
Gehrmann, Remiddi; Kptikov; Henn;…]

[Remiddi, Vermaseren; Vollinga, Weinzierl; Goncharov, Spradlin, Vergu, Volovich; Duhr, Gangl, Rhodes; Duhr; Duhr, Dulat; Mistlberger; Broedel, Duhr, Dulat, Tancredi; Abliger, Bluemlein, Round; Duhr, Tancredi; Panzer; Brown;…]

How far can one go with with reductions to master integrals?
Very far!

Higgs Rapidity Through N3LO

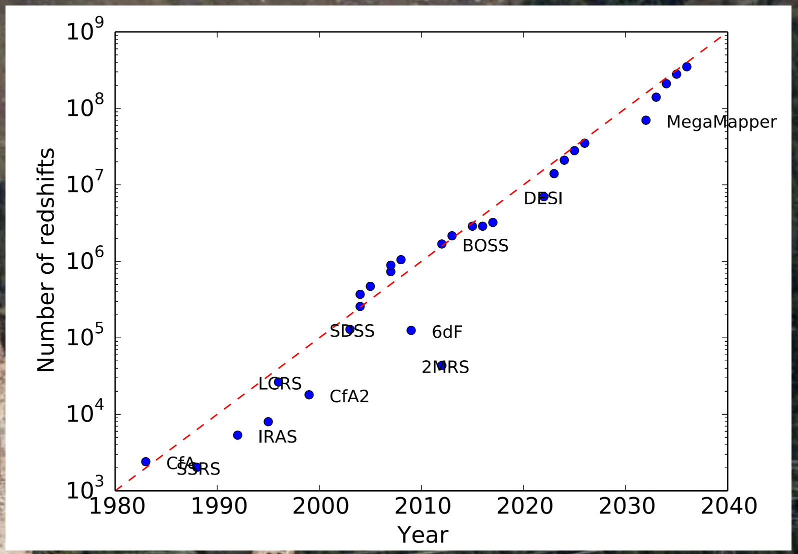
- Innovative deep expansion around Higgs threshold production (with two kinematic variables).
- Innovative reduction to master integrals (reconstruction of coefficients from numerics).
- High precision theoretical prediction.
- Awaiting data from the LHC at the high luminosity phase.



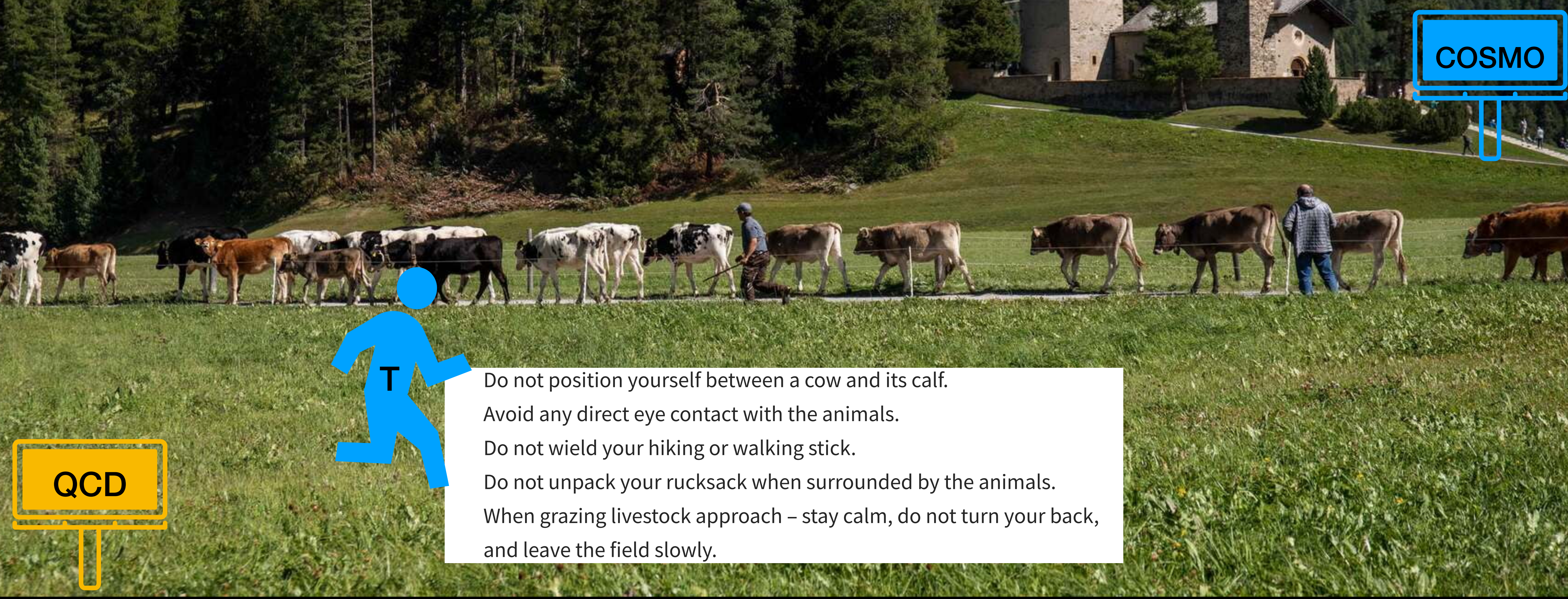
Dulat, Mistlberger, Pelloni, [1810.09462]

Rules of behaviour when crossing fields

Large scale structure with percent precision



Astro2020 APC White Paper [1907.11171]



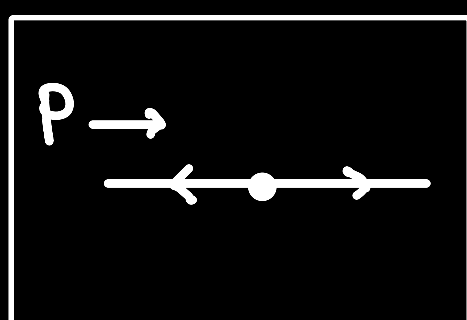
Do not position yourself between a cow and its calf.
Avoid any direct eye contact with the animals.
Do not wield your hiking or walking stick.
Do not unpack your rucksack when surrounded by the animals.
When grazing livestock approach – stay calm, do not turn your back, and leave the field slowly.

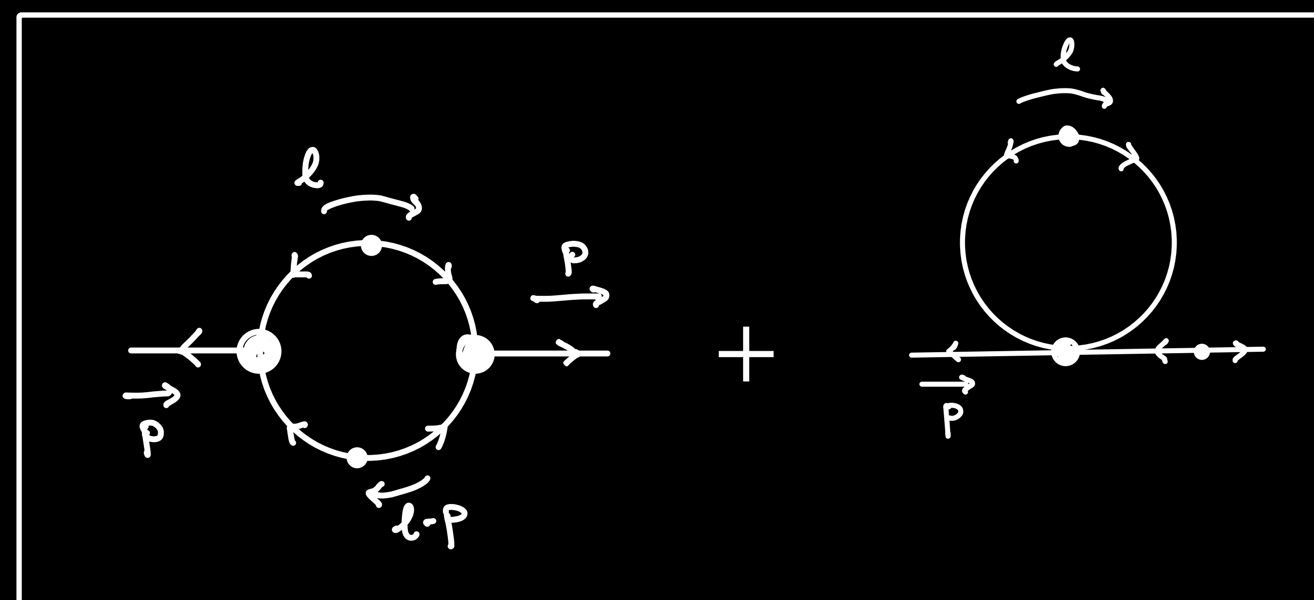
[source: Tourist Information, Engadin, Switzerland]

EFT of Large Scale Structure

Baumann, Nicolis, Senatore, Zaldarriaga [1004.2488]
Carrasco, Hertzberg,, Senatore [1206.2926]
Porto, Senatore, Zaldarriaga [1311.2168]
Senatore, Zaldarriaga [1404.5954]

$$\mathscr{P}(p) \equiv \int \frac{d^3\vec{r}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} \left\langle \left(\frac{\delta\rho}{\rho}\right)(\vec{x}) \left(\frac{\delta\rho}{\rho}\right)(\vec{x}+\vec{r}) \right\rangle$$

$\mathscr{P}(p) =$ 

$+$ 

$+ \ldots$

Solution of linearised equations. Depends on cosmological parameters, e.g. Hubble constant

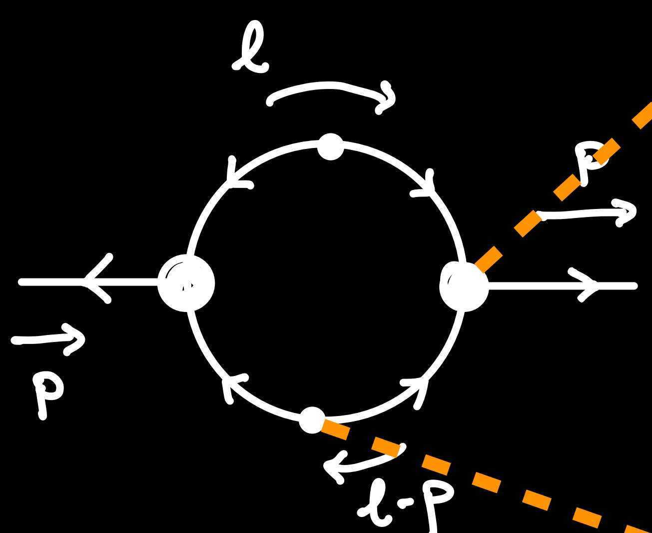
Small non-linearities as loop corrections. Cosmological parameters enter implicitly, through the propagator lines.

Loops in EFT of Large Scale Structure

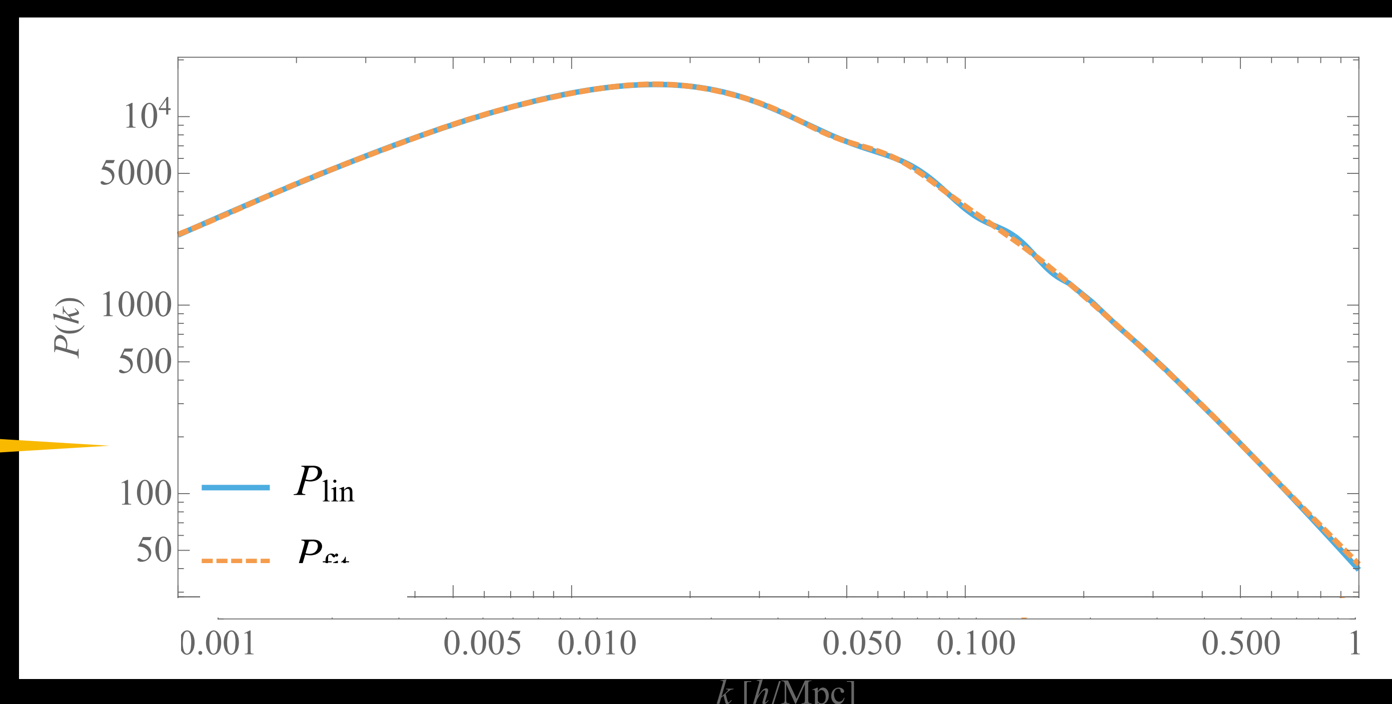
VERTICES

$$28\,F_2 = 10 + 3\,\frac{p^2}{l^2} + 3\frac{p^2}{(l-p)^2} - 5\frac{p^2}{(l-p)^2} - 5\frac{(l-p)^2}{p^2} + 2\frac{(p^2)^2}{l^2\,(l-p)^2}$$

Massless QCD-type propagator denominators

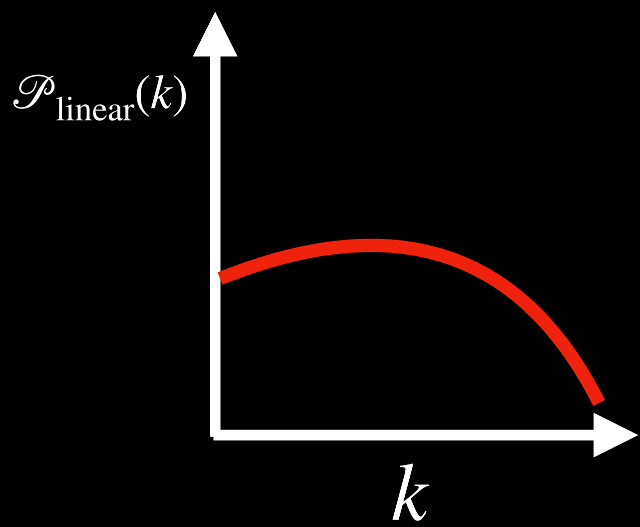


PROPAGATORS



A function of wavelength (loop momentum) and cosmological parameters

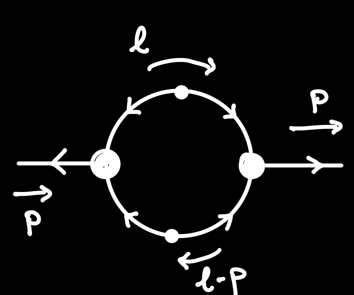
Mapping EFT of LSS correlators to QFT integrals



