Unified Halo-Independent Formalism for Direct Detection Experiments

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Based on JCAP12(2017)039, in collaboration with G. Gelmini, J.H. Huh

Statistical Motivation

Statistics:

- Given a dataset, we are interested in the determining viability of models, preferred parameters of these models, making inferences about a model, etc.

Use of likelihood dependent on model parameters

$$\mathcal{L}(\vec{\Theta}) = P(\vec{y}|\vec{\Theta})$$

$$\vec{\Theta} = (x_1, x_2, \cdots, x_n)$$

(model parameters)

But what happens if you model is a function of an infinite set of parameters?

Require simplifications, approximations, or tricks (or perhaps a very expensive computer and an apathetic attitude towards error)

Here, we study the case where observable is linear in unknown function

$$\mathcal{O} \propto f(x)$$

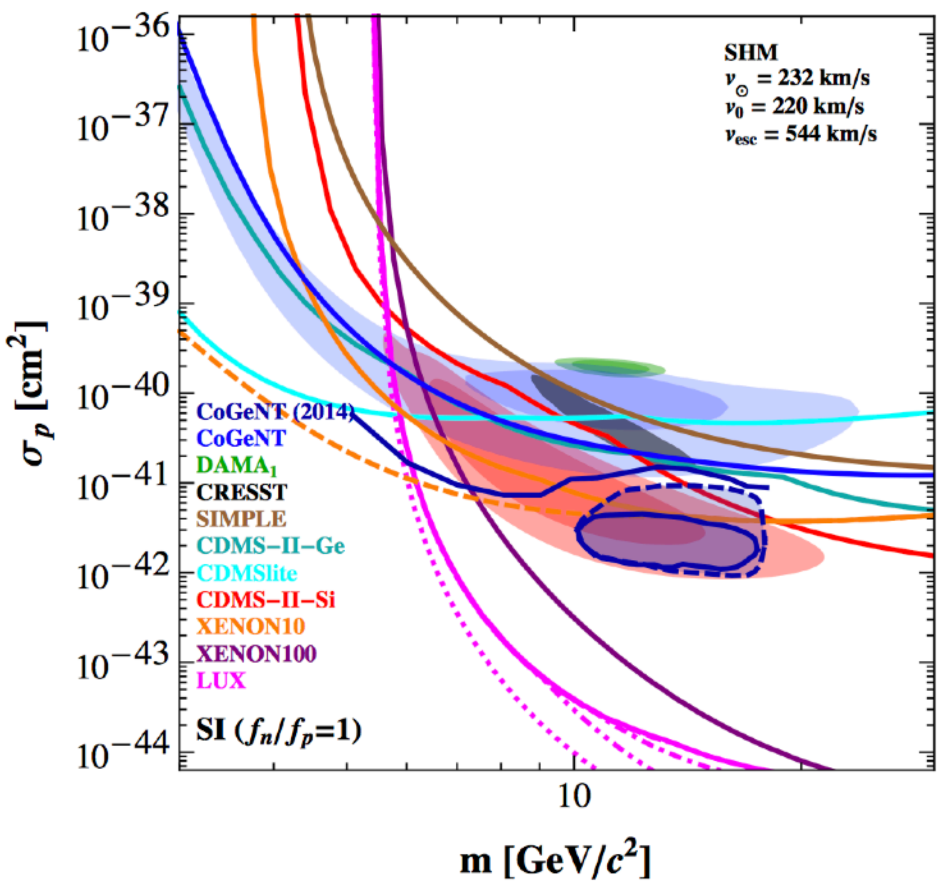
infinite parameter space

$$f(x) = \sum_{i=-\infty}^{\infty} c_i \, \delta(x - x_i)$$

Use tricks to show that, in parameter space of interest, f(x) takes on simplified form

Direct Detection Circa 2013

arXiv: 1311.4247



Various dark matter 'hints' juxtaposed against strong upper limits

- DAMA/LIBRA (~9 sigma annual modulation)
- CDMS-II-Si (~3 sigma scattering rate)
- CRESST (~4 sigma scattering rate)
- CoGeNT (~2 sigma annual modulation & scattering rate)

Viability of a given signal dependent upon various assumptions

$$\frac{dR}{dE_{\rm R}} = \frac{\rho_\chi C_T}{m_\chi m_T} \int_{v \geq v_{\rm min}(E_{\rm R})} d^3v\, f(\vec{v},t)\, v\, \frac{d\sigma_T}{dE_{\rm R}}(E_{\rm R},\vec{v})$$

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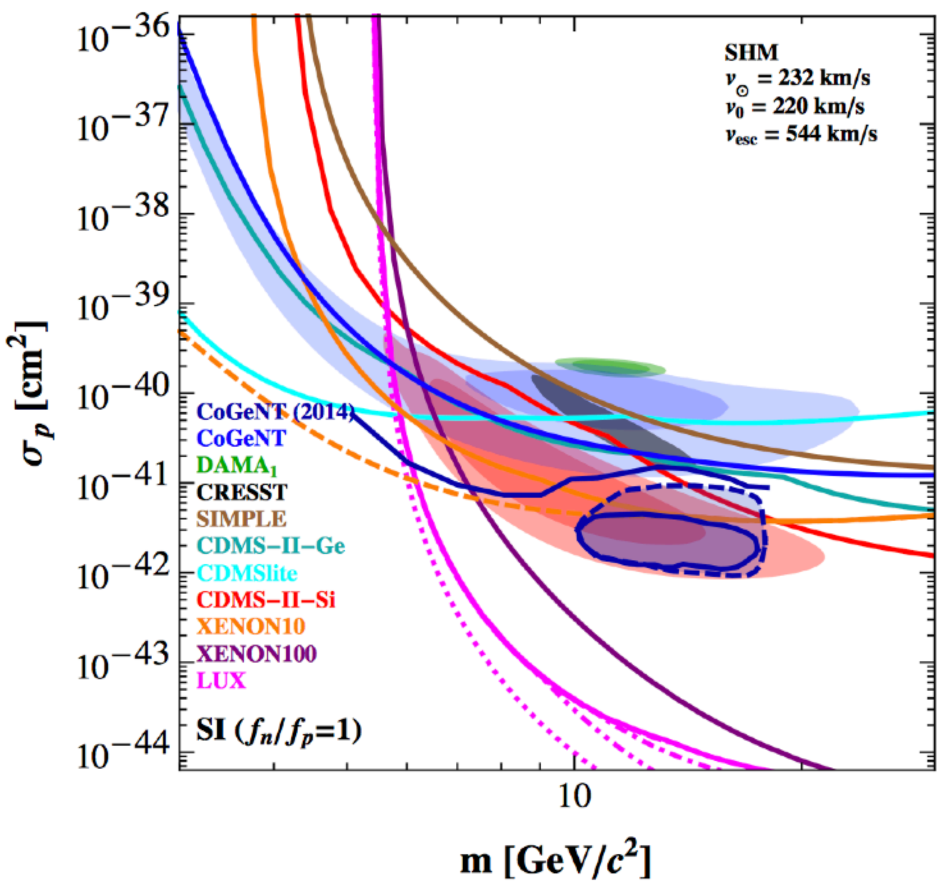

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Astrophysics

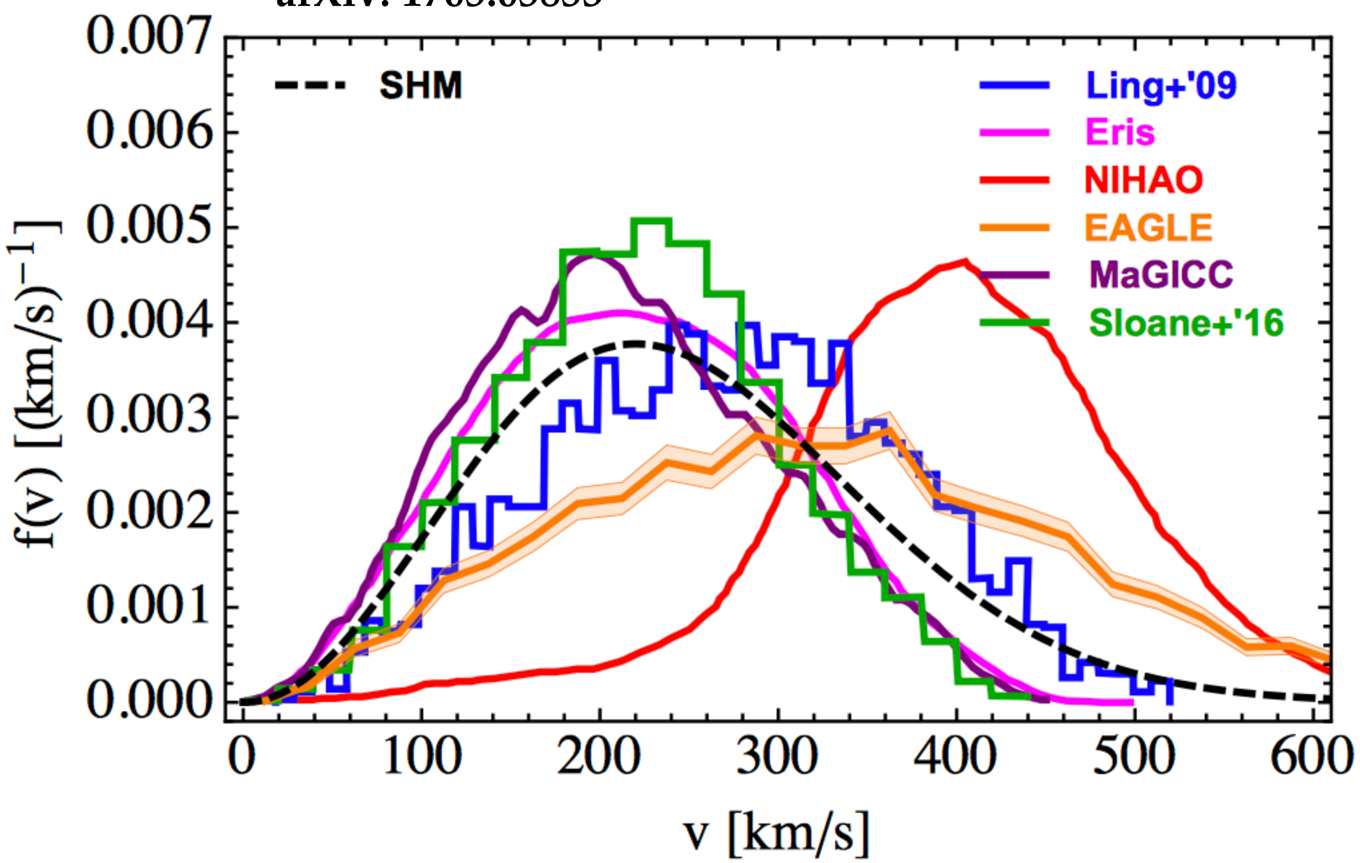
- Local dark matter density
- Dark matter velocity distribution

Particle Physics

- SI, SD, Magnetic (Electric) Dipole, etc.
- Proton/neutron couplings
- Scattering kinematics

Astrophysical Uncertainties

arXiv: 1705.05853



Much of what we know comes from simulations

Most problematic when experiments probe the tail of the distribution

- E.g. light WIMPs, inelastic scattering, etc

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Experiments sensitive to v > v_min(Target, DM mass)

Considering different halo functions (i.e. f(v)) can alter the sensitivity of an experiment by orders of magnitude…

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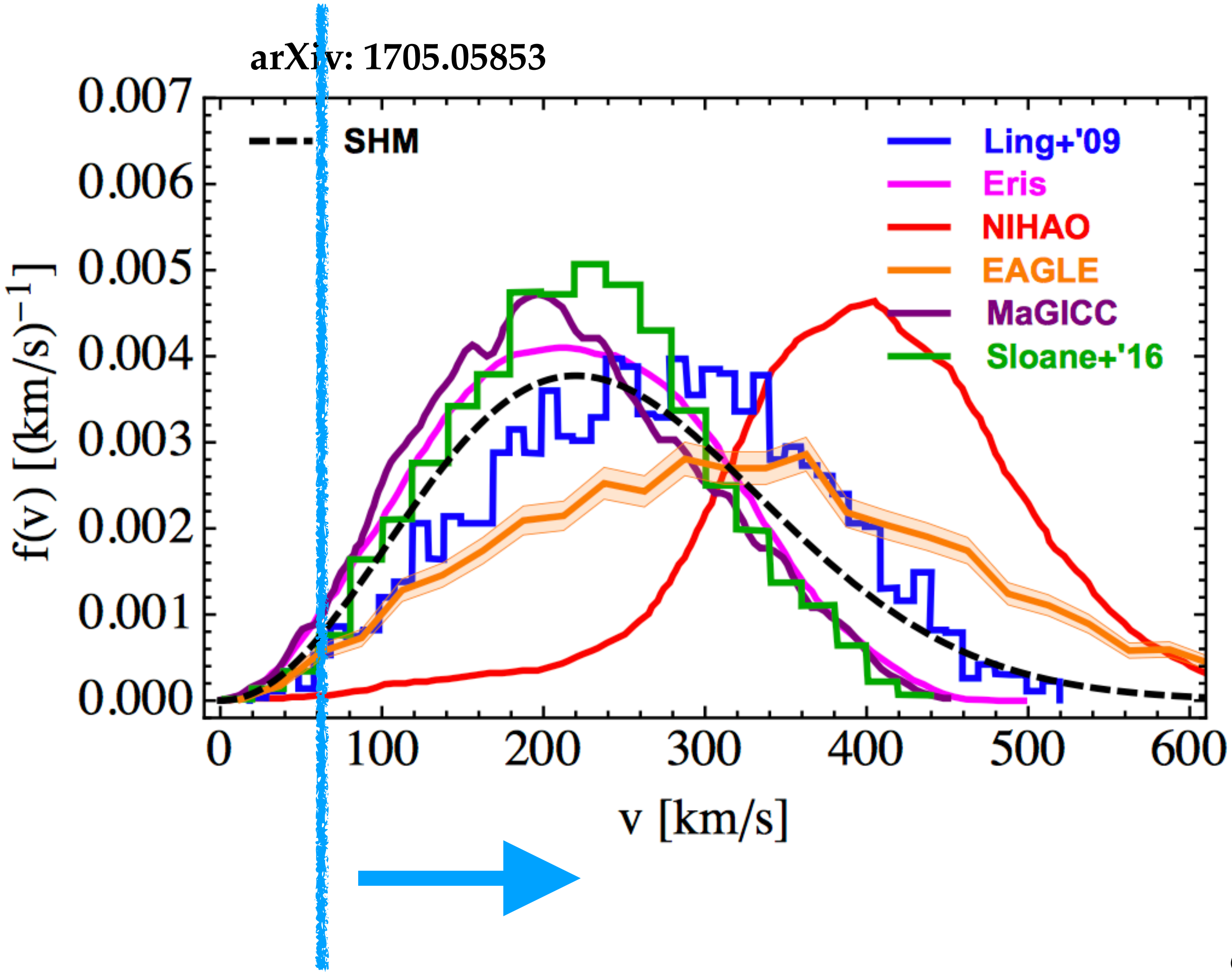

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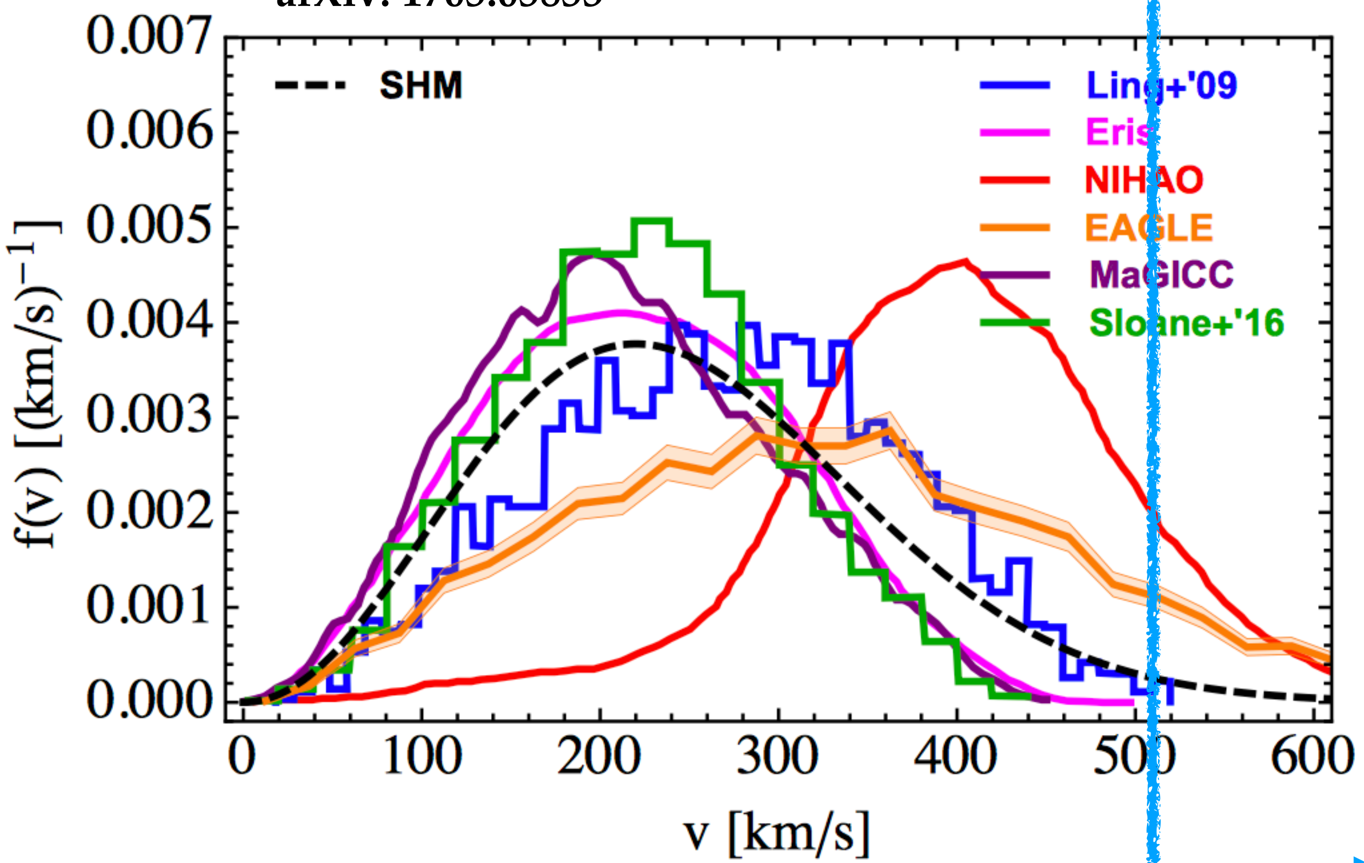

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Halo-Independent Analyses

Can we analyze direct detection data without making any assumptions on the underlying astrophysical distribution?

σ_p

Conventional
(Halo-Dependent)

Fix:

- Interaction
- Kinematics
- Astrophysics

m_χ

$\tilde{\eta}$

Halo-Independent

Fix:

- Interaction
- Kinematics
- Mass

$$\tilde{\eta} \equiv \frac{\rho\sigma}{m_\chi} \int_{v_{\min}} d^3v\, v\, f(\vec{v}, t)$$

$v_{\min}$

$$Rate = \int \tilde{\eta}(v)\mathcal{R}(v)dv$$

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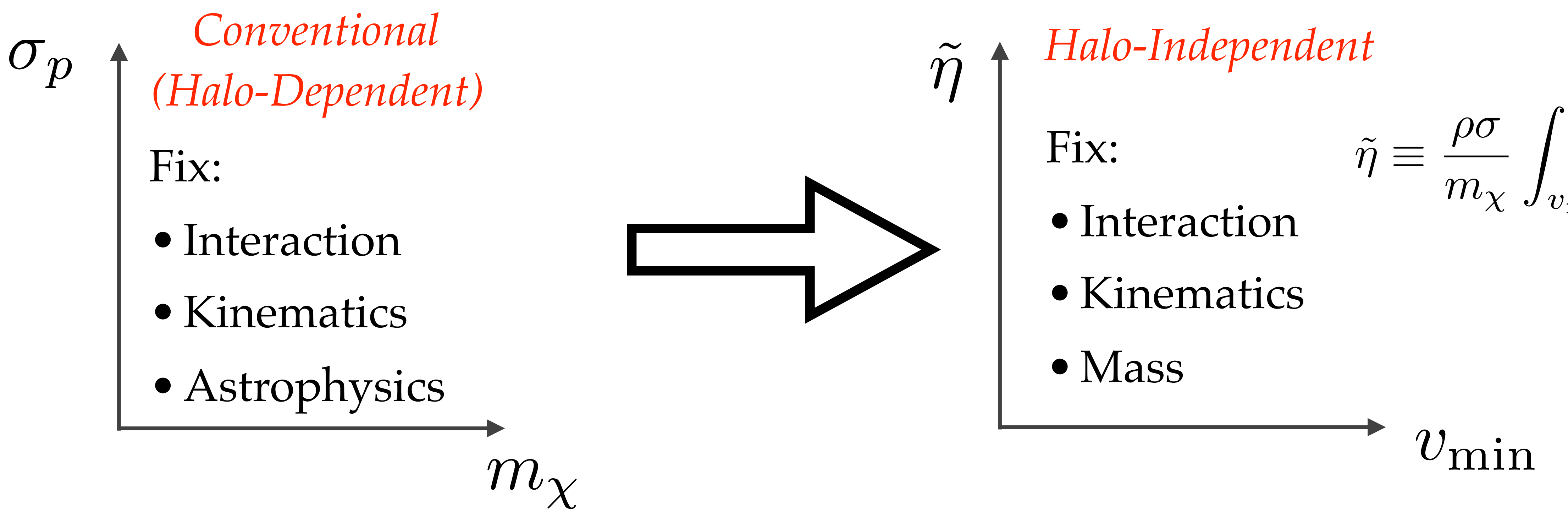

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Early Issues related to putative signals:

- Statistical interpretations often ambiguous (at best)
- Required unbinned measurements of data and background
- Could only be applied to time-averaged rate
  - (see Paolo Gondolo's Talk)

New Halo-Independent Formalism

(Derived from Convex Hulls)

Goal:

Develop a new halo-independent formalism that can be applied to any experiment/ dataset with a concrete and meaningful statistical interpretation

JCAP12(2017)039 Gelmini, Huh, SJW

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$$\mathcal{L}(R_1, R_2, \cdots)$$

(Frequentist method based on use of likelihood ratio)

e.g. R_1 is bin #1 (or experiment 1), and R_2 is bin #2 (or experiment 2)

Road Map:

1. Prove all likelihoods are necessarily strictly convex functions of the predicted rate
   - Likelihood maximized by $\hat{\vec{R}} = (\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{\mathcal{N}})$
2. Use theorems from convex geometry to argue that the set of rates that maximize the likelihood can always be obtained from very simple halo functions
   - Either $f_G(\vec{u}) = \sum_{i=1}^{\mathcal{N}} f_i \, \delta^3(\vec{u} - \vec{u}_i)$ or $F(v) = \sum_{i}^{\mathcal{N}} F_i \, \delta(v - v_i)$
3. Use point (2) to reduce the infinite dimensionality problem
   - Construct halo-independent confidence bands