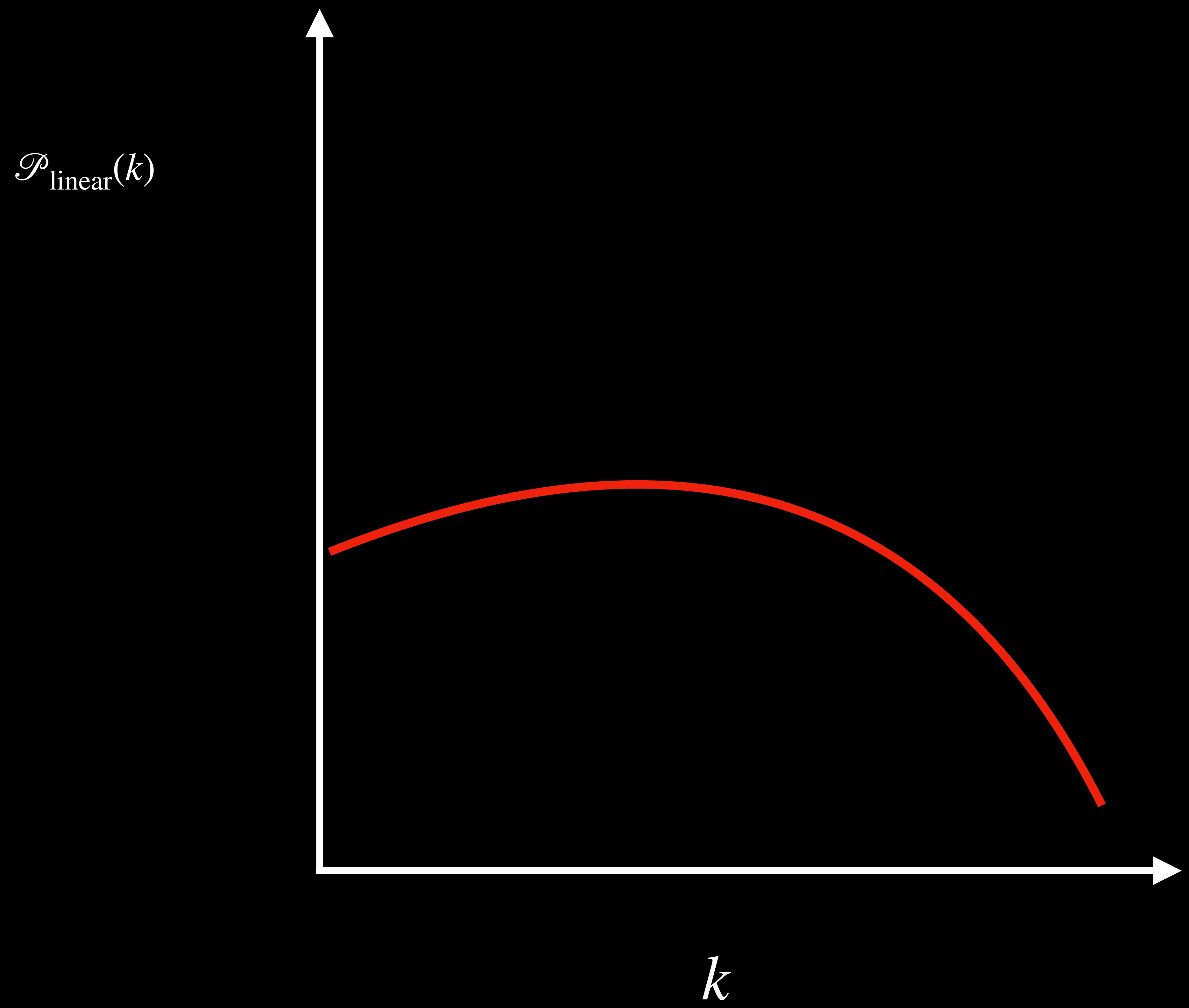
$$= \sum_n C_n\,(H, \Omega, \ldots)\, \frac{1}{\left(k^2\right)^{\nu + i\,\sigma_n}}$$

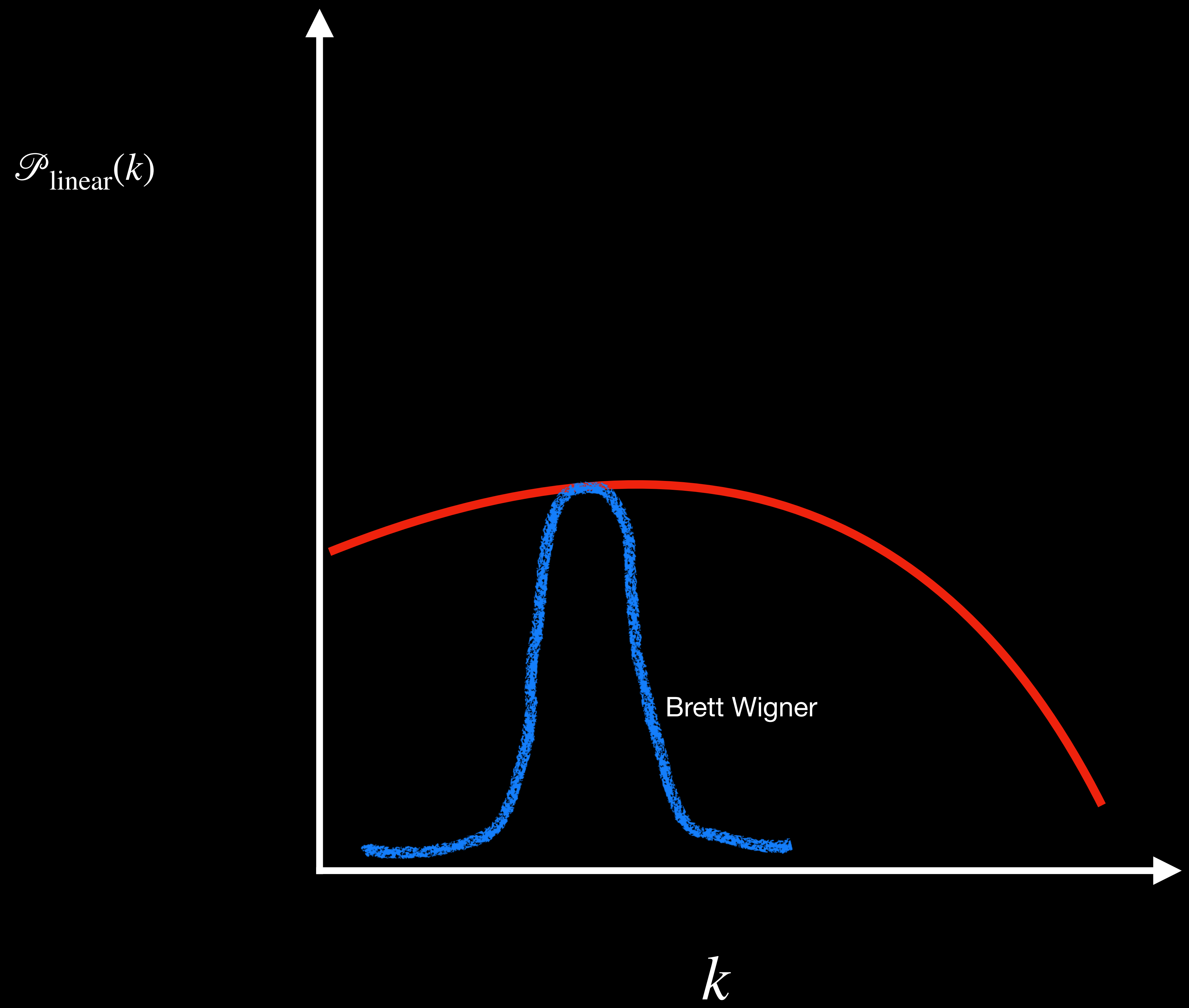
Simonovic, Baldauf, Zaldarriaga, Carrasco, Kollmeier [1708.08130]

- A fit of the linear power spectrum in a series of massless propagators raised to complex powers
- Integrations can be performed without reference to the values of the cosmological parameters.
- For the one-loop power spectrum:

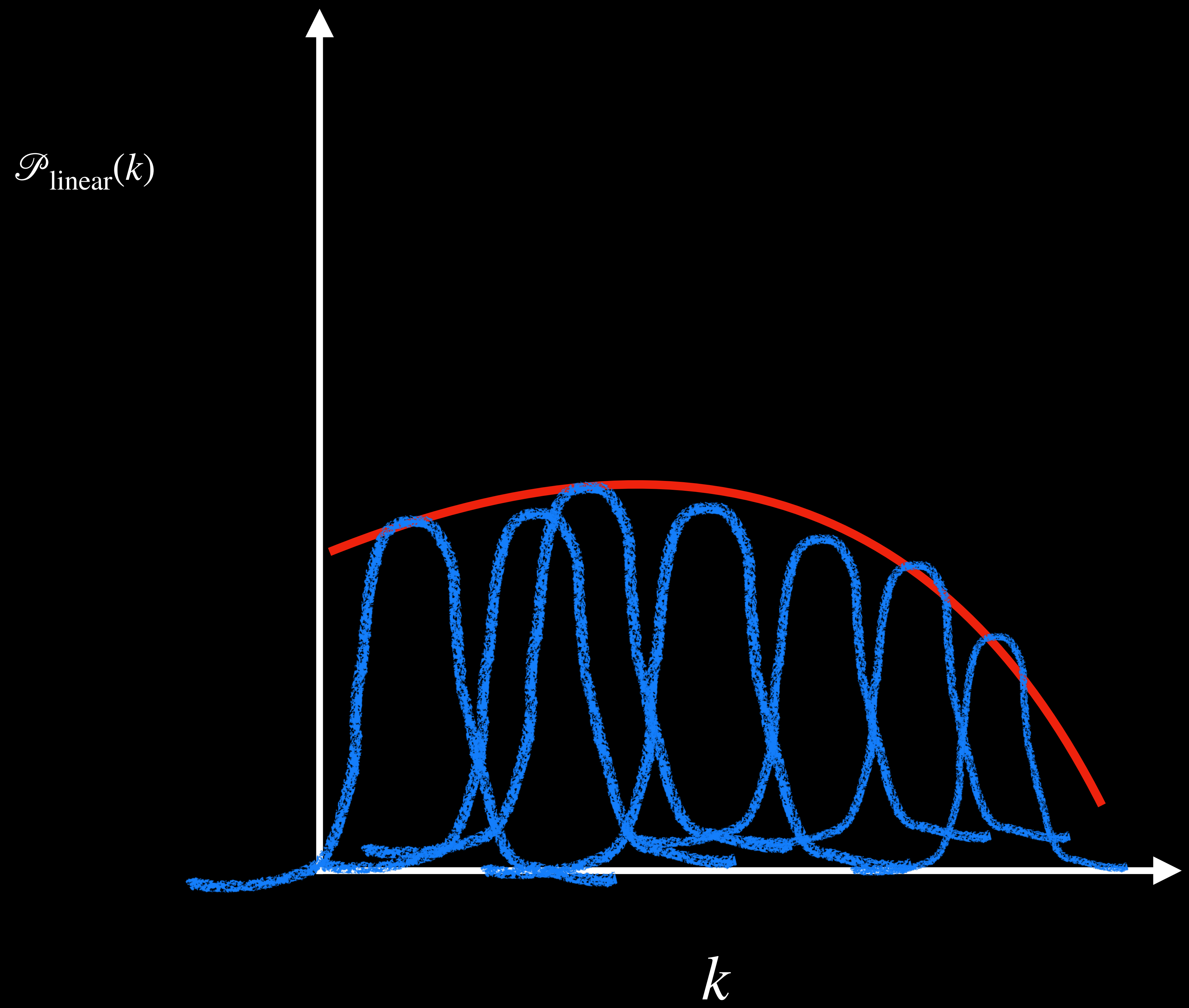


$$\rightarrow \sum_{n,m} C_m\,(H, \Omega, \ldots)\, C_n\,(H, \Omega, \ldots) \int d^3k \frac{1}{\left(k^2\right)^{\nu + i\,\sigma_m} \left((p-k)^2\right)^{\nu + i\,\sigma_n}} = \sum_{n,m} C_n C_m \ldots \frac{\Gamma\left(\frac{3}{2} - \nu - i\sigma_n\right)\Gamma\left(\frac{3}{2} - \nu - i\sigma_m\right)}{\Gamma\left(\frac{3}{-}2\nu - i\sigma_n - i\sigma\right)} \ldots$$

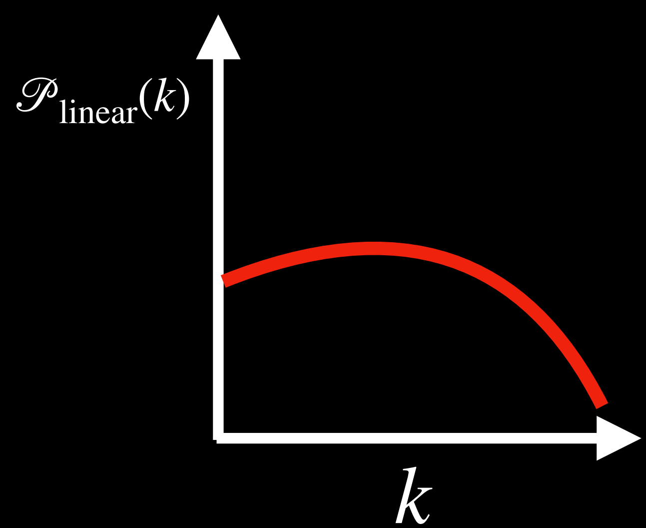

$\mathscr{P}_{\text{linear}}(k)$
k



$\mathscr{P}_{\text{linear}}(k)$
Brett Wigner
k


$\mathscr{P}_{\mathrm{linear}}(k)$
k

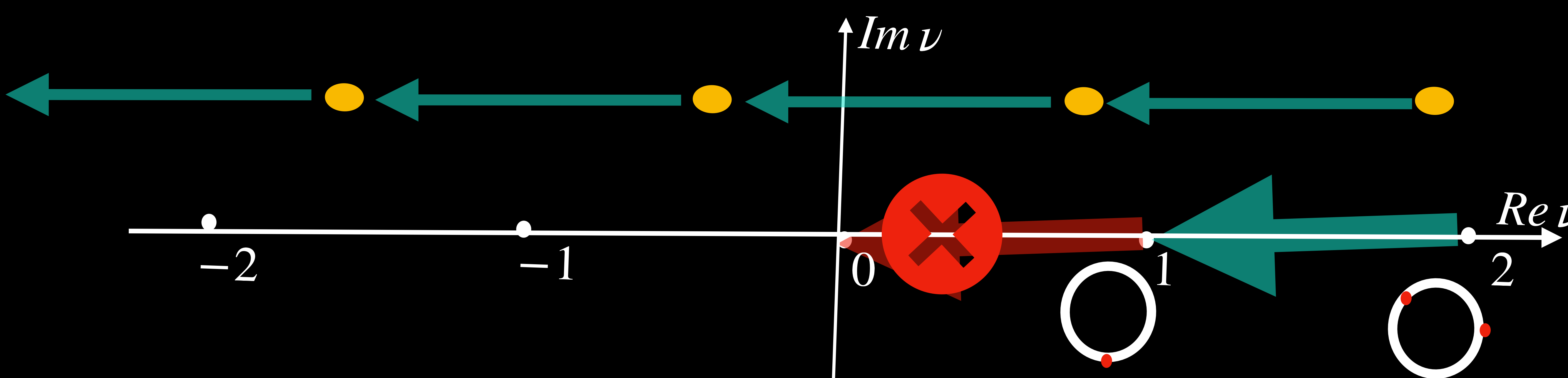
Mapping EFT of LSS correlators to QFT integrals differently