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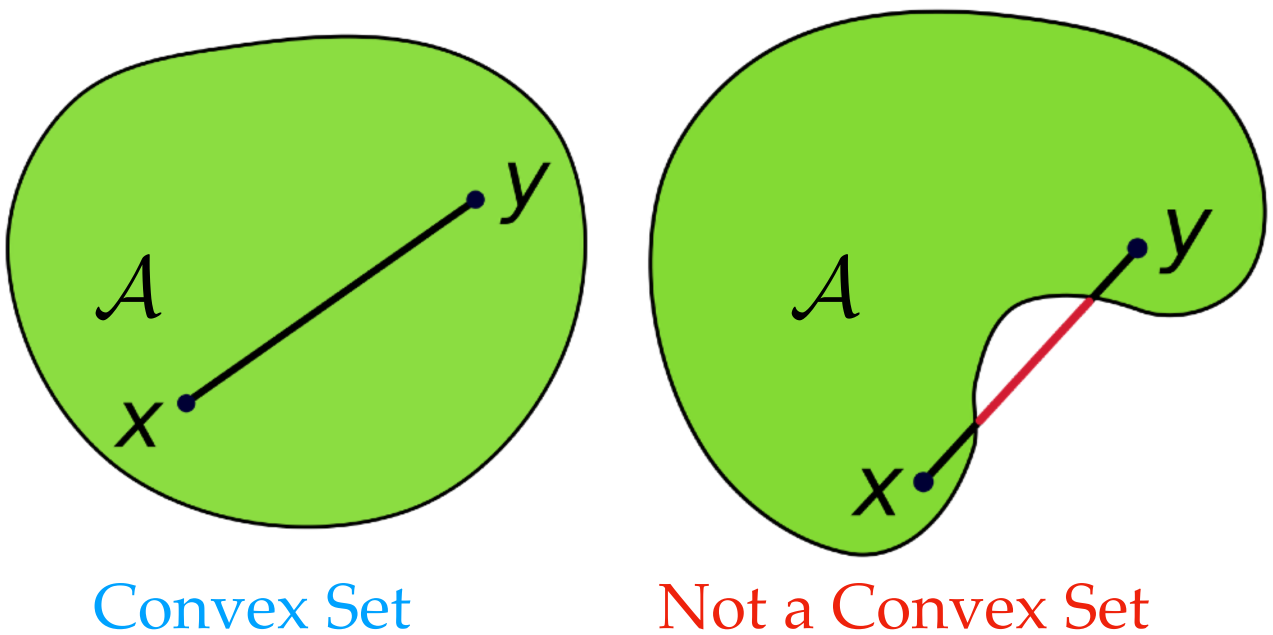
Aside into Convex Geometry

Convex Set

Let $\mathcal{A}$ be a convex set in a D-dimensional vector space.

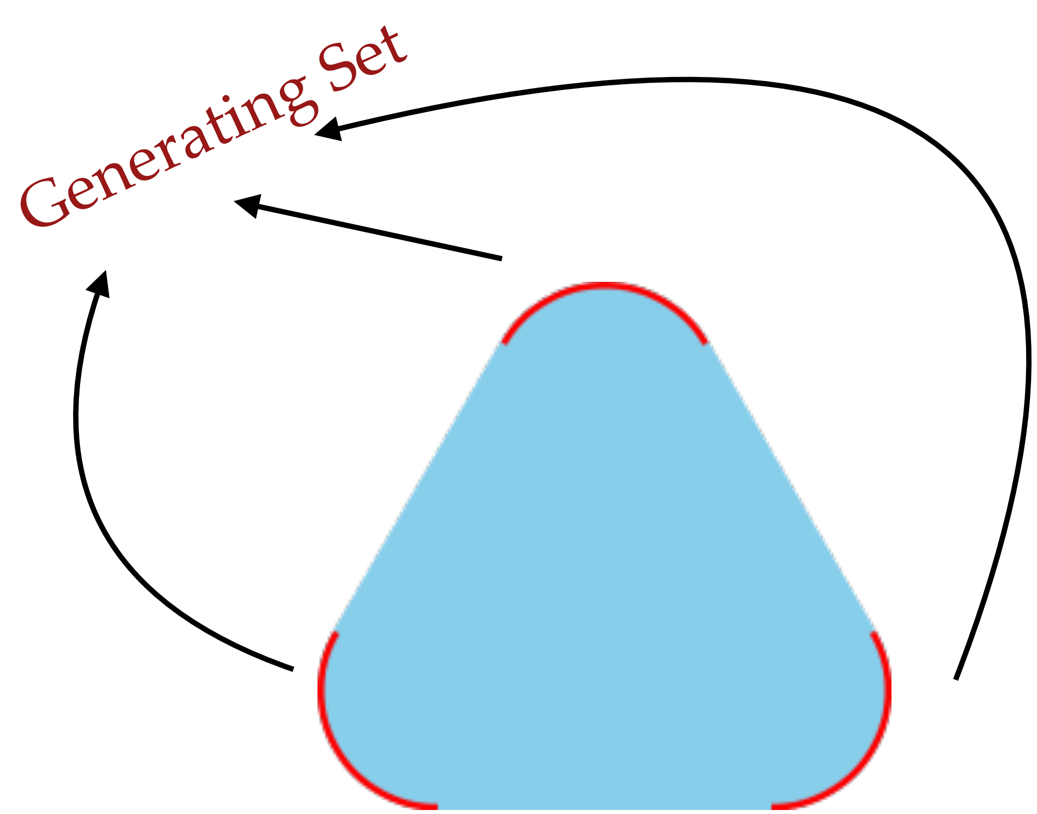
For any collection of $\vec{x}_i$ vectors in $\mathcal{A}$, and semi-positive definite coefficients λ_i w/ $\sum_i \lambda_i = 1$

$$\sum_i \lambda_i \vec{x}_i \in \mathcal{A}$$



Convex Hull

Given `generating set' Y, the convex hull is the minimal (unique) convex set containing Y



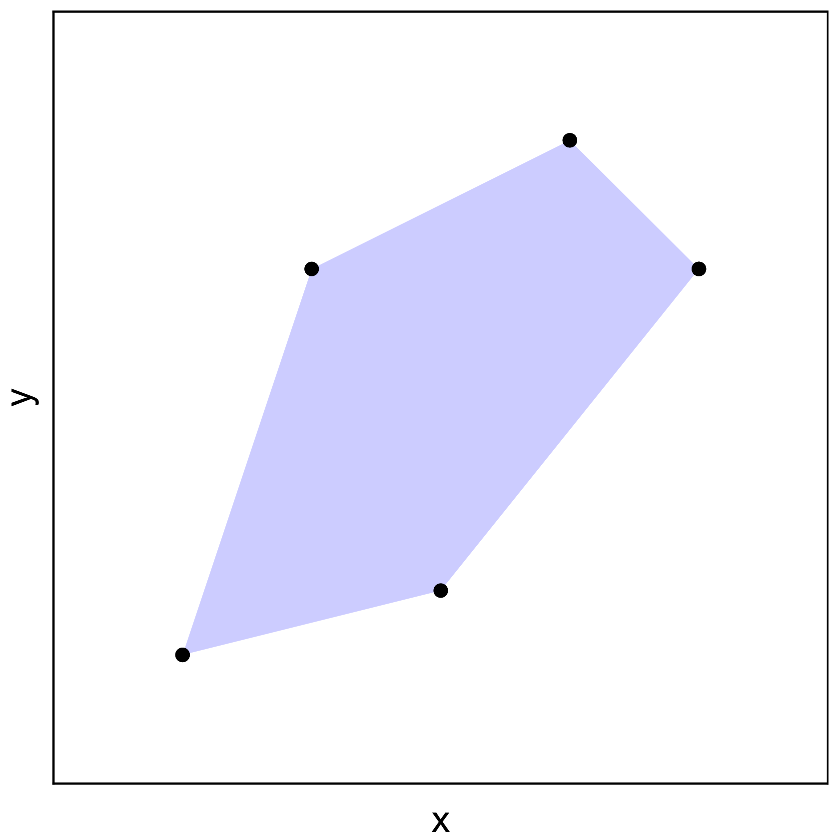
Images Courtesy of Wikipedia

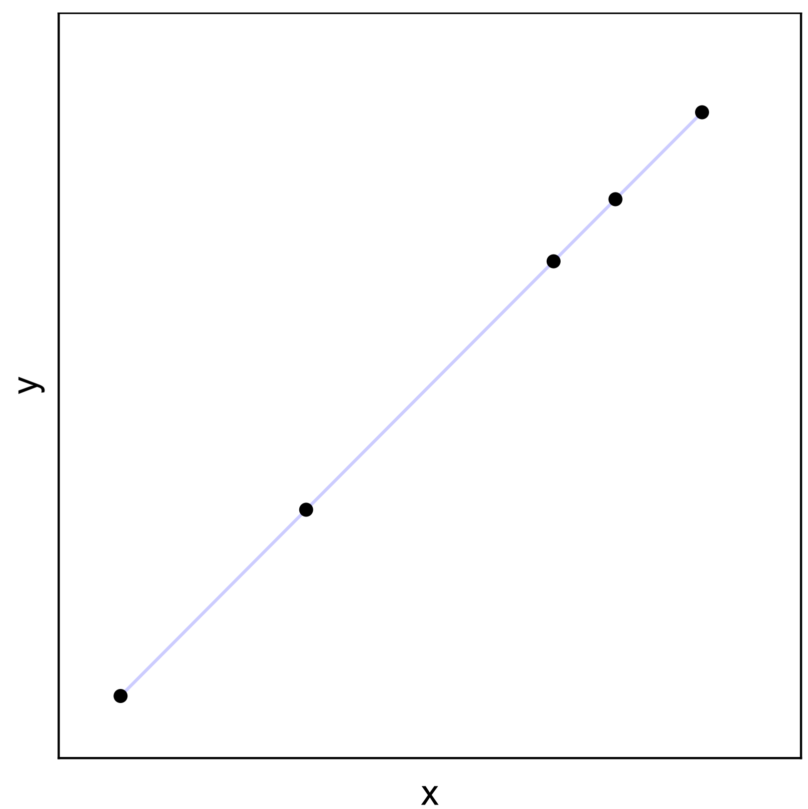
Caratheodory's Theorem (1907)

Lets say we have a convex hull in dimension D defined by generating set X

Any element in the convex hull can be expressed as a convex combination of *at most* (D+1) generating vectors

Caratheodory's Number





Reminder: Convex combination implies coefficients are semi-positive definite and sum to 1