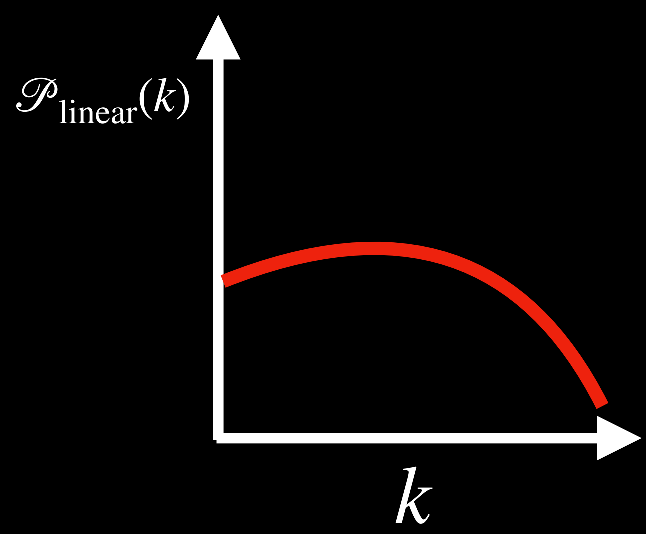
$$= \ldots = \sum_n C_{nm}(H, \Omega, \ldots) \frac{1}{\left(k^2 + M_n + i\Gamma_n\right)^{\nu_m}}$$

CA, Braganca, Senatore, Zheng [2212.07421]

- A fit of the linear power spectrum to a series of massive propagators raised to integer powers
- Integrations can be performed without reference to the values of the cosmological parameters.
- A reduction to FEWER master integrals $\left(\frac{1}{\Gamma(0)} = 0, \quad \frac{1}{\Gamma(0 + i\,\sigma)} \neq 0\right)$



Mapping EFT of LSS correlators to QFT integrals differently



$$= \sum_n C_{nm}\,(H, \Omega, \ldots)\,\frac{1}{\left(k^2 + M_n + i\Gamma_n\right)^{\nu_m}}$$

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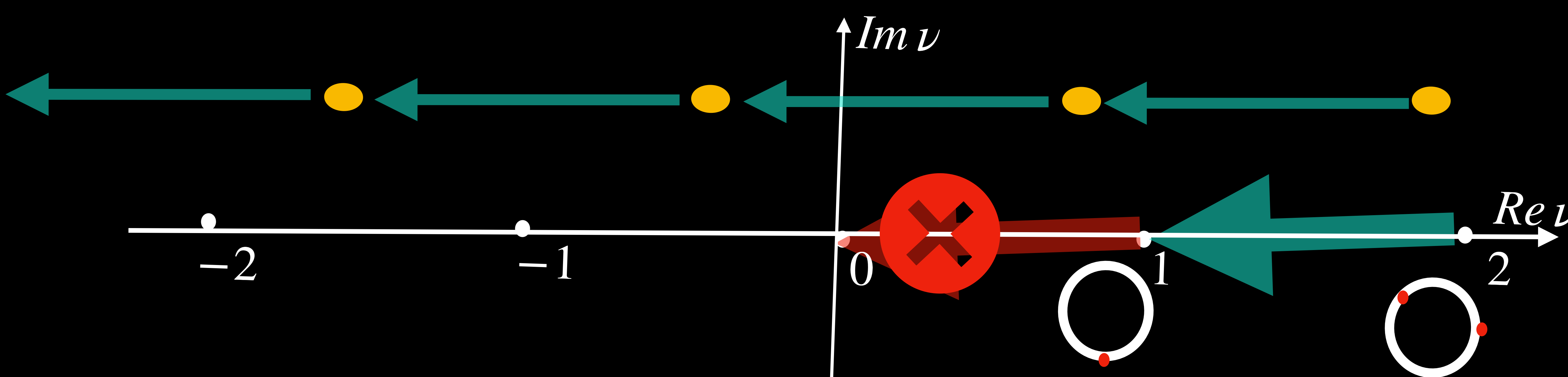
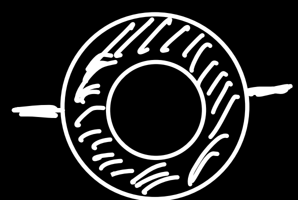
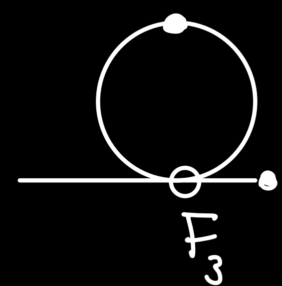
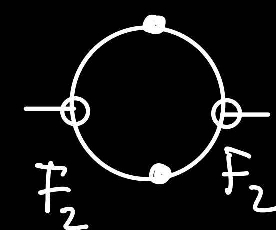


One-loop power spectrum in EFT of LSS

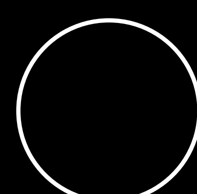
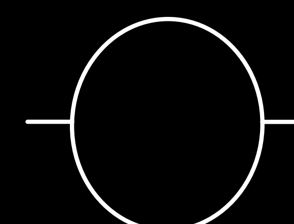
Cosmology
dependent coefficients, obtained by fit to leading order (linear) solution

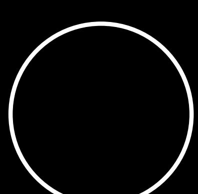
 = 

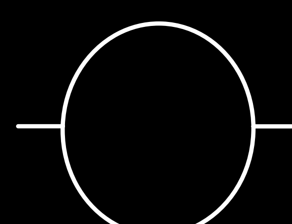
\+ 

$$= \sum_n C^{(n)}_{H,\Omega,\ldots} \mathscr{I}^{(2)}_n$$

1-loop QFT
integrals with massive propagators

$\mathscr{I}^{(2)}_n = \quad d_n^{tadp.}$  $+ \quad d_n^{bubble}$ 

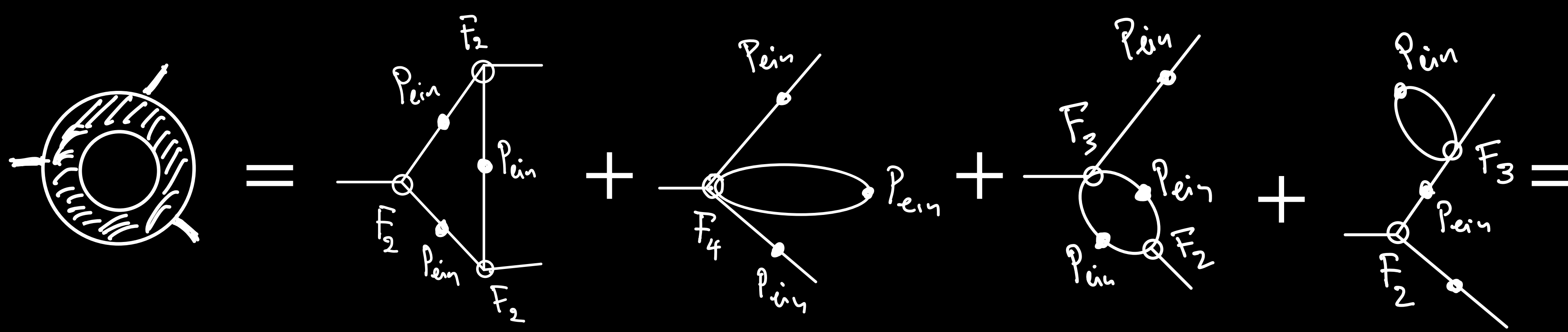
 $= -\sqrt{4\,\pi\,M}$



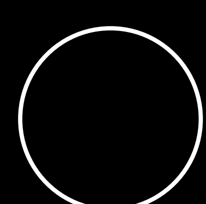
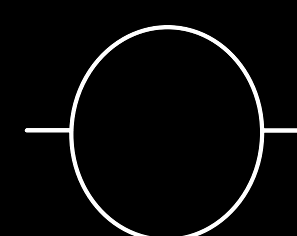
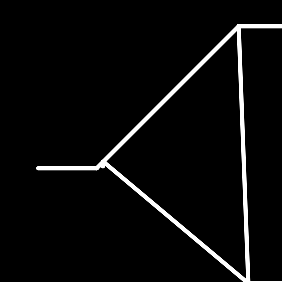
$$= \frac{\sqrt{\pi}}{k} i[\log\left(A(1,m_1,m_2)\right) - \log\left(A(0,m_1,m_2)\right)$$
$$- 2\pi i H(\mathrm{Im}\,A(1,m_1,m_2))H(-\mathrm{Im}\,A(0,m_1,m_2))]\,,$$

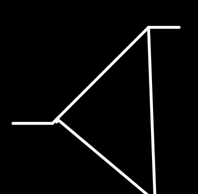
$$A(0,m_1,m_2) = 2\sqrt{m_2} + i(m_1 - m_2 + 1)$$
$$A(1,m_1,m_2) = 2\sqrt{m_1} + i(m_1 - m_2 - 1)$$

One-loop bispectrum in EFT of LSS



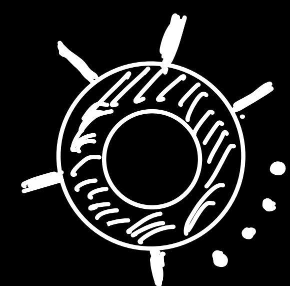
$$= \sum_n C^{(n)}_{H,\Omega,\ldots} \mathcal{I}^{(3)}_n$$

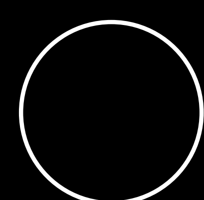
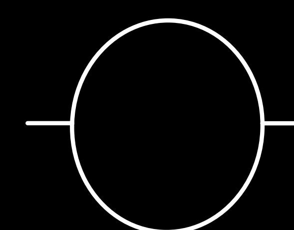
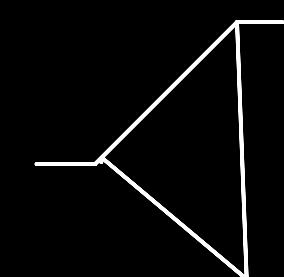
$\mathcal{I}^{(3)}_n = d_n^{tadp.}$  $+ \; d_n^{bubble}$  $+ \; d_n^{trian.}$ 

 $= \left[c_1 F_{\rm int}(R_2, z_+, z_-, x_+) + c_2 F_{\rm int}(R_2, z_+, z_-, x_-)\right]_{y=0}^{y=1}$

$$F_{\rm int}(R_2, z_+, z_-, x_0) = s(z_+, -z_-)\frac{\sqrt{\pi}}{\sqrt{|R_2|}} \left. \frac{\arctan\left(\frac{\sqrt{z_+-x}\sqrt{x_0-z_-}}{\sqrt{x_0-z_+}\sqrt{x-z_-}}\right)}{\sqrt{x_0-z_+}\sqrt{x_0-z_-}} \right|_{x=0}^{x=1} - \text{discontinuitie}$$

Generic N-point correlators in EFT of LSS.

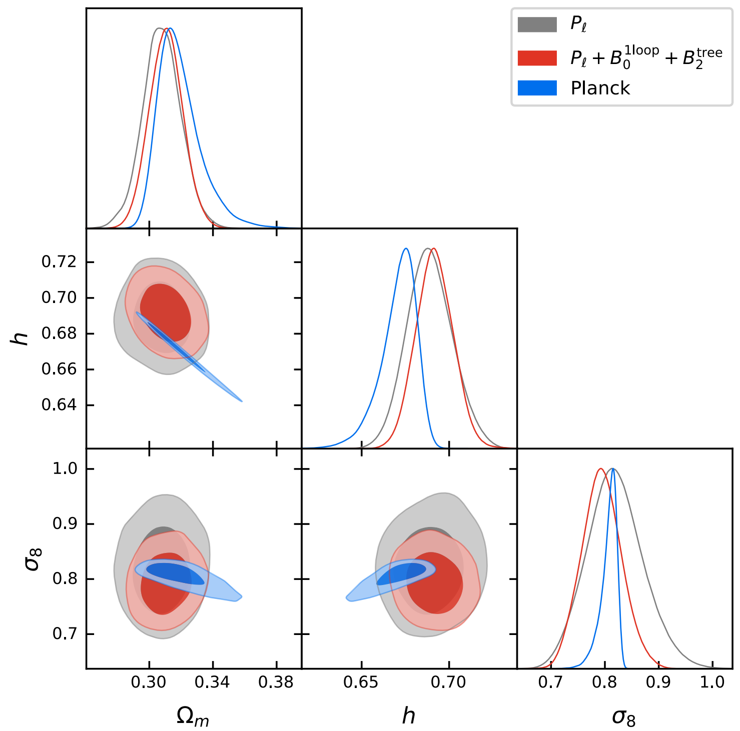
 $= \sum_n C^{(n)}_{H,\Omega,\ldots} \mathcal{I}^{(N_p)}_n$

$\mathcal{I}^{(N_p)}_n = d_n^{tadp.}$  $+ \; d_n^{bubble}$  $+ \; d_n^{trian.}$ 

No box, pentagon, hexagon,… master integrals in three dimensions

Van Neerven, Vermaseren [Phys.Lett.B 137 (1984) 241-244]

- At one-loop, in $D = 3 - 2\epsilon$, all loop integrals are free of $1/\epsilon$ poles.
- Reduction to master integrals, with memoization in arbitrary arithmetic precision, numerically (setting D=3 exactly)
- Fast evaluation of integrals, permitting an efficient inference of cosmological parameters comparing with data.