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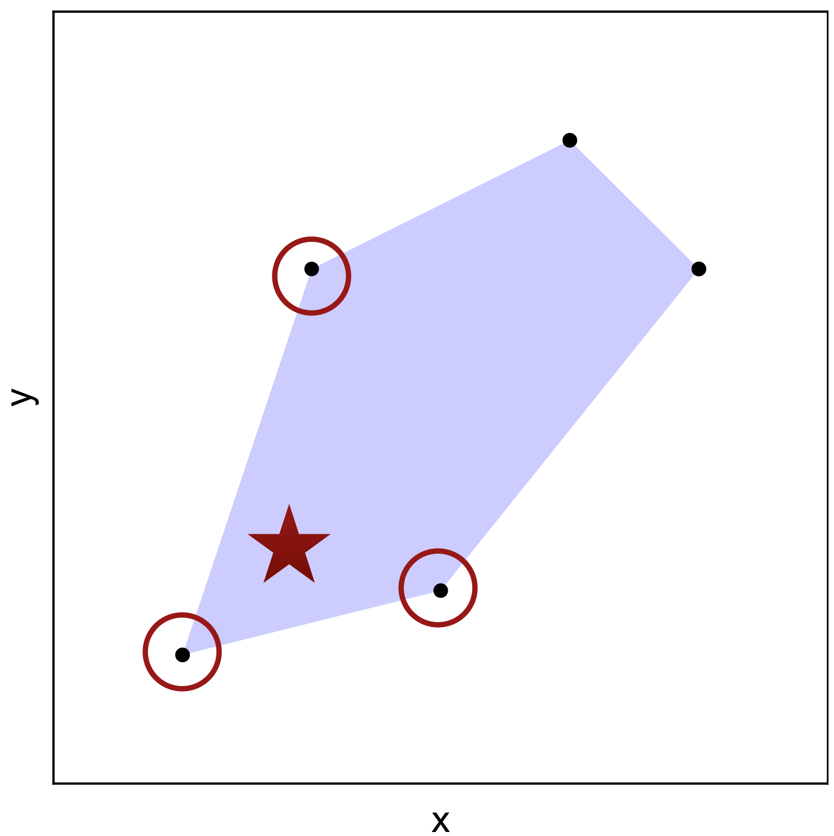
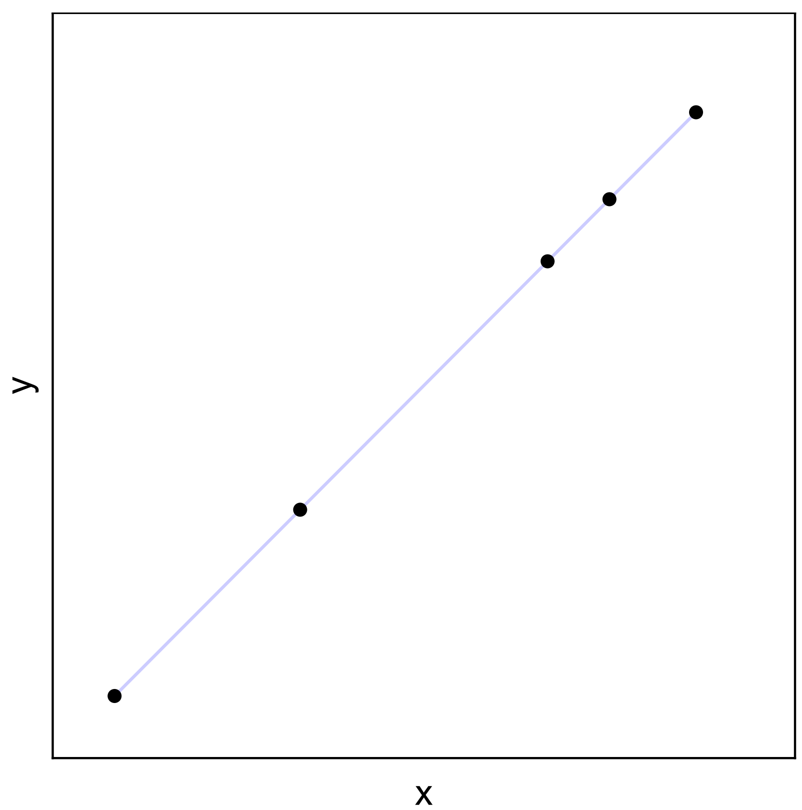


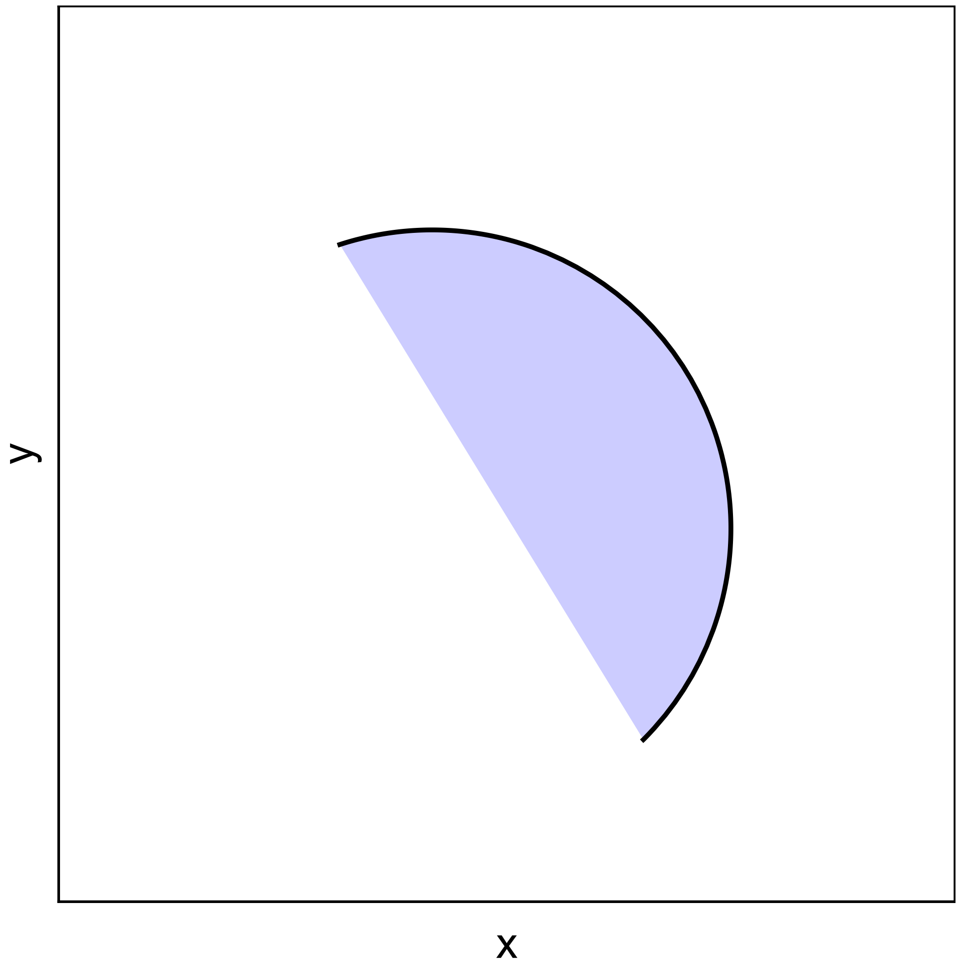


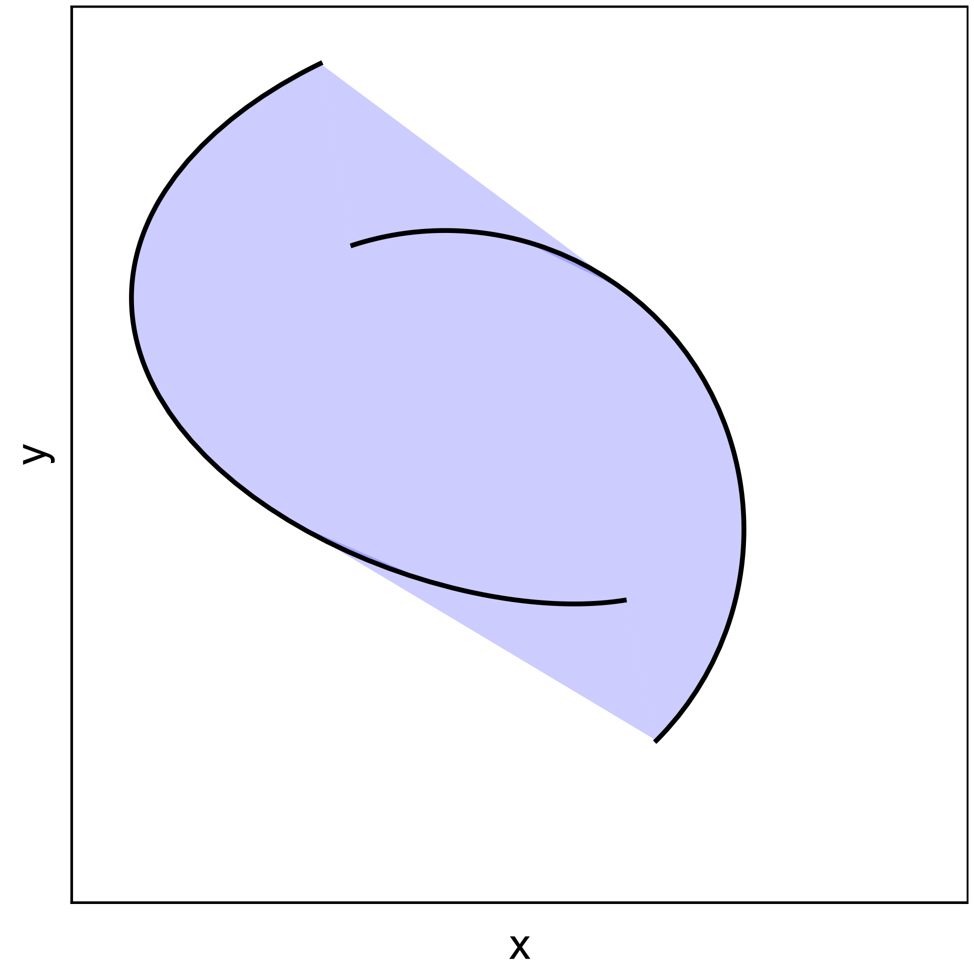
Reminder: Convex combination implies coefficients are semi-positive definite and sum to 1

Fenchel-Eggleston Theorem (1953/58)

Consider Caratheodory's theorem, but in the limiting case where the generating set consists of at most D connected sets

Caratheodory's number is reduced from (D+1) to D





(Also developed additional proof to reduce this to D-1 for some cases)

Forming the Convex Hull

Define a convex hull all possible rate vectors using the infinite generating set:

$$\left\{ \mathcal{C} \frac{\vec{\mathcal{H}}(v_i)}{v_i} \right\} \in \mathcal{A}$$

$$\vec{R} = \mathcal{C} \int_0^\infty dv \frac{\vec{\mathcal{H}}(v)}{v} F(v) \to \sum_i \mathcal{C} \frac{\vec{\mathcal{H}}(v_i)}{v_i} F(v_i) dv_i$$

Rate vector maximizing likelihood is contained in convex hull $\hat{\vec{R}} = (\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_\mathcal{N})$

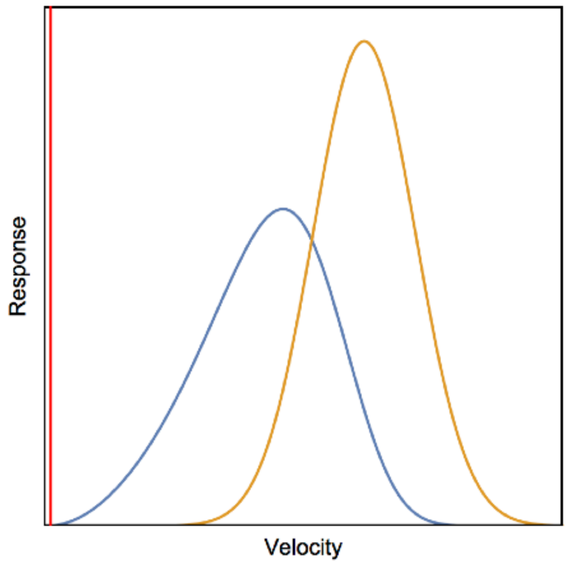
Previous theorems guarantee: $\hat{\vec{R}} = \sum_i \lambda_i \times \mathcal{C} \frac{\vec{\mathcal{H}}(v_i)}{v_i}$ with $\sum_i \lambda_i = 1$

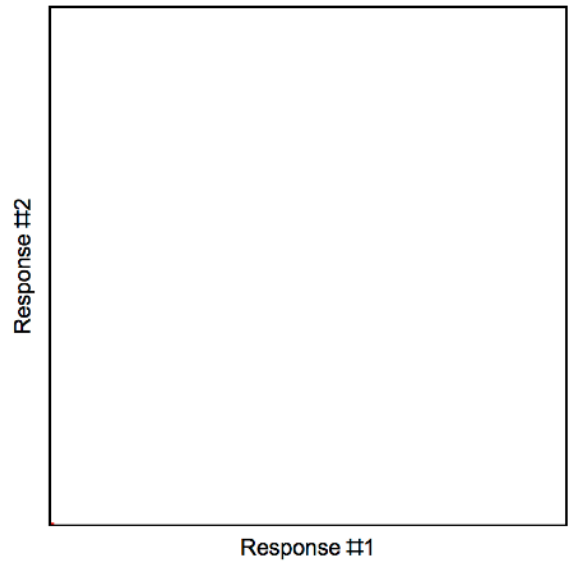
Compare to: $\vec{R} = \sum_i \mathcal{C} \frac{\vec{\mathcal{H}}(v_i)}{v_i} F(v_i) dv_i$ with $\forall v \,,\, F(v) \geq 0$ $\left(\sum_i dv_i F(v_i) = 1 \right)$

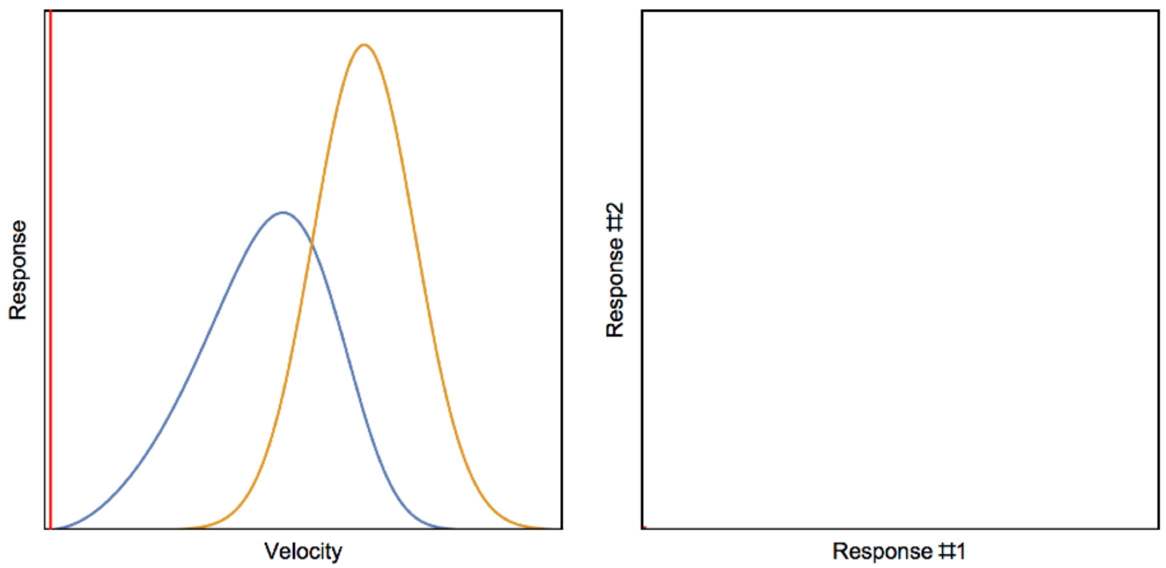
Consequently:

$$F(v) = \sum_i^{\mathcal{N}} F_i \, \delta(v - v_i)$$

Successfully reduced parameter space to manageable size


Response
Velocity


Response ♯2
Response ♯1


Response
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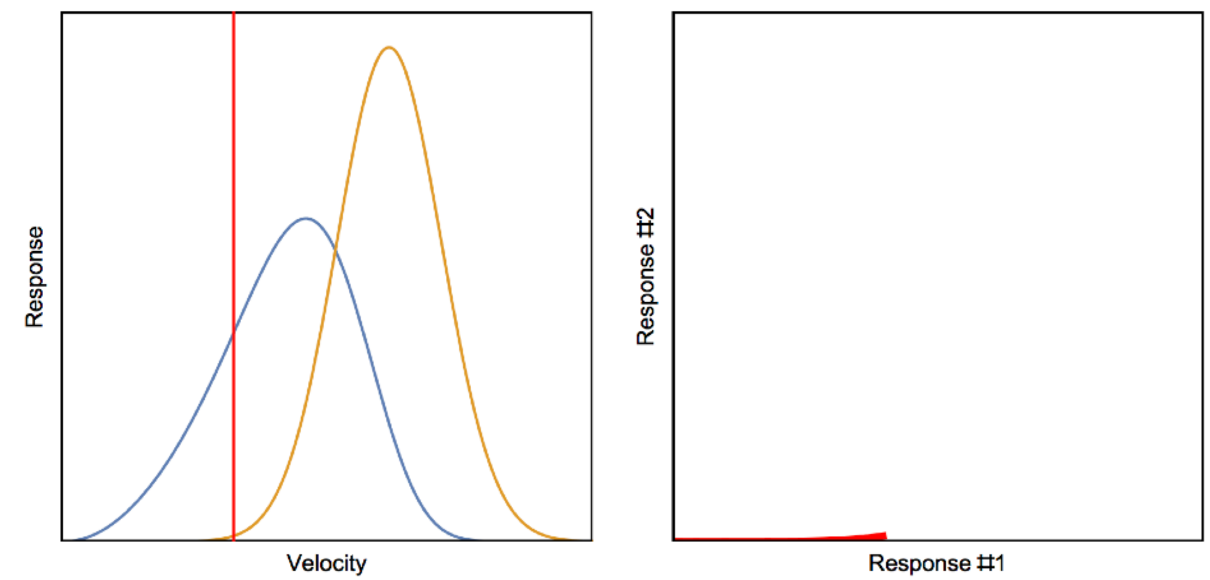
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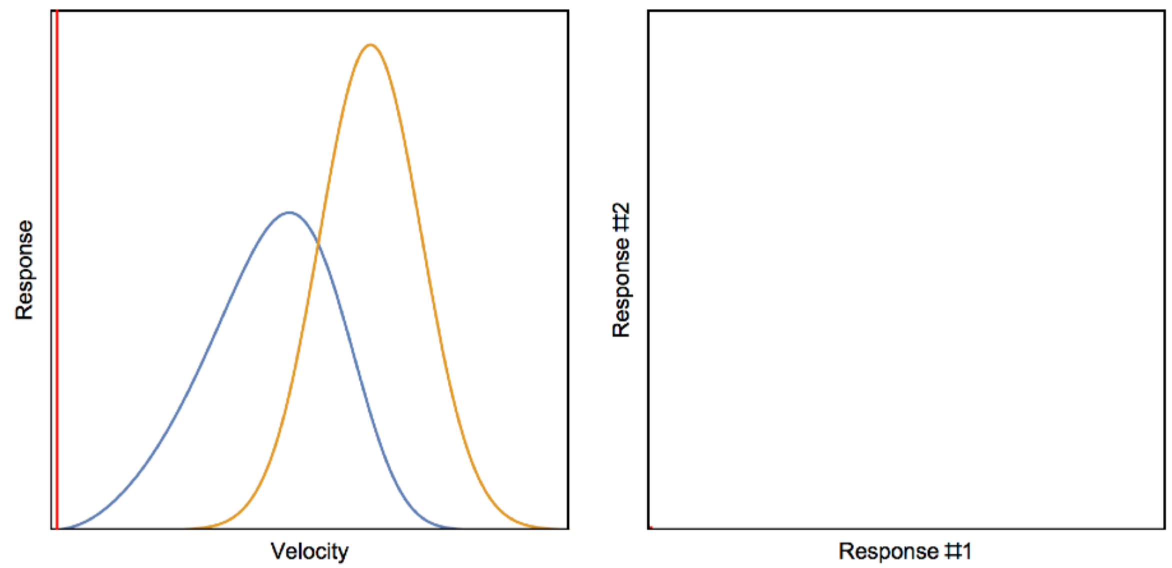
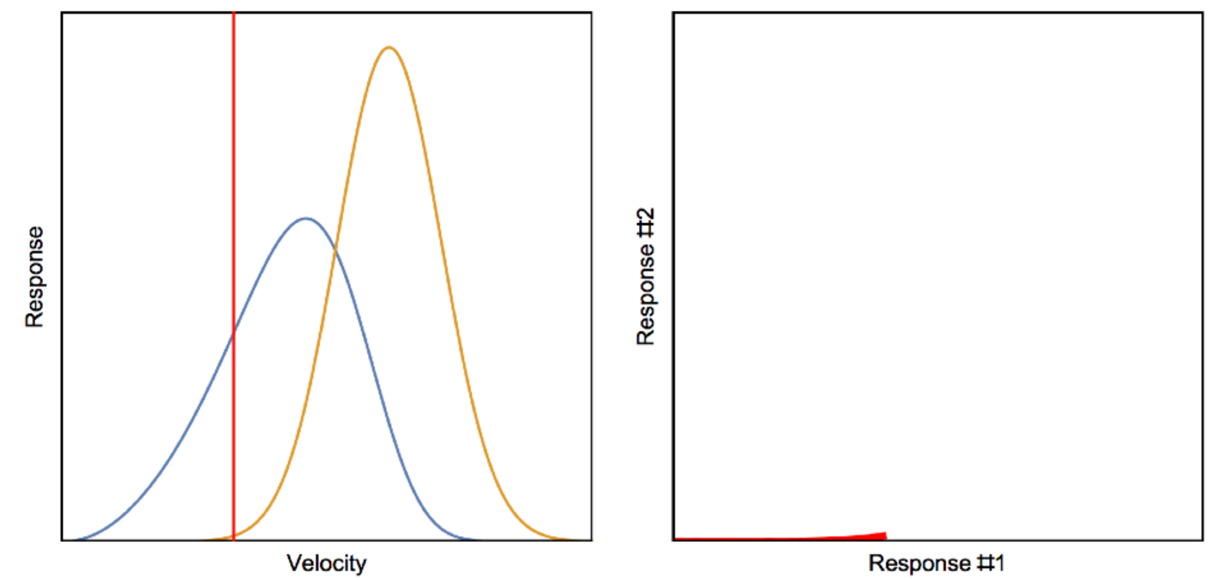