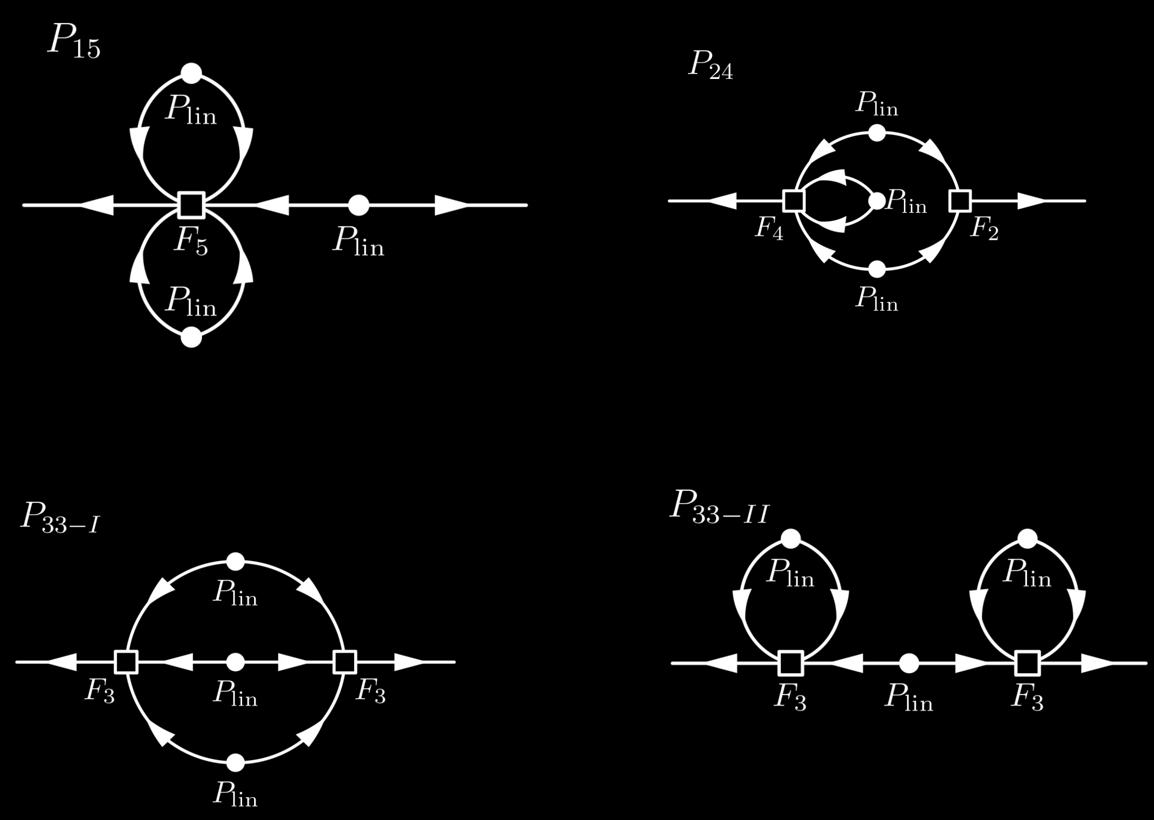
best-fit mean $\pm\,\sigma$	Ω_m	h	σ_8	ω_{cdm}	$\ln(10^{10}A_s)$	S_8
P_ℓ	0.2984 0.308 ± 0.012	0.6763 $0.689^{+0.012}_{-0.014}$	0.8305 $0.819^{+0.049}_{-0.055}$	0.1143 0.1232 ± 0.0075	3.123 3.02 ± 0.15	0.8283 $0.830^{+0.051}_{-0.060}$
$P_\ell + B_0^{\rm tree}$	0.3101 0.309 ± 0.011	0.6907 0.691 ± 0.012	0.8063 0.804 ± 0.049	0.1248 0.1246 ± 0.0058	2.98 2.97 ± 0.13	0.8197 $0.816^{+0.050}_{-0.057}$
$P_\ell + B_0^{\rm 1loop}$	0.3210 0.314 ± 0.011	0.6956 0.693 ± 0.011	0.7882 $0.790^{+0.033}_{-0.037}$	0.1331 0.1278 ± 0.0061	2.82 2.90 ± 0.11	0.8153 $0.807^{+0.037}_{-0.043}$
$P_\ell + B_0^{\rm 1loop} + B_2^{\rm tree}$	0.3082 0.311 ± 0.010	0.6928 0.692 ± 0.011	0.7856 0.794 ± 0.037	0.1258 0.1255 ± 0.0057	2.88 2.94 ± 0.11	0.7962 0.808 ± 0.041
Planck	$0.3191^{+0.0085}_{-0.016}$	$0.671^{+0.012}_{-0.0067}$	$0.807^{+0.018}_{-0.0079}$	0.1201 ± 0.0013	3.046 ± 0.015	0.832 ± 0.013

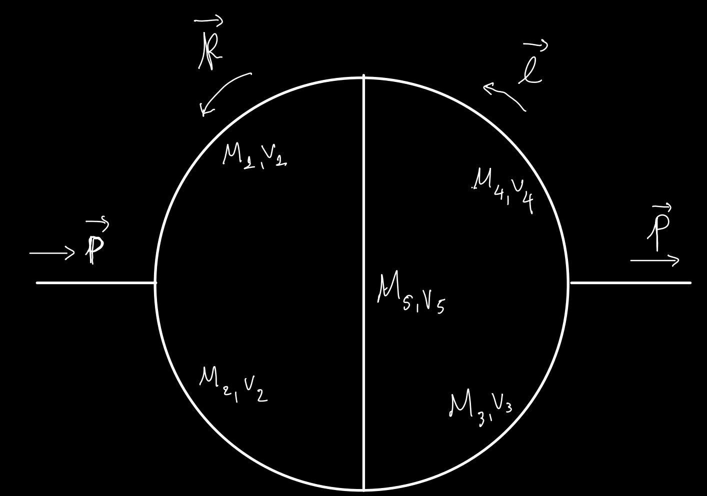
Figure 1: Triangle plots, best-fit values, and relative 68%-credible intervals of base cosmological parameters measured from the analysis of BOSS power spectrum multipoles P_ℓ, $\ell = 0, 2$, at one-loop, bispectrum monopole B_0 at tree or one-loop level, and bispectrum quadrupole B_2 at tree-level. Planck $\nu\Lambda$CDM results are shown for comparison.

D' Amico, Donath, Lewandowski, Senatore, Zhang [2206.08327]

Two-loop power spectrum in EFT of Large Scale Structure

$$\sum_{\{\nu_i,M_i\}} C\left(\{\nu_i, M_i\}\right) \int \frac{d^D k\, d^D l}{\left[k^2+M_1\right]^{\nu_1}\left[(k+p)^2+M_2\right]^{\nu_2}\left[(l+p)^2+M_3\right]^{\nu_3}\left[l^2+M_4\right]^{\nu_4}\left[(k-l)^2+M_5\right]^{\nu_5}}$$



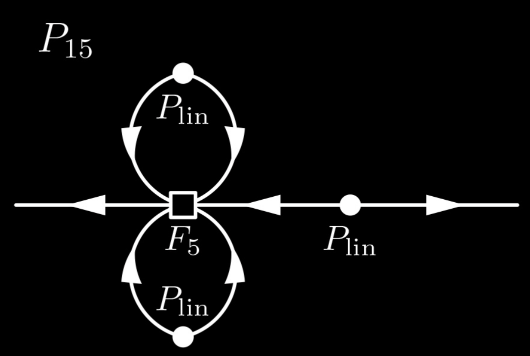


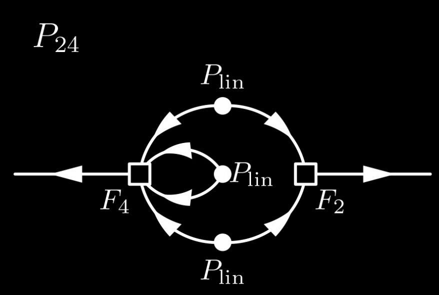
CA, Favorito, Senatore, Mistlberger, Zheng, in progress

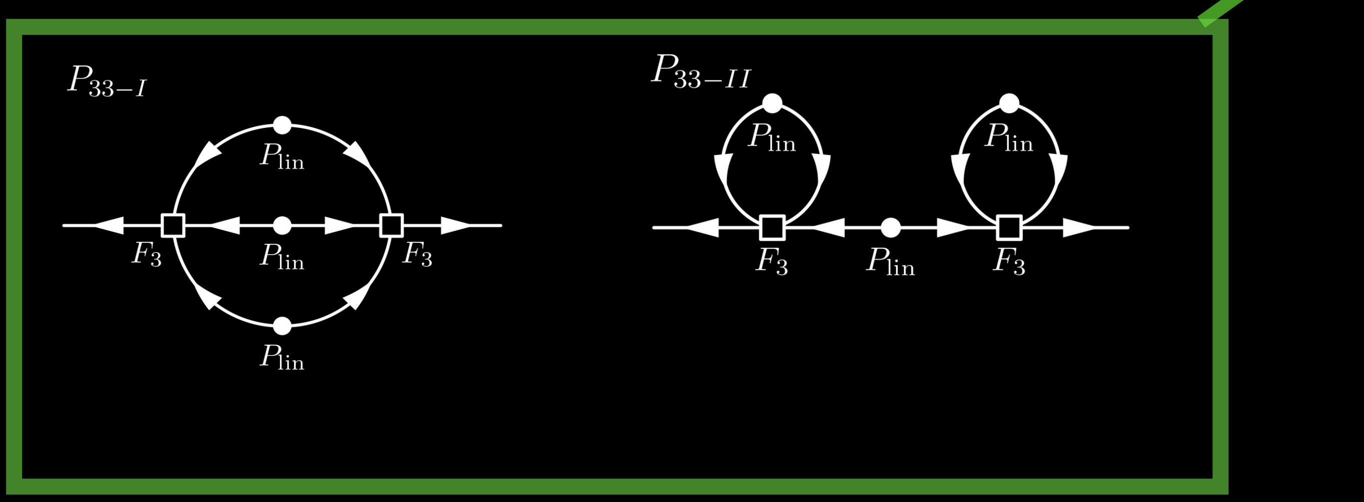
Two-loop master integrals are (I believe...)
intractable analytically. On the contrary,
they are especially simple with a direct integration in three-momentum space.

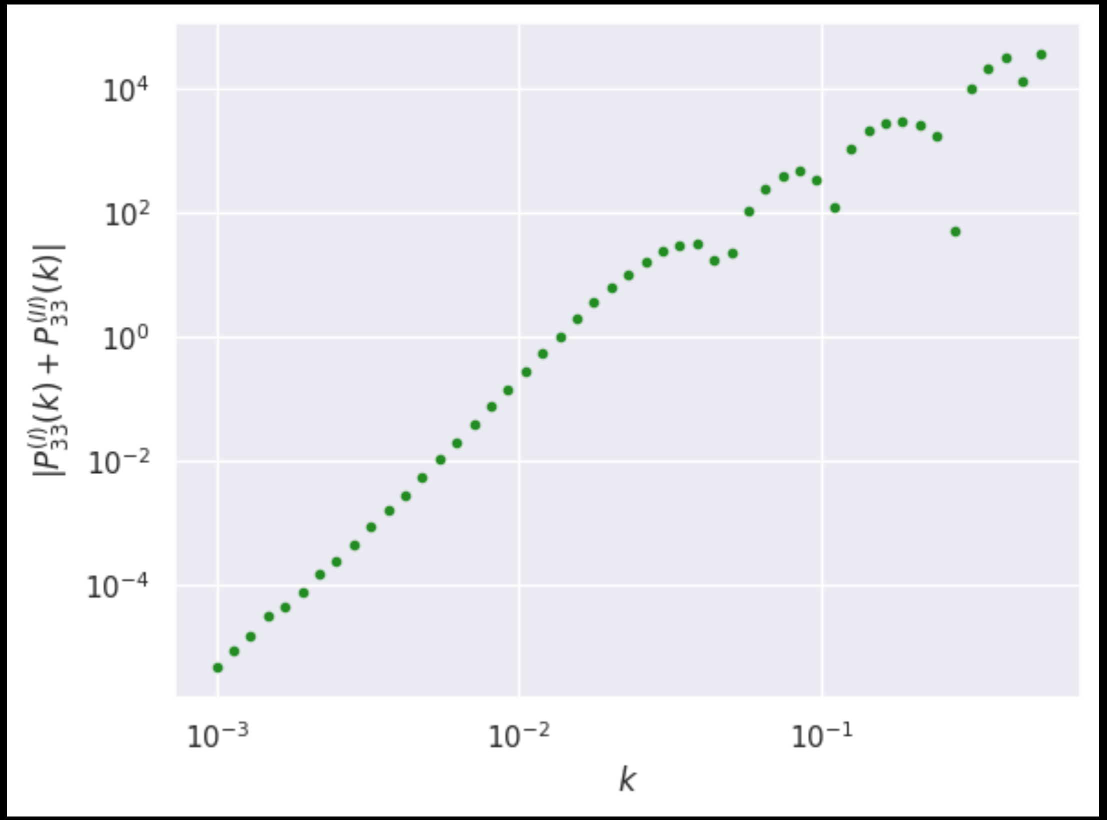
Partial results from numerical integration at two loops

The 2 Loops diagrams are:



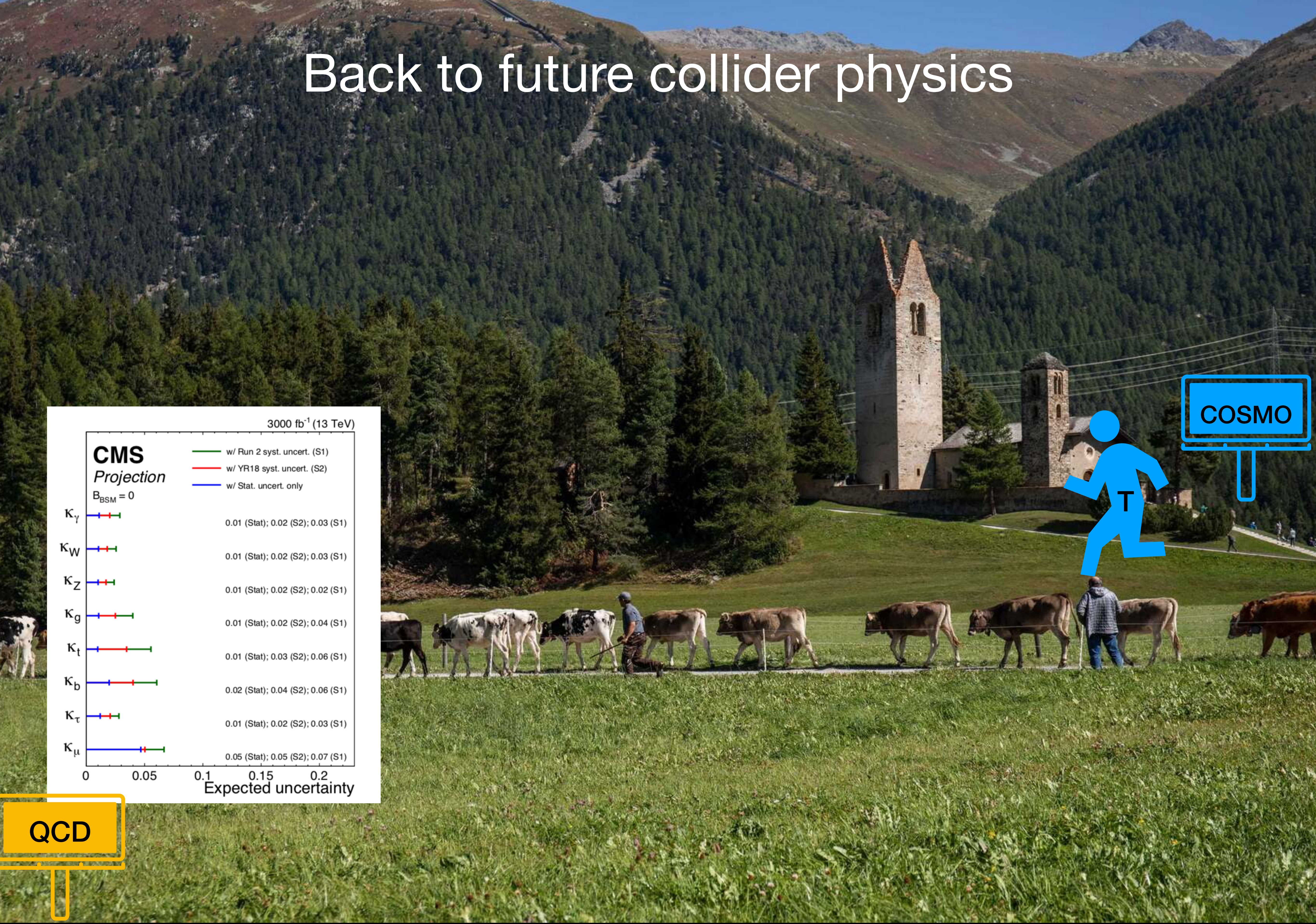






(preliminary) Andrea Favorito, et al

Back to future collider physics

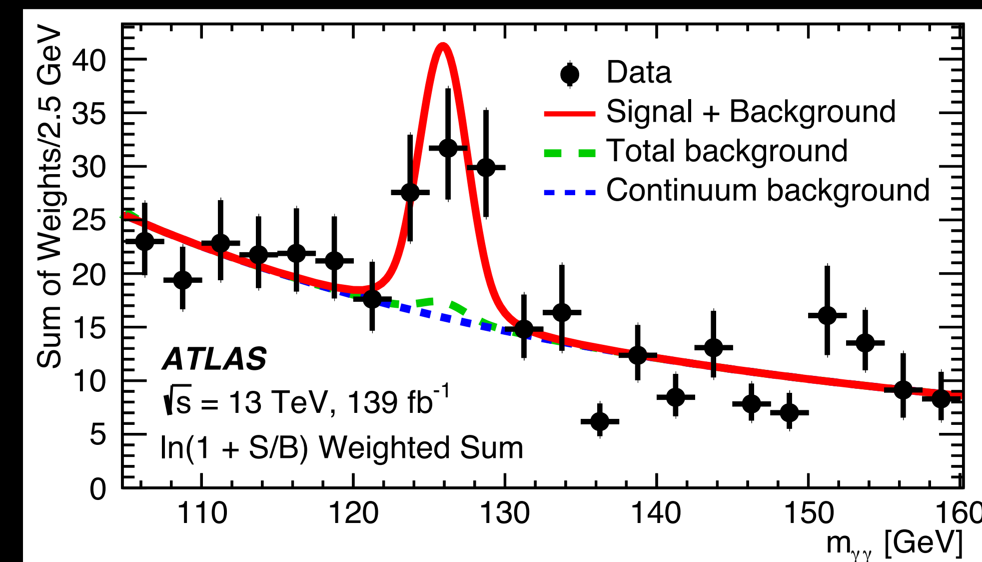


[source: Tourist Information, Engadin, Switzerland]