Response
Velocity
Response #2
Response #1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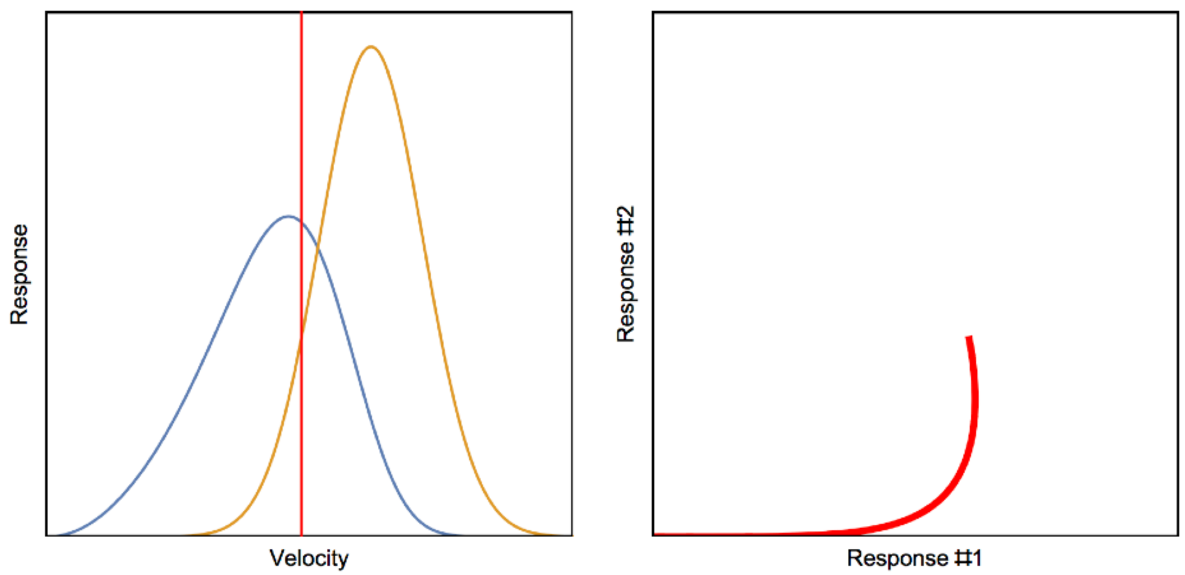
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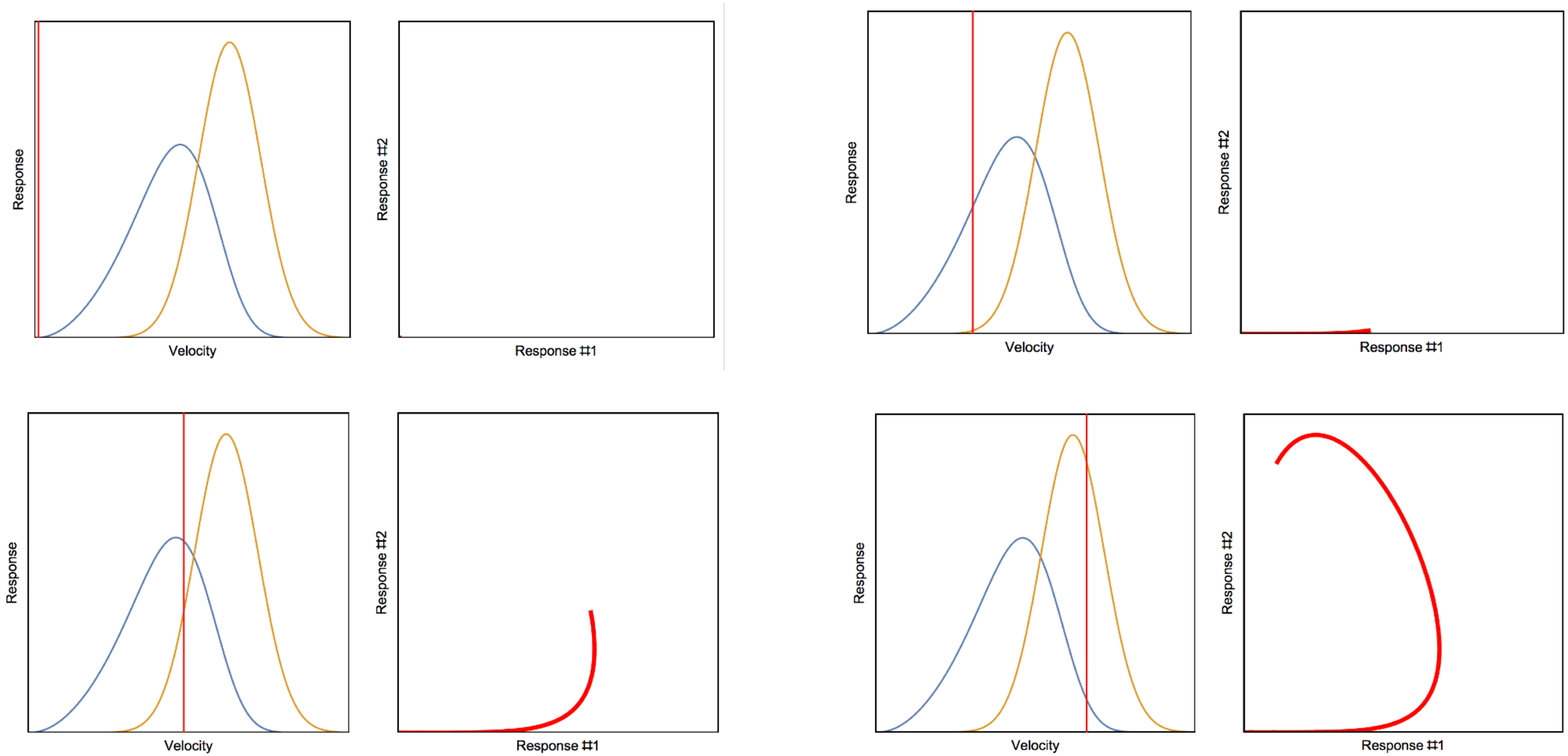
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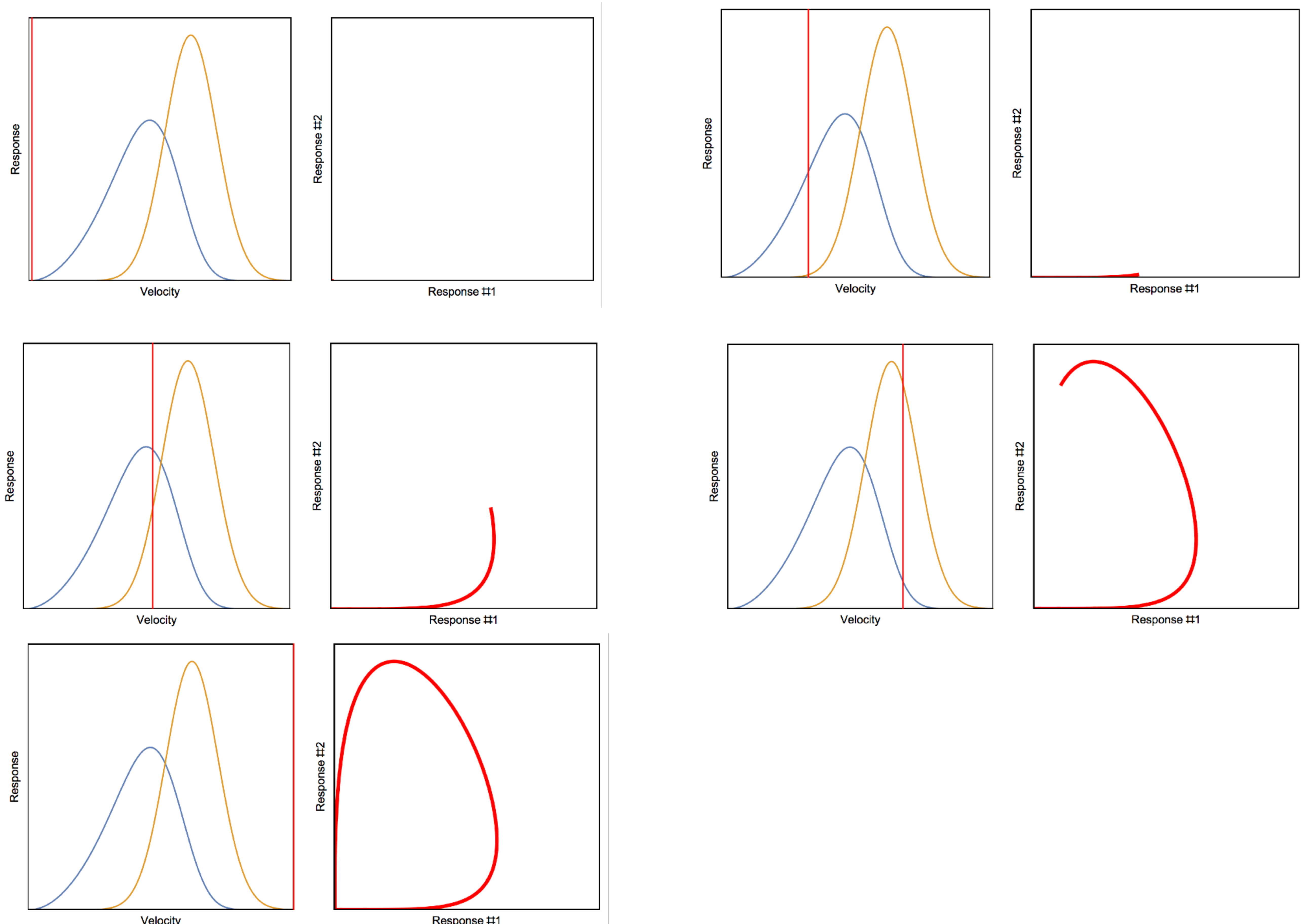
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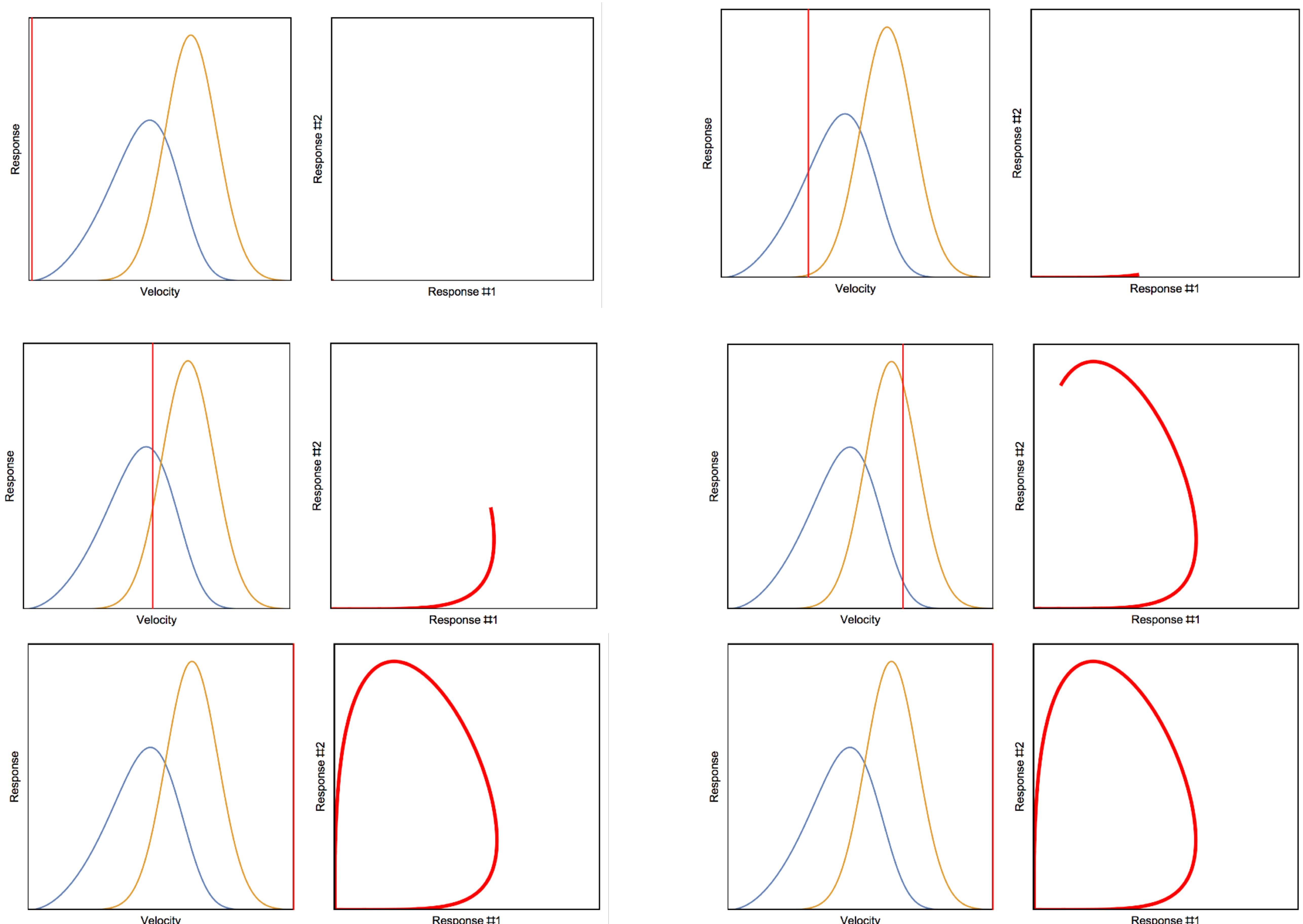
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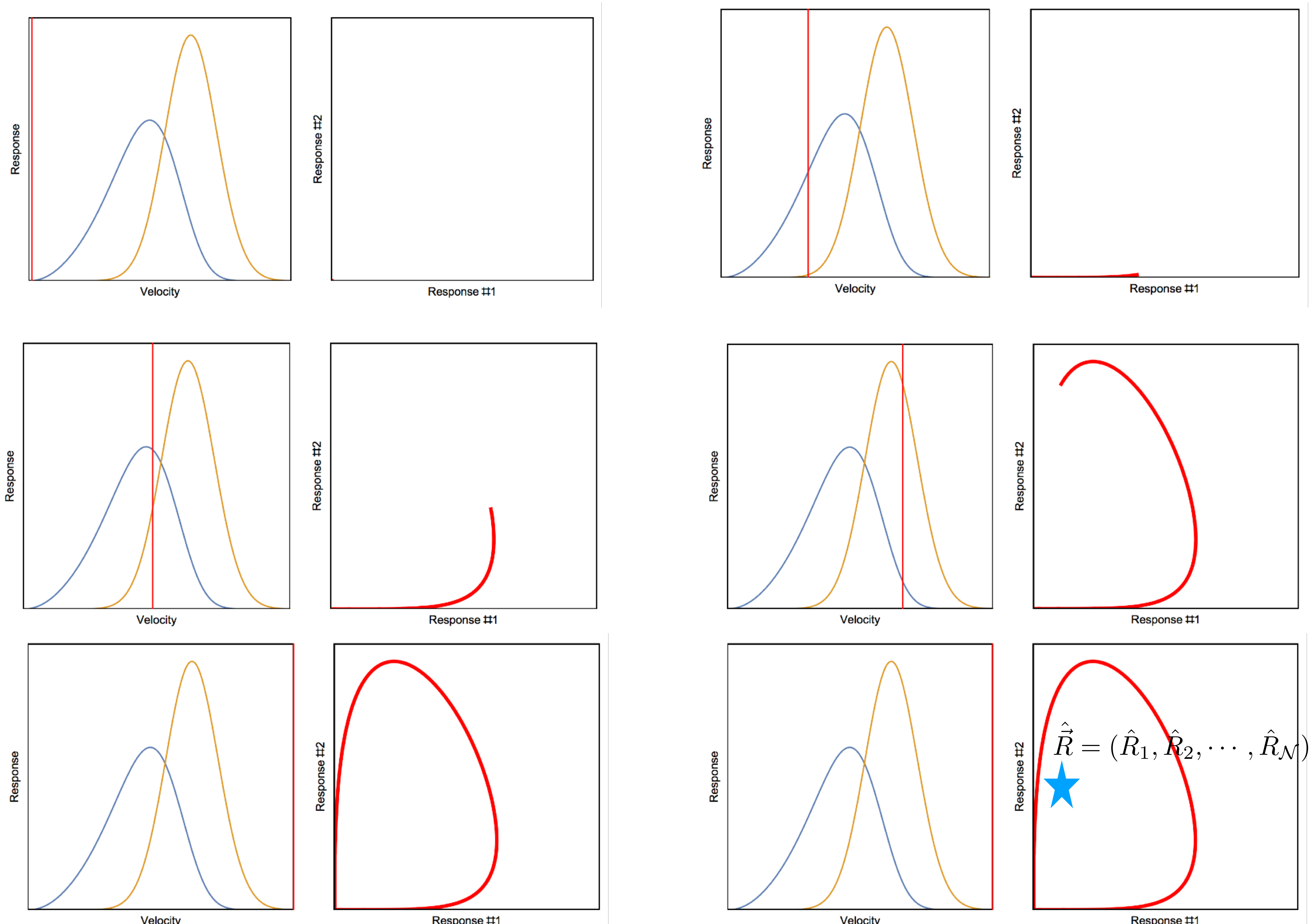
Response
Velocity
Response ⌗2
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Towards a Confidence Band