Experimental advances

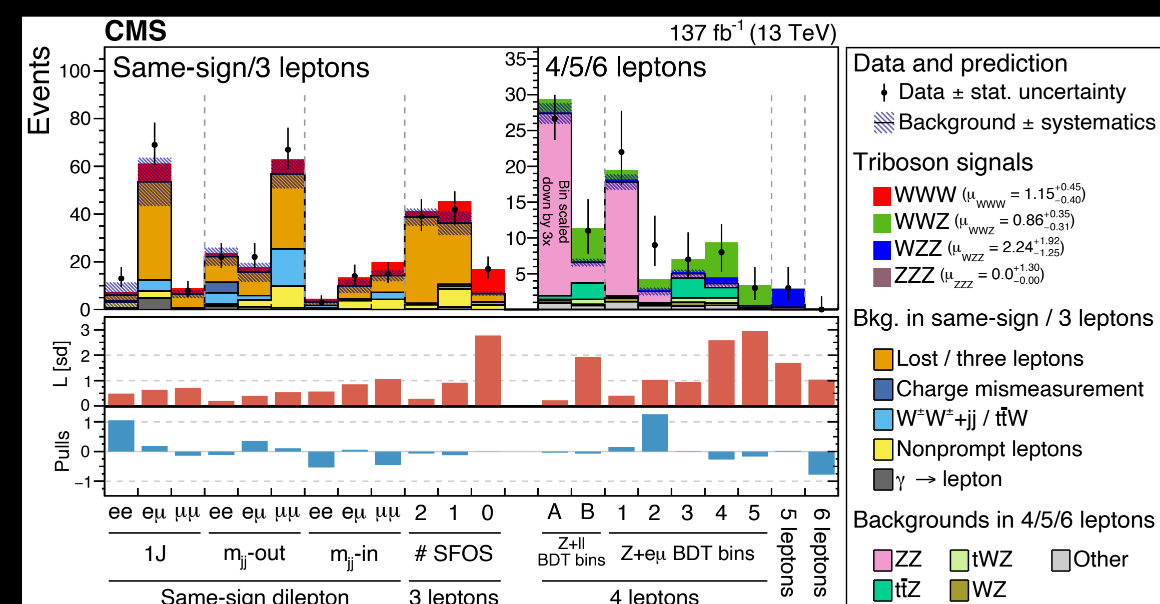
“Rare” LHC processes

$$\sigma\left[pp \to ttH\left(\to \gamma\gamma\right)\right]$$



- ATLAS and CMS observed
  1. triple weak gauge boson production
  2. Higgs production associated with top pairs.

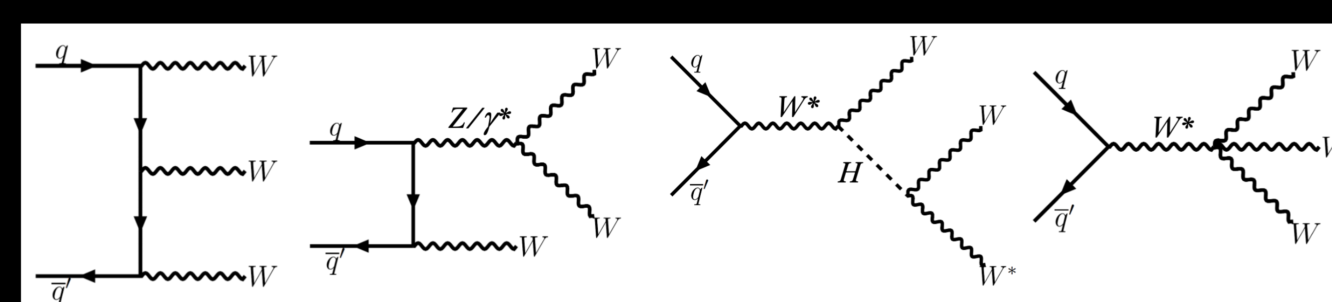
$$\sigma\left[pp \to VVV\right] \quad V = W, Z$$



- These processes are valuable for testing the electroweak sector of the Standard Model.

- With 10 times more data until the end of the LHC physics programme, they will be measure precisely.

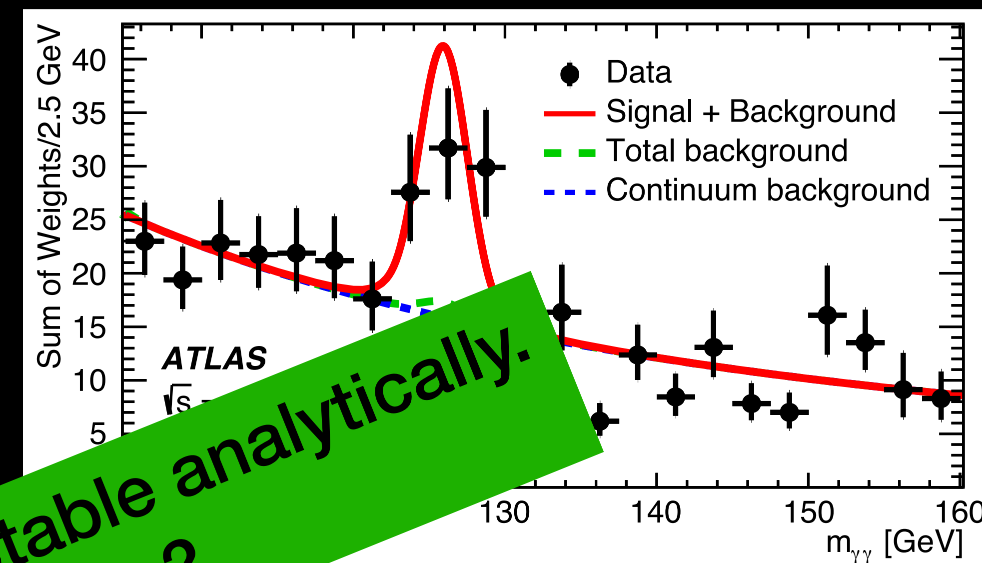
- Can we make predictions for such processes?



Experimental advances

“Rare” LHC processes

$$\sigma\left[pp \to ttH\left(\to \gamma\gamma\right)\right]$$



- ATLAS and CMS observed
  1. triple weak gauge boson production
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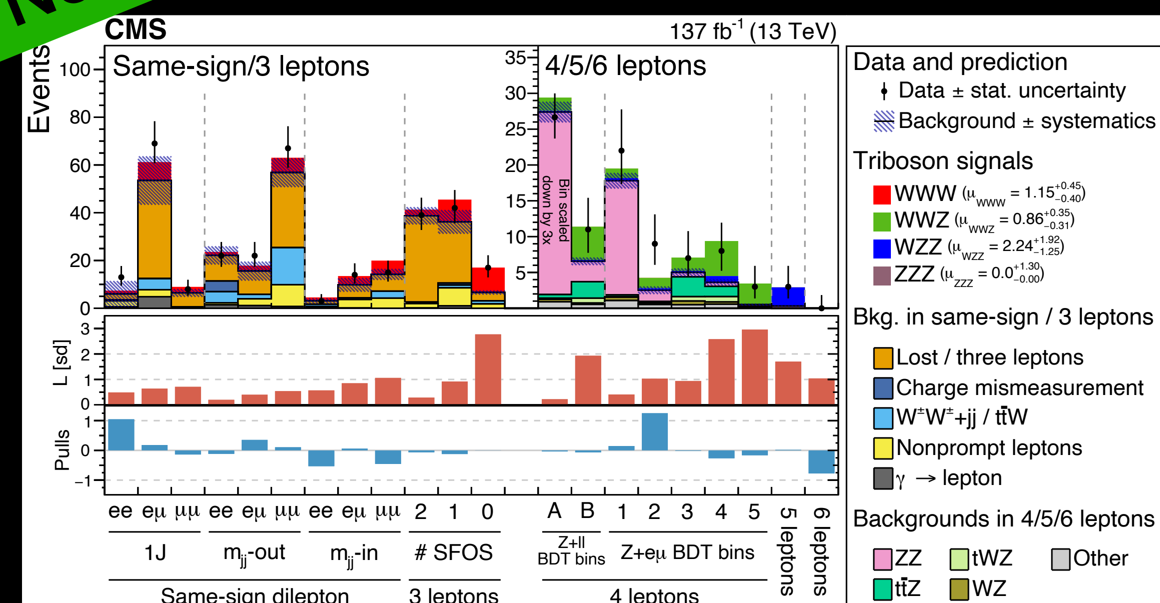
Two loop amplitudes are intractable analytically.
Challenging! New ideas?

- These processes are valu
  testing the electrow
  Standard Mod

$$\sigma\left[pp \to VVV\right] \quad V = W, Z$$



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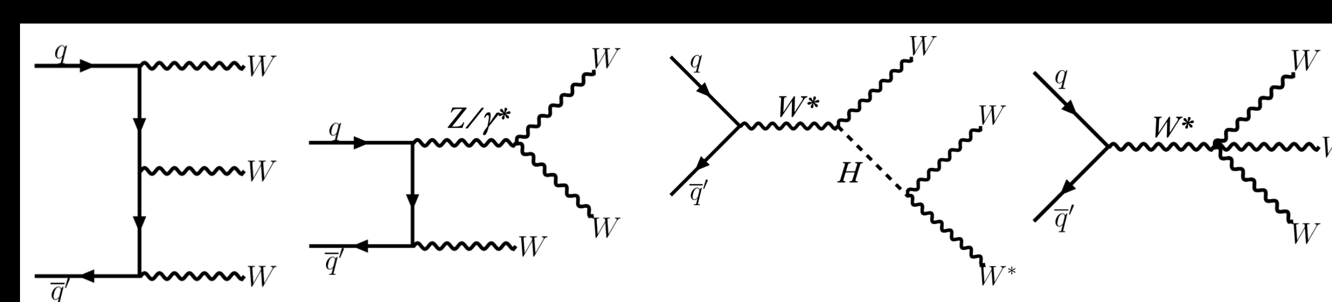


- Can we make predictions for such processes?

Two loop amplitudes with direct integration

- Two-loop amplitudes with direct integration over loop momenta?
- Number of integrals is SIX.
- … for all two-loop amplitudes and kinematic configurations.
- Understand fully the singular structure of QCD amplitudes at two loops.

$$A_2\left(\left\{p_{ext_i}\right\},\{M_i\}\right)$$

$$= \int d^dk \int d^dl \, \mathscr{A}_2\left(k, l, \left\{p_{ext_i}\right\}, \{M_i\}\right)$$

Monte-Carlo Integration?

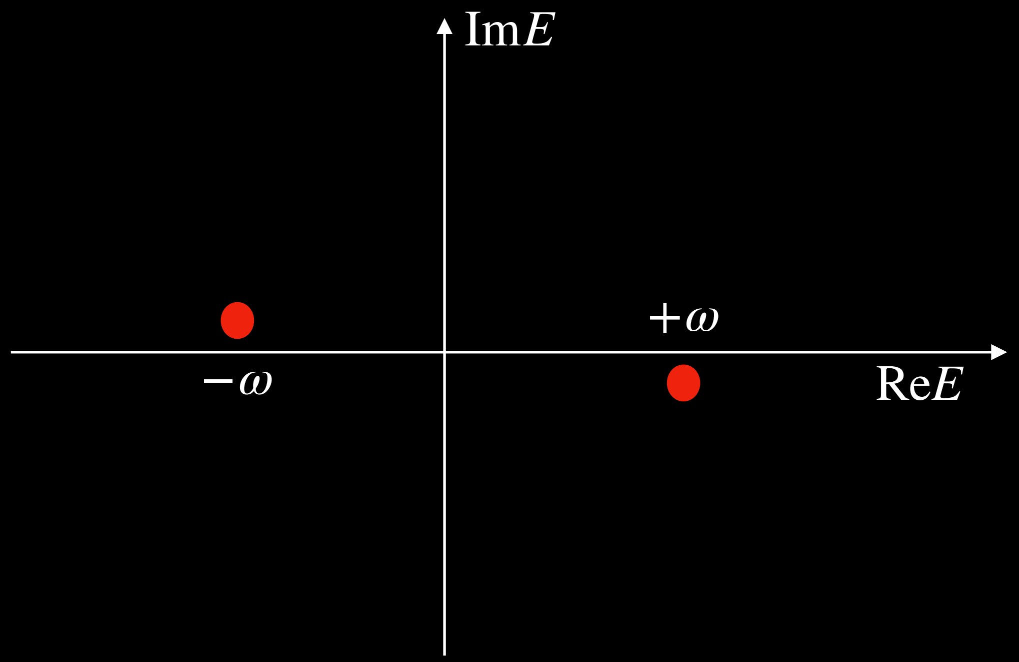
$$d \longrightarrow 4?$$

Singularities

Singularities of Feynman diagrams and scattering amplitudes

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega^2 + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.

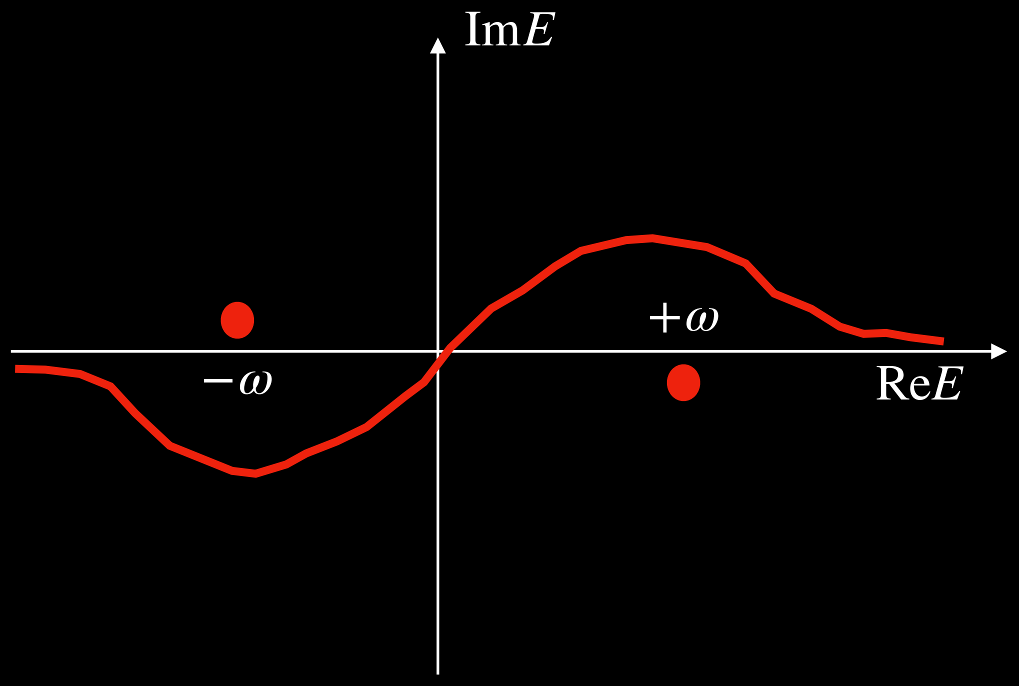


$\omega \to \omega - i\delta$ with $\delta \to 0$

Integrable Singularities

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega^2 + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.
- If we can deform the path of integration away from the poles, then they lead to no singularities

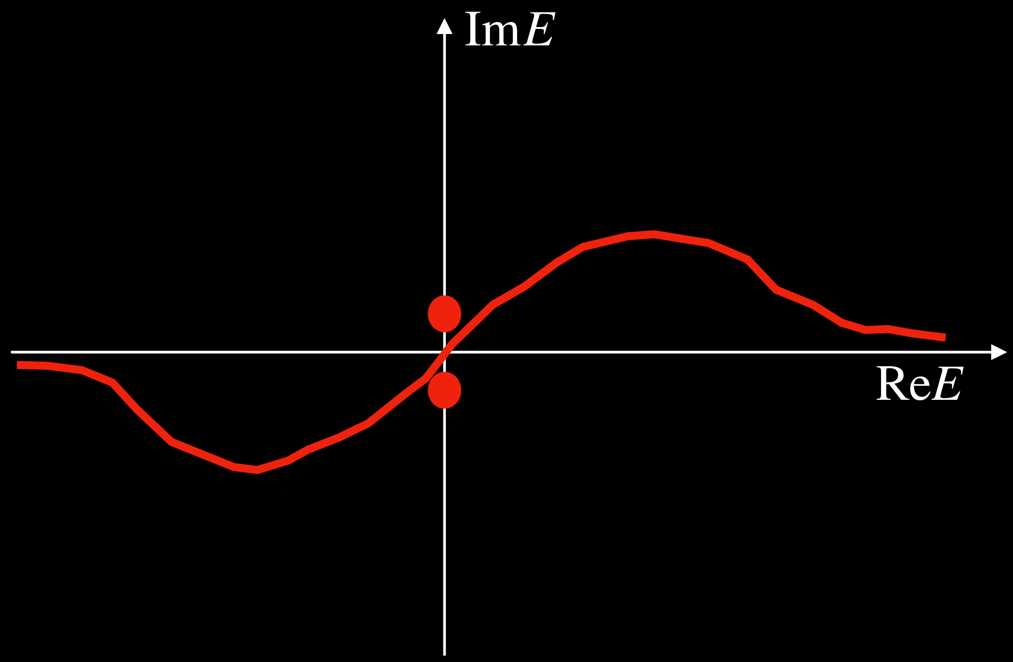


$\omega \to \omega - i\delta$ with $\delta \to 0$

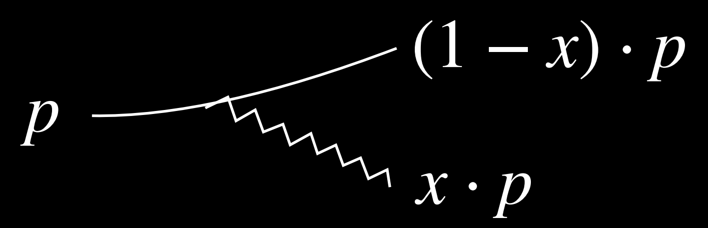
Soft massless particles

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{(E + i\delta)\,(E - i\delta)}$$

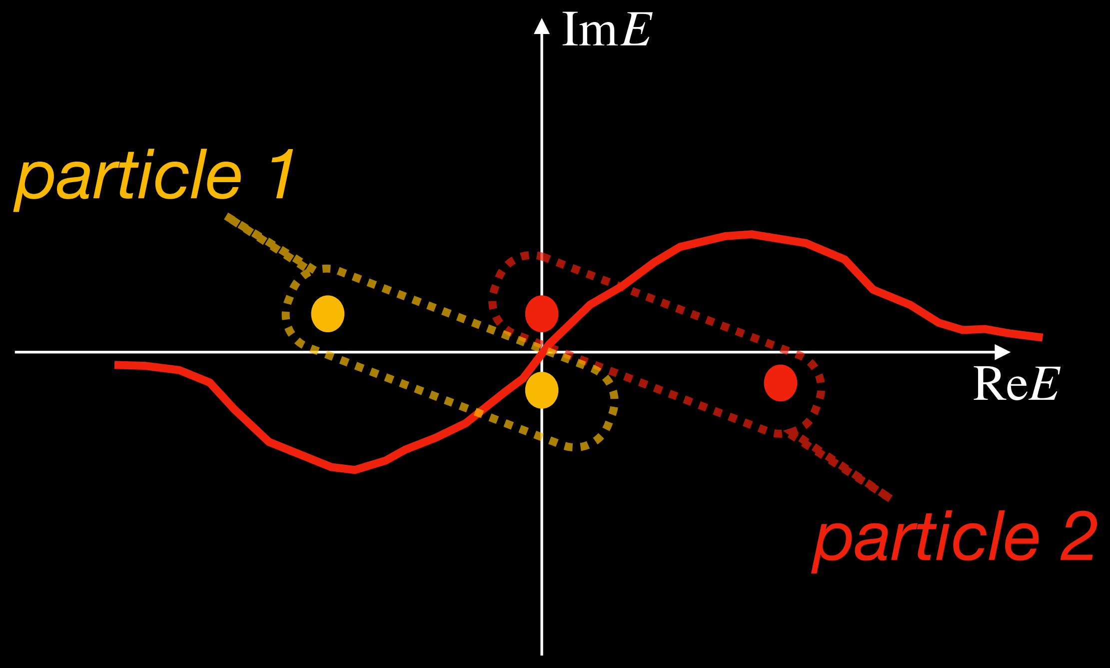
- Poles due to soft massless particles.
- These singularities pinch the integration path from both sides.
- Condition for a TRUE INFINITY



Collinear massless particles



- A second source of infinities due to massless collinear particles.
- A singularity of one particle in the lower half-plane lines up with the singularity of a collinear particle in the higher half-pane.
- The singularities pinch the integration path from both sides.
- We cannot deform the path, a condition for a TRUE INFINITY!



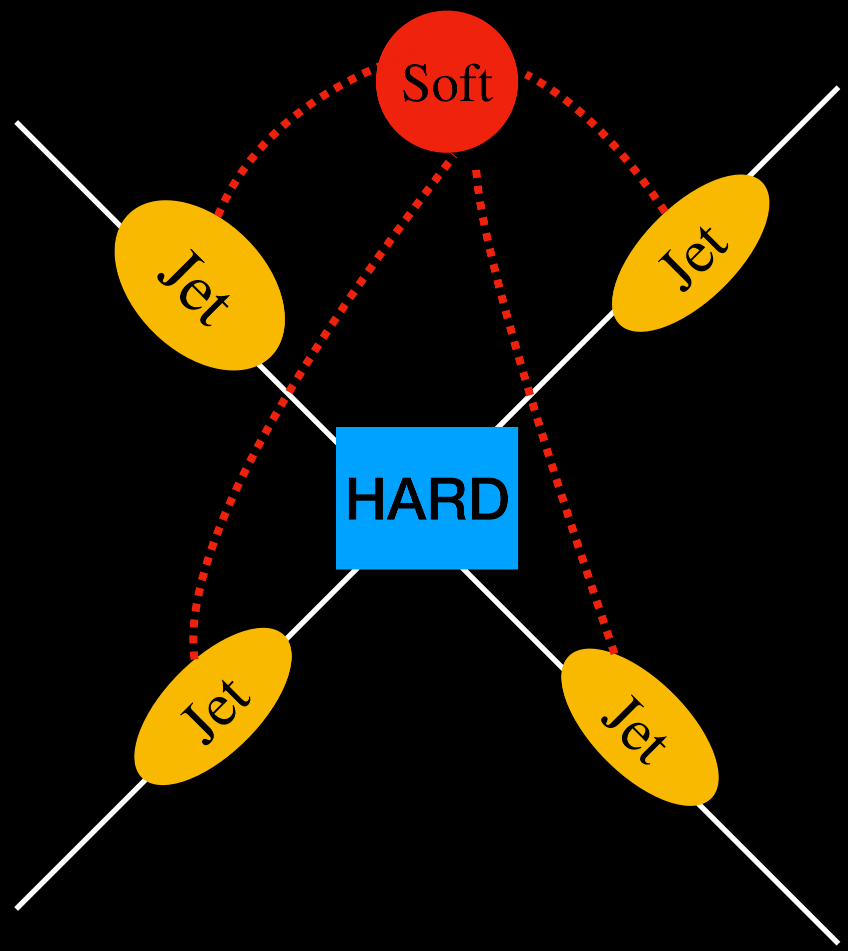
Infrared amplitude factorization

- UV Renormalized scattering amplitudes for well-separated final-states take a simple factorized form

$$Amplitude = \text{hard} \cdot \text{soft} \cdot \prod_i \text{Jet}_i \,.$$

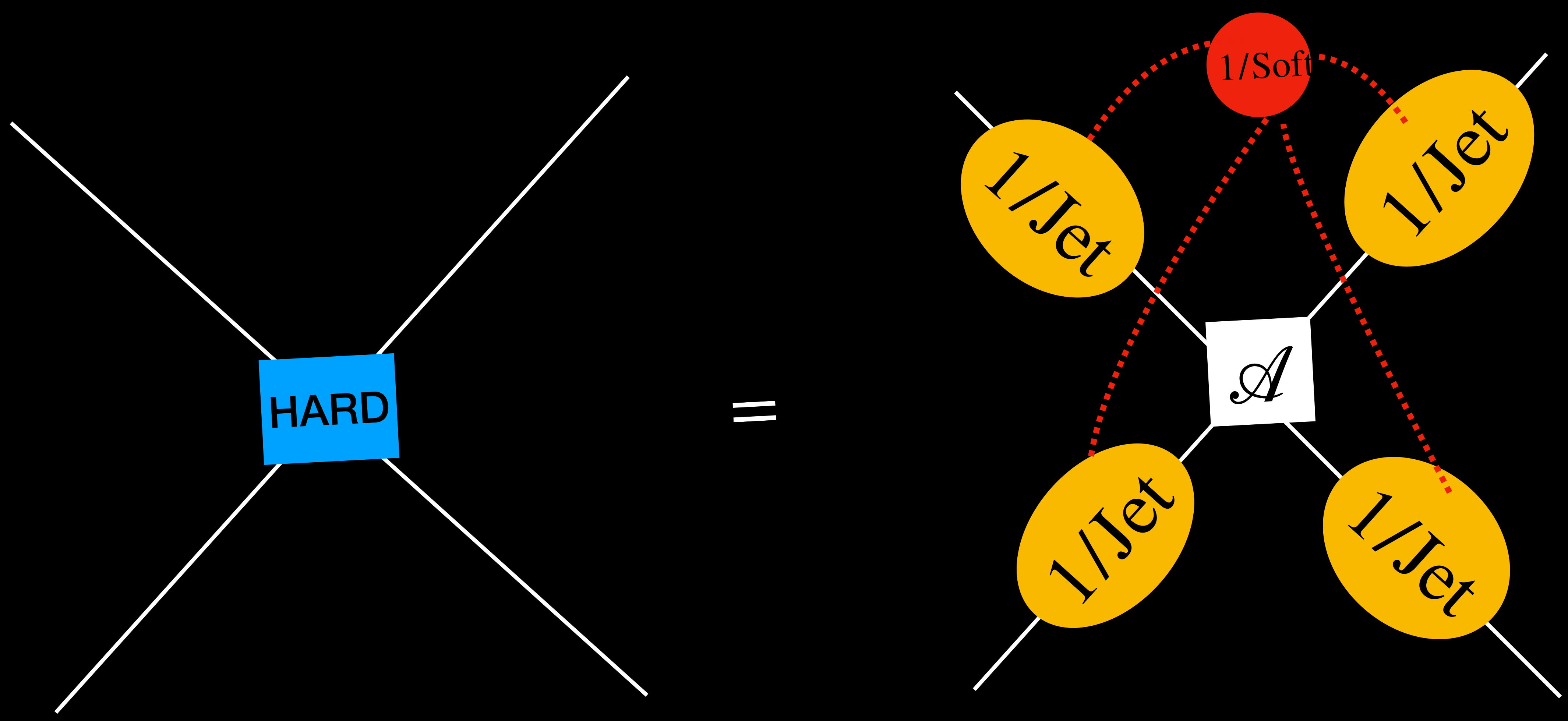
  - "soft" and "jet" functions contain all divergences.

- These are universal functions. For any new process we should need to compute only the "hard" function.
- So far, we do not have a way to compute the "hard" function directly.
- But, what if we did?



Ma;
Erdogan, Sterman;
Schwartz;
Collins

How could we imagine using factorisation?



An inverted factorization theorem

How could we imagine using factorization?

$$A = \int [dk]\, \mathcal{A}(k) = \int \mathcal{S} \prod_i \mathcal{J}_i \cdot \int [dk]\, \mathcal{A}(k) \cdot \mathcal{S}^{-1}(k) \cdot \prod_i \mathcal{J}_i^{-1}(k)$$

soft/collinear — Divergent

Hard — Finite

Analytic Integration in $D = 4 - 2\epsilon$, known to at least three-loops

Numerical integration in exactly $D = 4.$

Universal

Process dependent

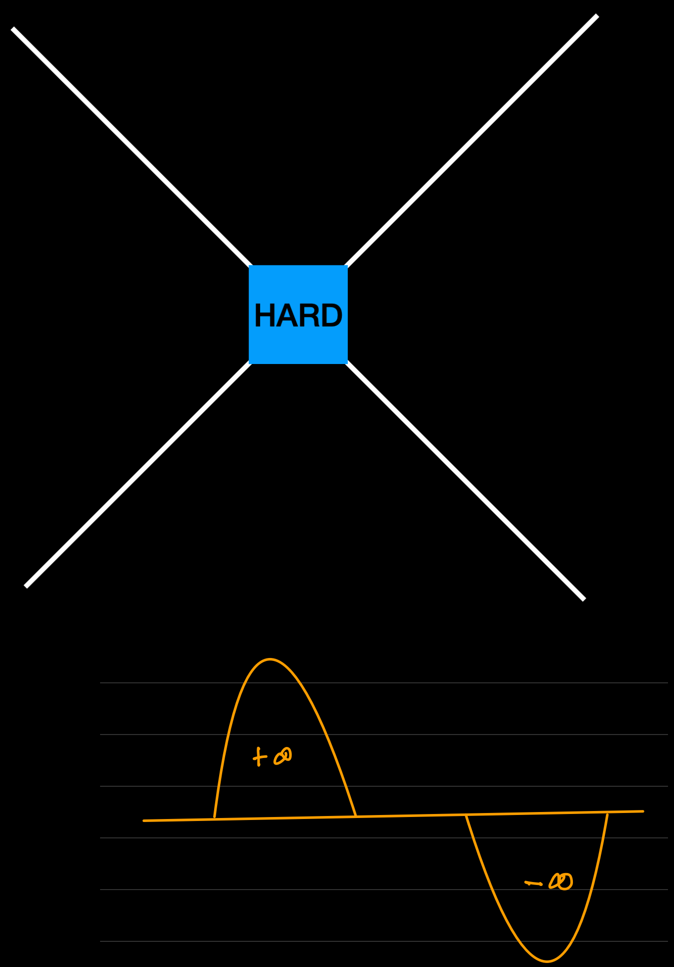
This procedure is universal…could be applied to any process, irrespectively of the complexity of its final state.