We have shown that the likelihood is always maximized by $$f(x) = \sum_{i=1}^{\mathcal{N}} c_i \, \delta(x - x_i)$$

But in statistics the best-fit is rather meaningless…

Conventional Neyman-Pearson Likelihood Ratio: $$\lambda \equiv -2 \ln \left[\frac{\mathcal{L}(x = x_0)}{\mathcal{L}(\hat{x})} \right]$$

Again, working in infinite dimensional parameter space makes this impossible…

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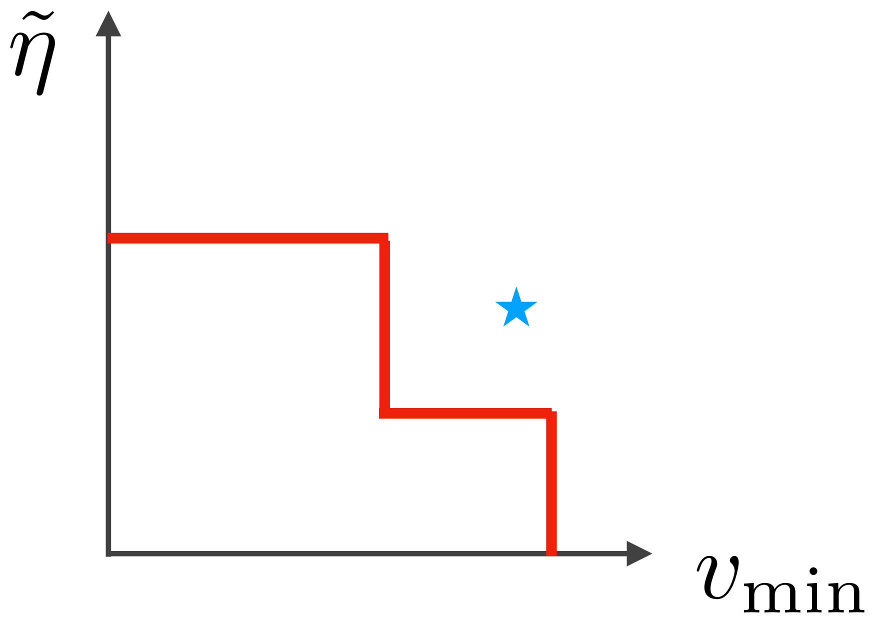
New Question: Does there exist at least one halo function passing through (v, \eta) compatible at the desired CL?



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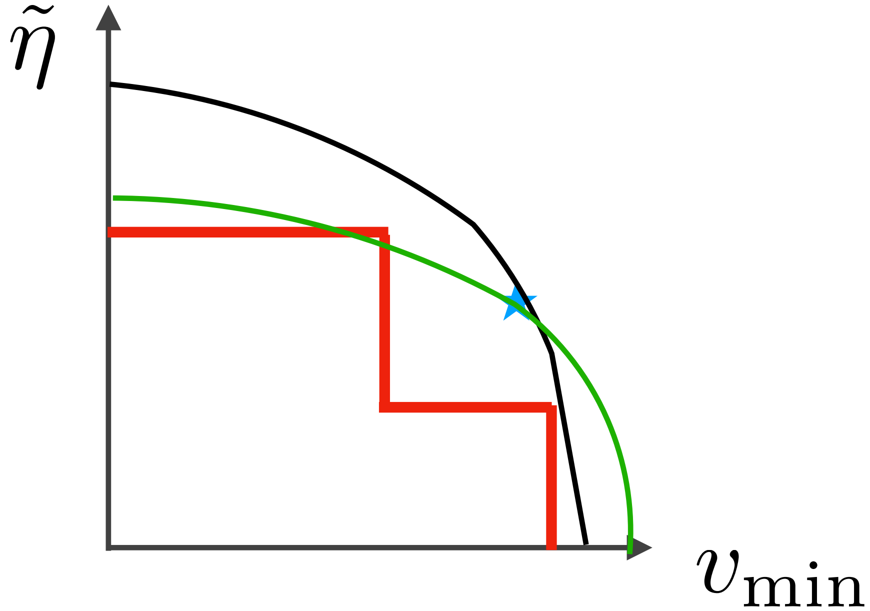

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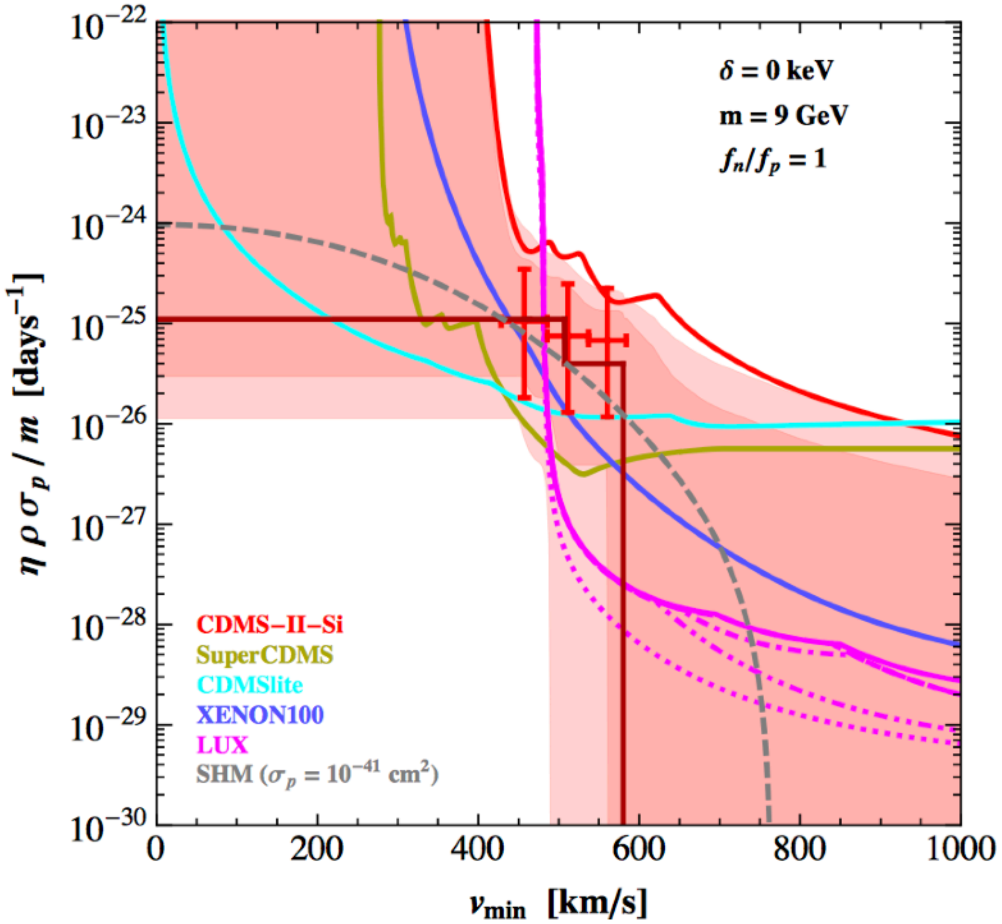
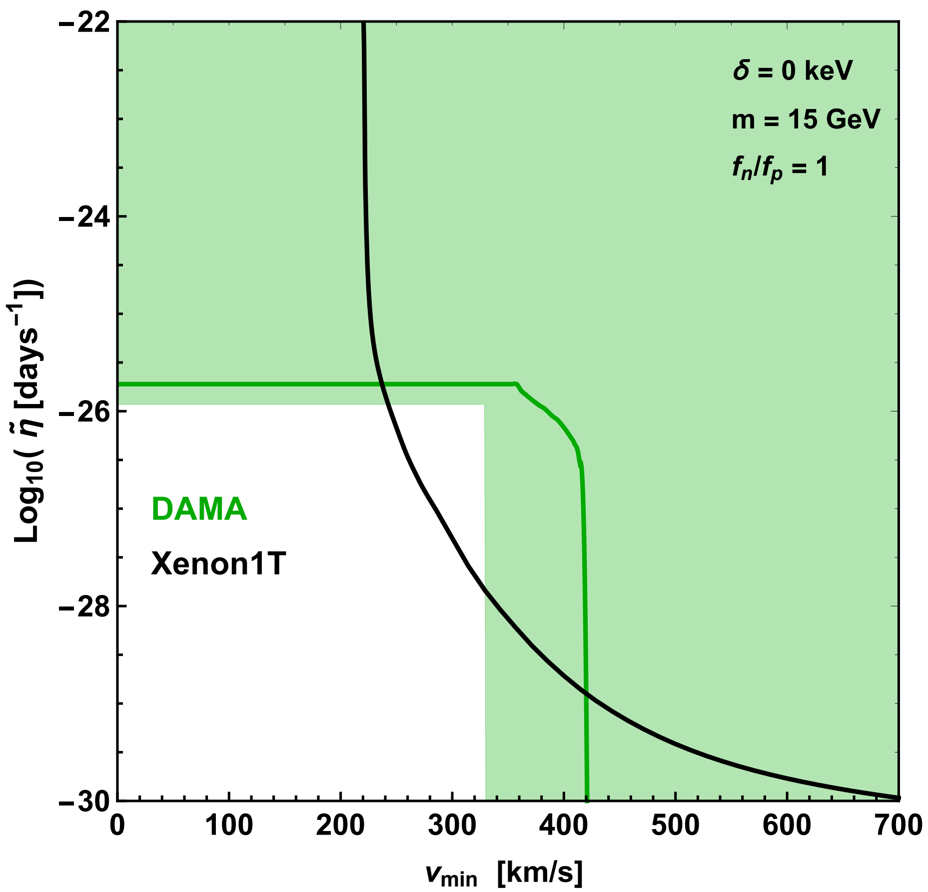
Convex hull arguments can be applied to this 'constrained maximization' as well