From factorisation we could identify, remove and integrate separately the singular parts of amplitudes order by order in perturbation theory:

$$\mathcal{H}^{(0)} = \mathcal{A}^{(0)} \qquad \mathcal{H}^{(1)} = \mathcal{A}^{(1)} - \mathcal{J}^{(1)}\mathcal{H}^{(0)} - \mathcal{S}^{(1)}\mathcal{H}^{(0)} \qquad \mathcal{H}^{(2)} = \mathcal{A}^{(2)} - \mathcal{J}^{(1)}\mathcal{H}^{(1)} - \mathcal{S}^{(1)}\mathcal{H}^{(1)} - \mathcal{J}^{(2)}\mathcal{H}^{(0)} - \mathcal{S}^{(2)}\mathcal{H}^{(0)} + \mathcal{J}^{(1)}\mathcal{S}^{(1)}\mathcal{H}^{(0)} \quad .$$

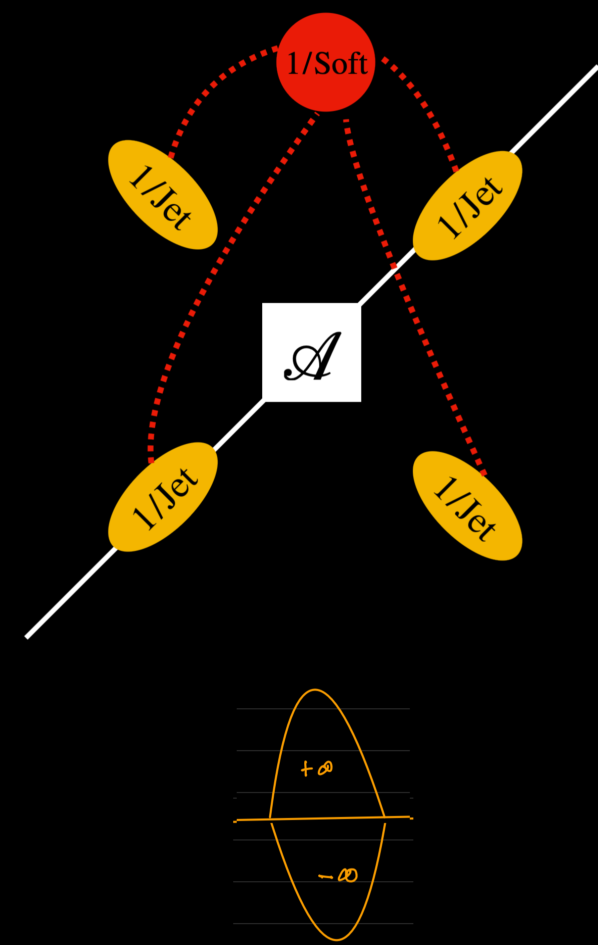
Factorisation and locality

Is it an obstacle for a meaningful invertion of the factorization theorem?



Non-local cancellations

=

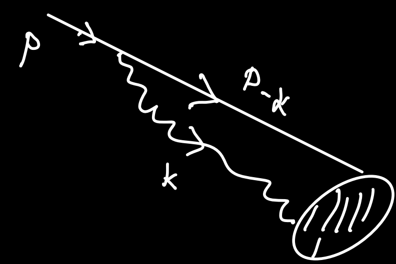


Local cancellations
Numerically integrable

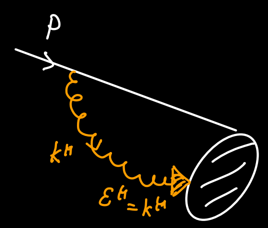
- In the integral expression of the process dependent “HARD” function, we need singularities to be cancelled locally, AT THE INTEGRAND.
- A naive construction leads to non-local cancellations.
- Integrands with non-local cancellations cannot be integrated numerically.
- To enable Monte-Carlo integration methods, can we ensure that ALL soft, collinear and ultraviolet singularities cancel point by point in the integrand?
- A challenge!

Ingredients of factorization

- Collinear gluons acquire longitudinal (non-physical) polarisations.
- Gauge symmetry and the Ward identities derived from it, guarantee that contributions from unphysical gluons almost cancel…
- … leaving a factored correction to external legs

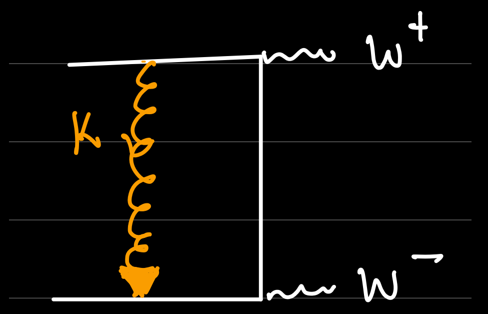


$$= \ldots \frac{(p-k)\gamma^\mu u(p)}{k^2(k+p)^2} \xrightarrow{k = x\,p, \quad x = \frac{k\cdot\eta}{p\cdot\eta}} \ldots u(p)\frac{2(1-x)}{x}\frac{k^\mu}{k^2(k+p)^2}$$

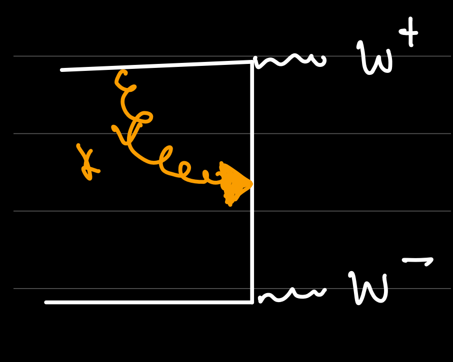


The collinear gluon carries a longitudinal polarization.

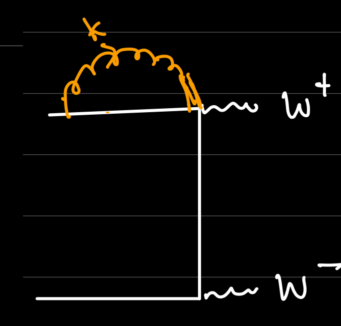
$$\lim_{k\|p_1} \mathscr{A}^{(1)} =$$



$+$



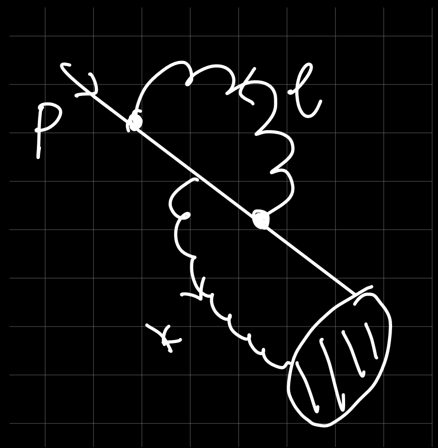
$= \; -$



Collinear limit is a self-energy correction to an external state. Factorized.

Ingredients of factorization are “almost” local!

- Collinear gluons off one-loop vertices acquire random polarisations.

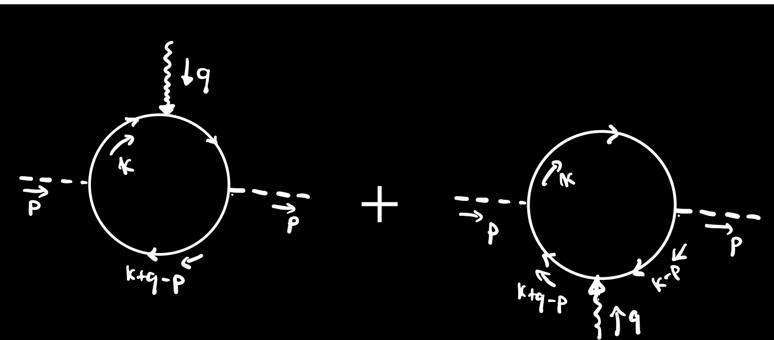


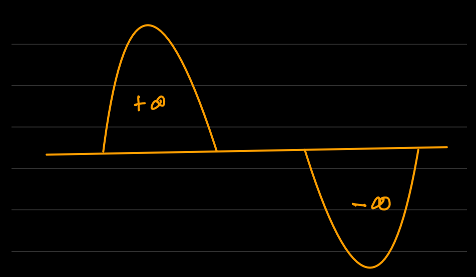
$$\xrightarrow[l \sim hard]{k \to x p} \frac{\ldots k^\mu + \ldots l^\mu}{k^2(k+p)^2 l^2 (l+p)^2 (k+l+p)^2}$$

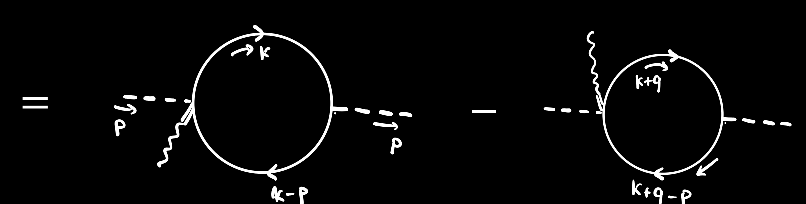
Longitudinal polarization

Loop polarization

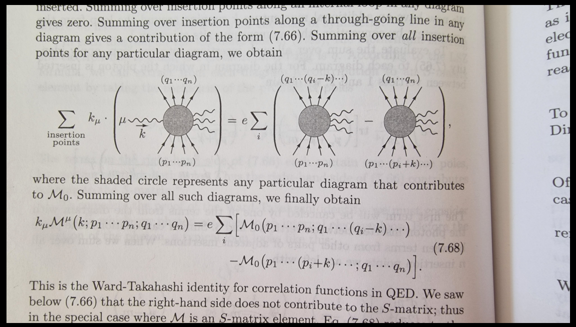
- Ward identities generate non-local zeros.







$\overset{\int}{=} 0 .$


gives zero. Summing over insertion points along a through-going line in any diagram gives a contribution of the form (7.66). Summing over *all* insertion points for any particular diagram, we obtain

$$\sum_{\substack{\text{insertion}\\ \text{points}}} k_\mu \cdot \left(\ldots \right) = e \sum_i \left(\ldots - \ldots \right),$$

where the shaded circle represents any particular diagram that contributes to $\mathcal{M}_0$. Summing over all such diagrams, we finally obtain

$$k_\mu \mathcal{M}^\mu(k; p_1 \cdots p_n; q_1 \cdots q_n) = e \sum_i \Big[\mathcal{M}_0(p_1 \cdots p_n; q_1 \cdots (q_i - k) \cdots) - \mathcal{M}_0(p_1 \cdots (p_i + k) \cdots; q_1 \cdots q_n) \Big]. \quad (7.68)$$

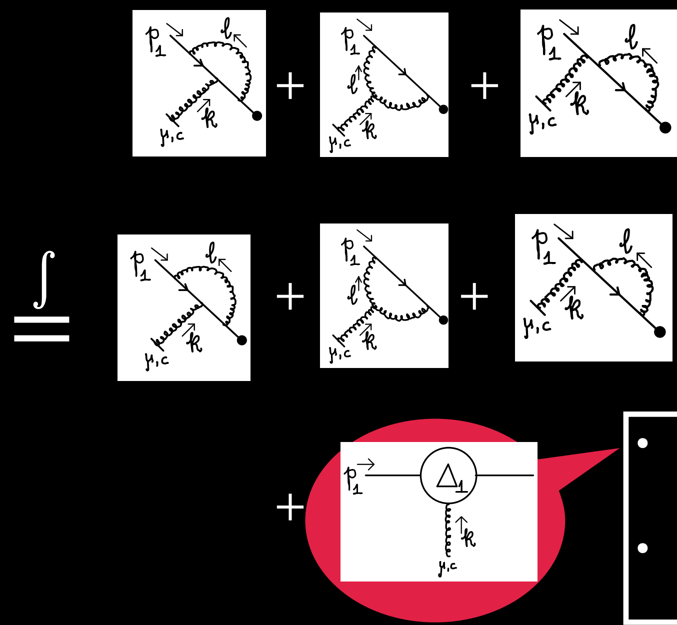
This is the Ward-Takahashi identity for correlation functions in QED. We saw below (7.66) that the right-hand side does not contribute to the S-matrix; thus in the special case where $\mathcal{M}$ is an S-matrix element,

...from Peskin and Schroeder

Factorization at the integrand

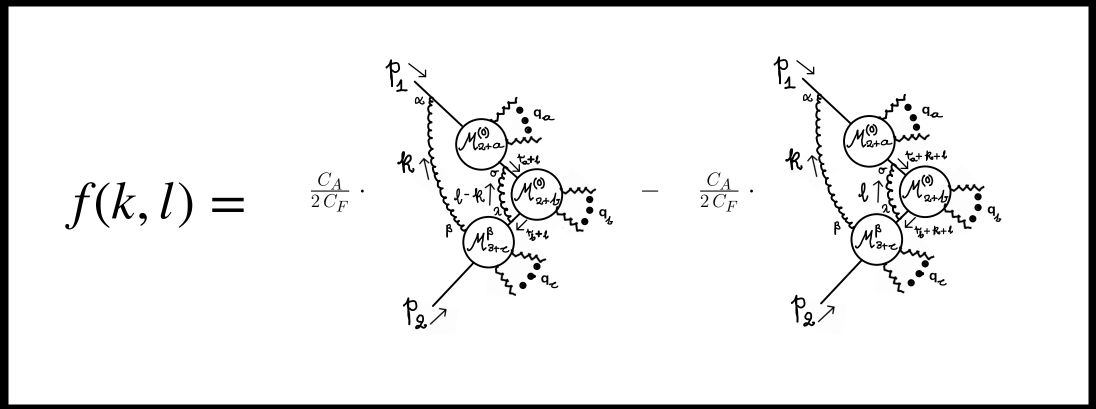
Amplitude construction

- Assign correlated momentum flows to all diagrams.
- Cure "loop polarizations" with additional vertices at external legs.
- Cure non-local remnants of Ward identities with "shift counterterms" which integrate to zero
- Locally subtract ultraviolet singularities respecting Ward identities and the above integrand modifications.



- Integrates to zero
- Eliminates loop polarizations

$$\mathscr{A}^{(2)} \to \mathscr{A}^{(2)} + f(k,l) \quad \textbf{with} \int d^dk\,d^dl\, f(k,l) = 0.$$



- Integrates to zero
- Removes non-local remnants of Ward Identities

Locally finite integrands for a class of two-loop QCD amplitudes (gluon fusion)

$$g + g \to V_1 + V_2 + \ldots V_n\,, \quad V_i = Higgs, W, Z, \gamma^*$$ ✔

$$\mathcal{M}^{(2)}_{n,\text{finite}} = \mathcal{M}^{(2)}_{n,\,\text{UV-finite}} - \frac{1}{2}\mathcal{F}^{(1)}_{ss,\,\text{UV-finite}}\left(\widetilde{\mathcal{M}}^{(1)}_{n,\,\text{finite}}(l) + \widetilde{\mathcal{M}}^{(1)}_{n,\,\text{finite}}(l+k)\right)$$

CA, Julia Karlen, George Sterman, Ani Venkata (to appear)

Locally finite integrands for a class of two-loop QCD amplitudes (quark fusion)

$$q + \bar{q} \to V_1 + V_2 + \ldots V_n\,, \quad V_i = W, Z, \gamma^*$$ ✔

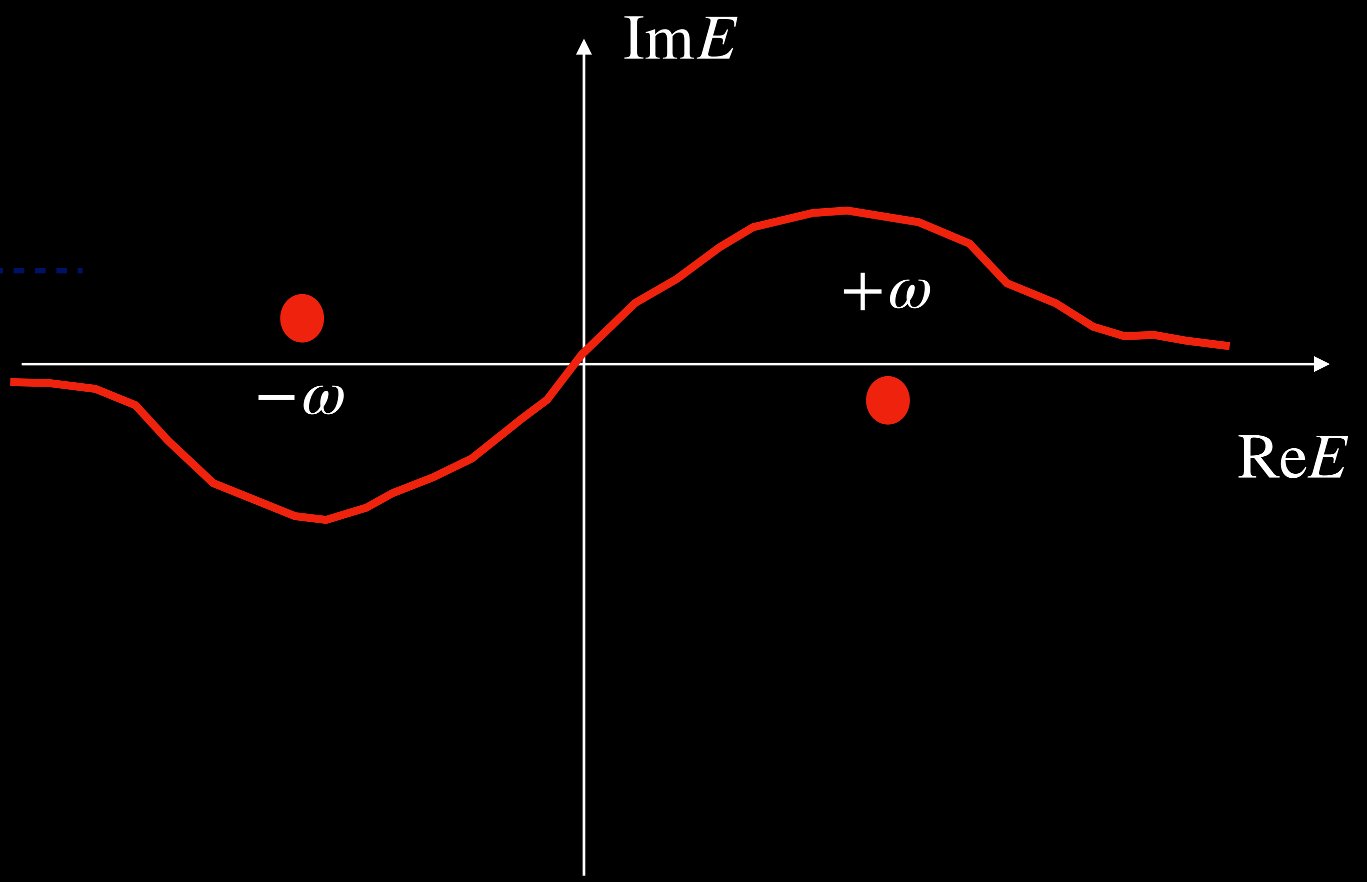
CA, George Sterman

$$\mathscr{H}_{1-loop}(k) = \mathscr{A}_{1-loop} - \mathscr{F}^{(1)}\left[\mathscr{A}_0\right]$$

$$\mathscr{H}_{2-loop}(k,l) = \mathscr{A}_{2-loop} - \mathscr{F}^{(2)}\left[\mathscr{A}_0\right] - \mathscr{F}^{(1)}\left[\mathscr{H}_{1-loop}\right]$$

Numerical integration

- Can such IR subtractions be used for evaluating loop amplitudes numerically?
- They are an important ingredient! They remove “pinch” singularities.
- Other singularities which can be avoided with appropriate contour-deformations are equally important.
- Breakthroughs and excellent ideas.



Numerical integration of $\mathscr{A}^{1-loop}_{q\bar{q}\to\gamma\gamma\gamma}$

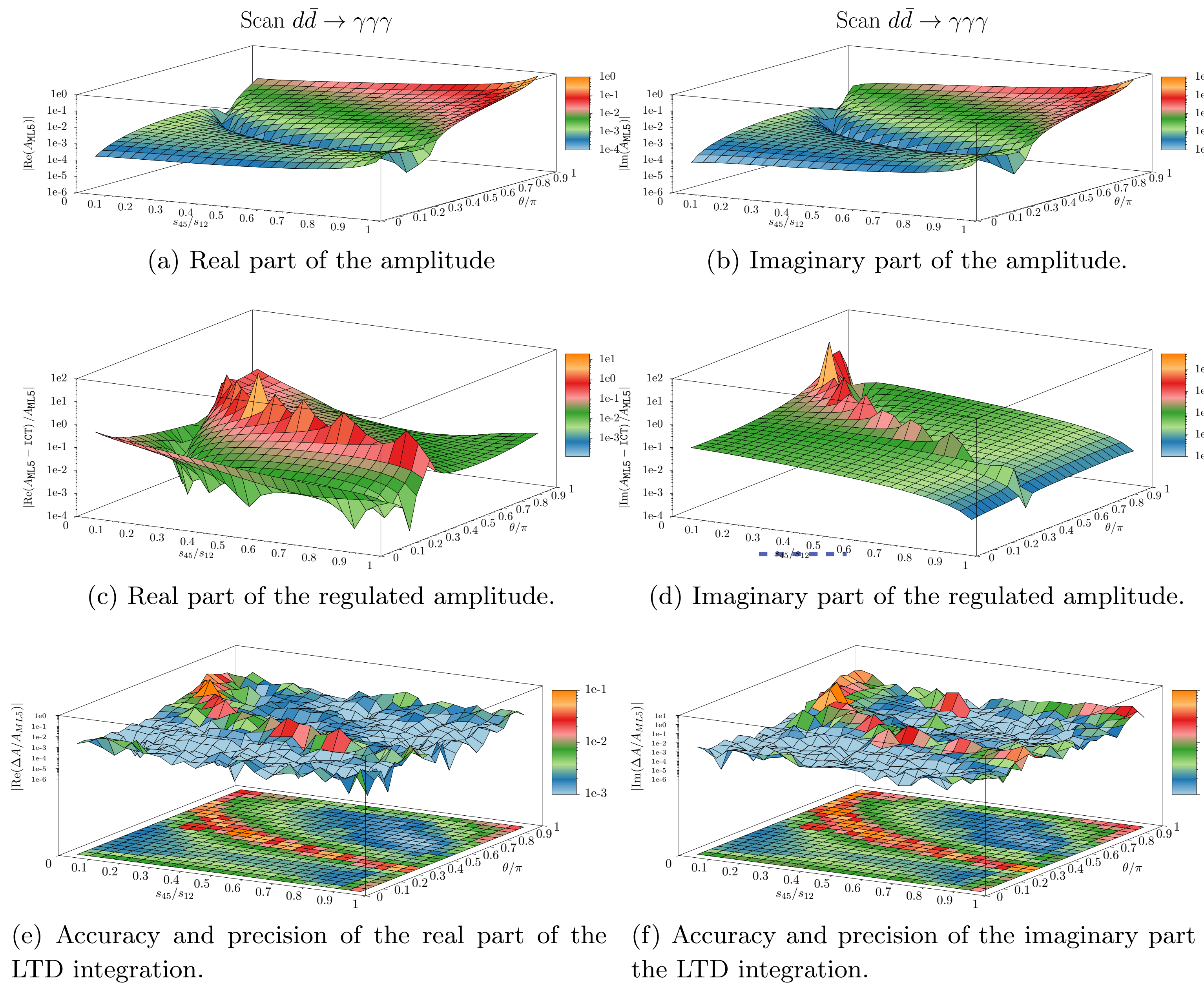


Figure 23: A scan for $d\bar{d} \to \gamma_1\gamma_2\gamma_3$. The results are absolute values plotted on a log scale The first row (a – b) shows the real and the imaginary part of the amplitude computed wit ML5. The second row (c – d) shows the relative difference between the analytic expressio and the integrated counterterms. The last row (e – f) shows the LTD integration. They ar a combination of two plots: the surface above shows the relative error of the central valu compared with the analytic expression, the flat surface below shows the Monte Carlo erro for the point right above.

With a novel contour deformation method

Capatti, Hirschi, Kermaschah, Pelloni, Ruijl [1912.0929]

New ideas and schemes for numerical integration

Exposing the threshold structure of loop integrals

Zeno Capatti*
Institute for Theoretical Physics, ETH Zürich,
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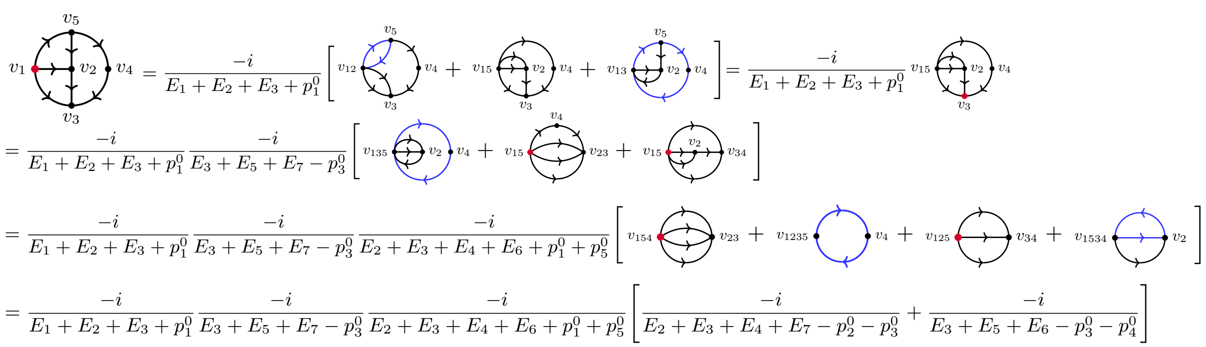
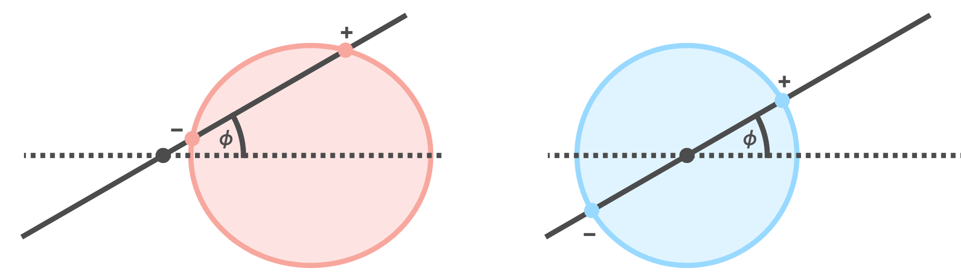
(Dated: November 18, 2022)



Numerical integration of loop integrals through local cancellation of threshold singularities

D. Kermanschah
ETH Zürich,
Rämistrasse 101, 8092 Zürich, Switzerland



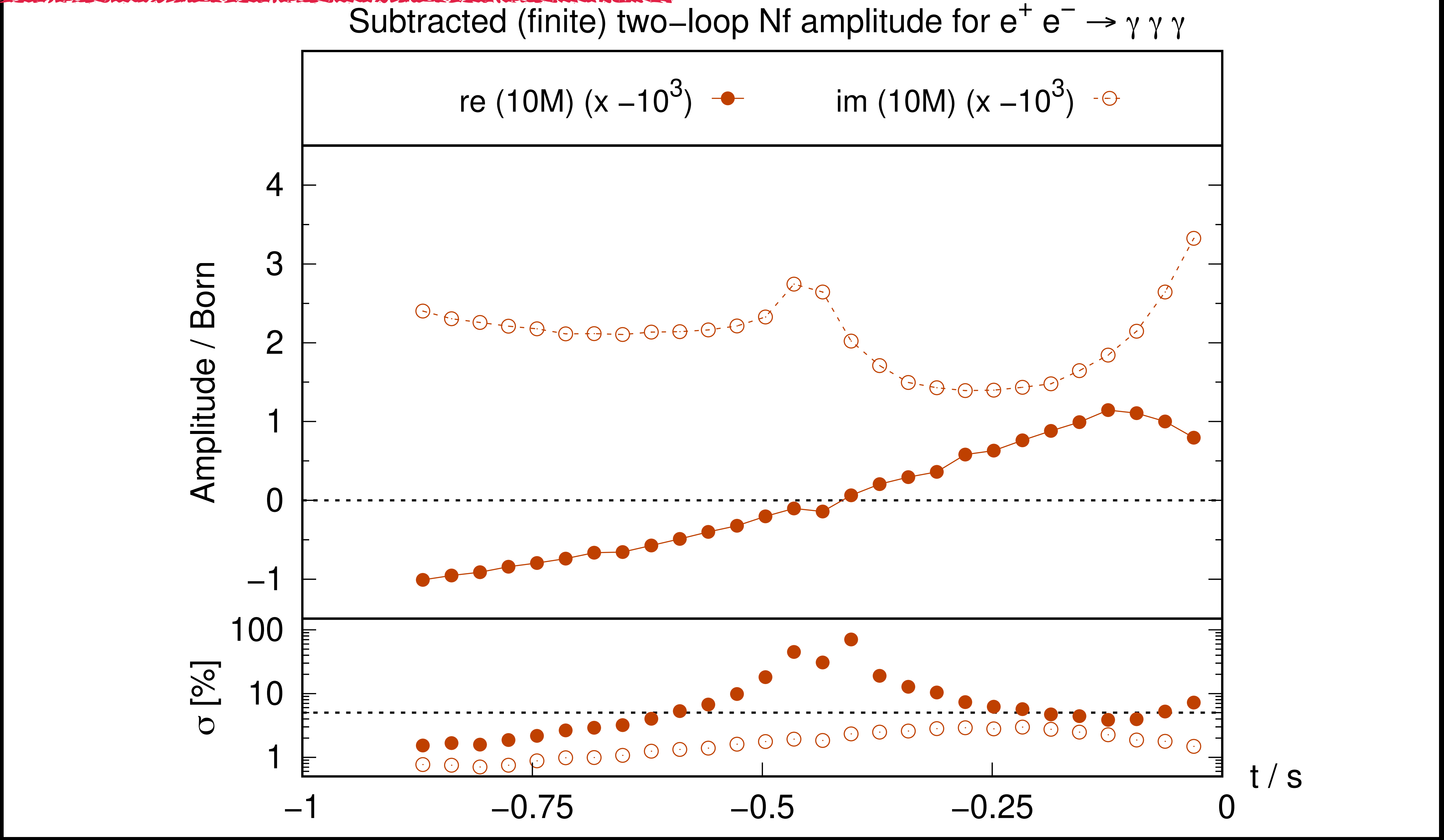
Integrates out the energy component of the loop momenta. An alternative to Time Ordered Perturbation Theory or Loop Tree Duality. Avoiding the introduction of spurious singularities

Devises counterterms to subtract integrable singularities from cuts/thresholds.
A shift of paradigm away from "contour deformation".

Numerical integration of $\mathscr{A}^{2-loop,N_f}_{q\bar{q}\to\gamma\gamma\gamma}$

With a novel threshold subtraction method

Very Very Preliminary!!!



Kermanshah, Lazopoulos, Vicini

Conclusions

- We have witnessed rapid progress in perturbative QCD, matching the precision of the LHC experiments.
- Perturbative QCD methods find application to other areas of physics.
- New formalism, utilising perturbative QCD methods, for computing correlators in the EFT of Large Scale Structure.
- Can we keep up improving precision? A need to keep reinventing our field and understanding perturbation theory at deeper levels.
- Infrared Factorization can turn into a new computational method for next generation problems in precision collider physics.