Conclusions

- Presented technique that allows one to infer statistically interesting information when in the presence of unknown background function
- Generalized halo-independent analyses such that they are now applicable to all types of data
  - Likelihoods always maximized by speed/velocity distributions written as sum over small number of deltas

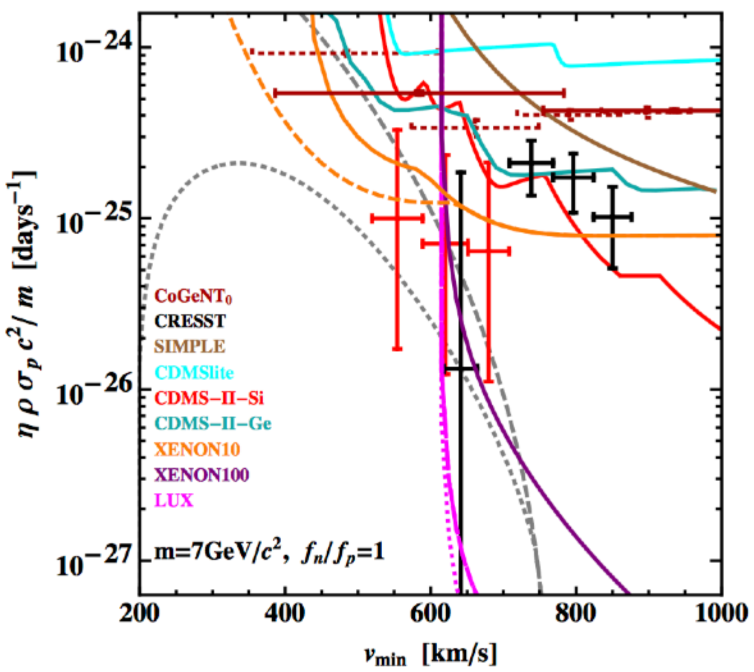
Method also allows for joint analysis with solar annihilation (see e.g. Ibarra and Rappelt 2017)

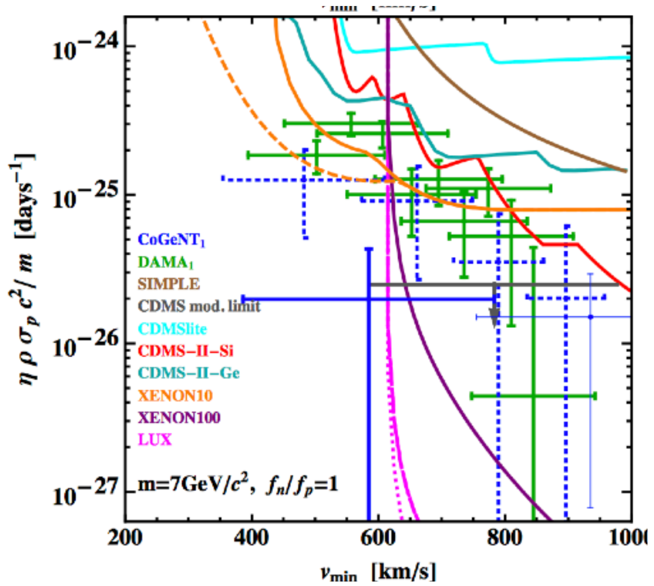




Back-Up Slides

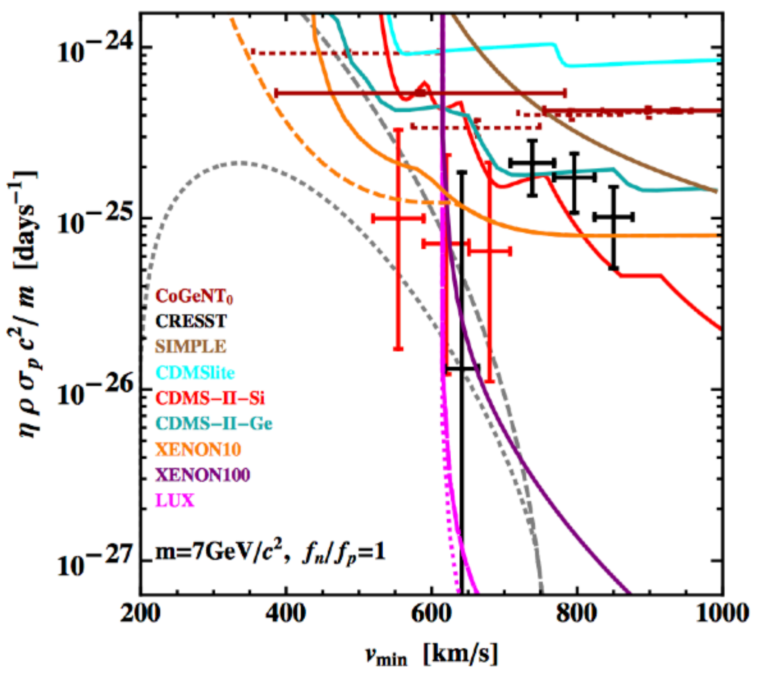
Prior Methods

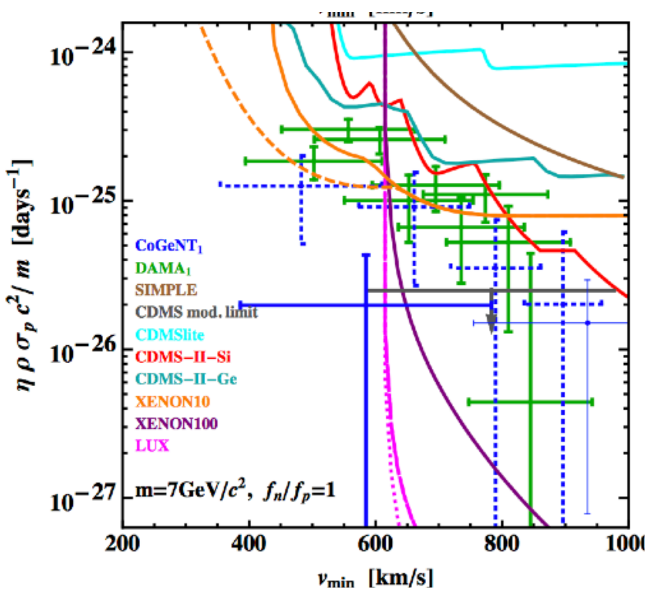


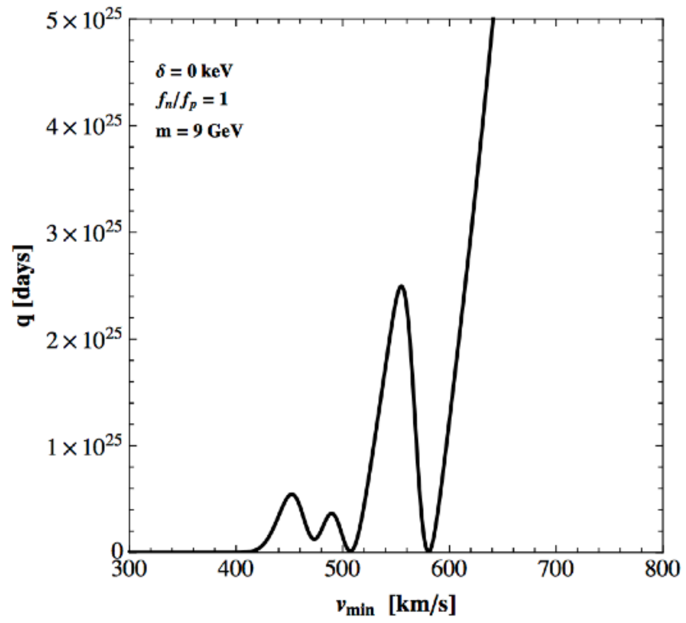


Interpretation of crosses ambiguous

Prior Methods







Interpretation of crosses ambiguous

Minimize likelihood functional with respect to halo function
(enforcing monotonically decreasing requirement with KKT multipliers)

Karush-Kuhn-Tucker Conditions

$$L[\tilde{\eta}] \equiv -2\ln\mathcal{L}[\tilde{\eta}]$$

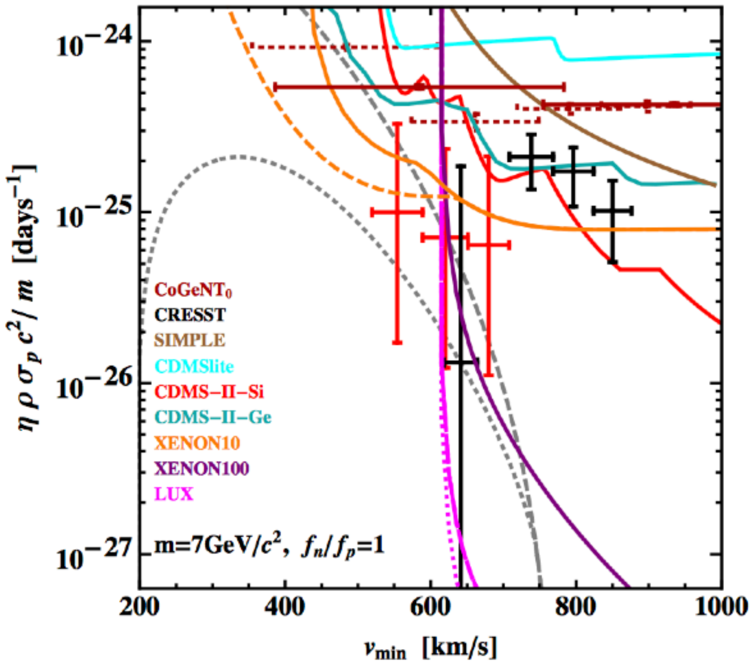
$$q(v_{\rm min}) = \int_{v_\delta}^{v_{\rm min}} dv\,\frac{\delta L}{\delta\tilde{\eta}(v)}$$

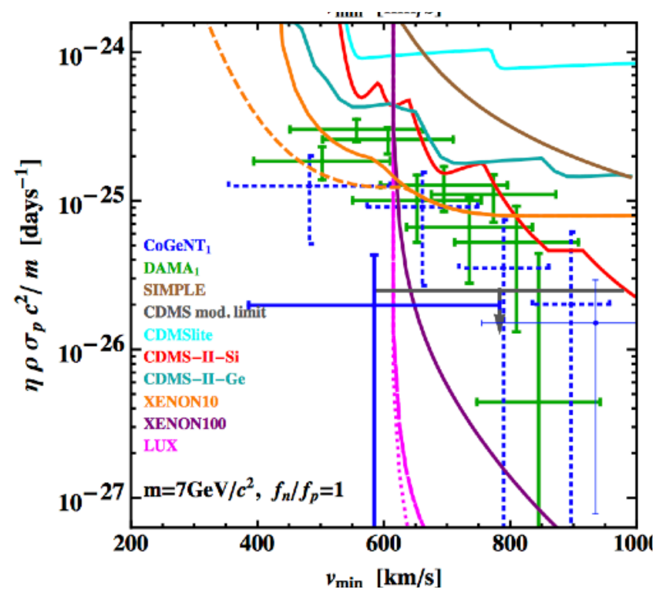
Defines KKT multiplier

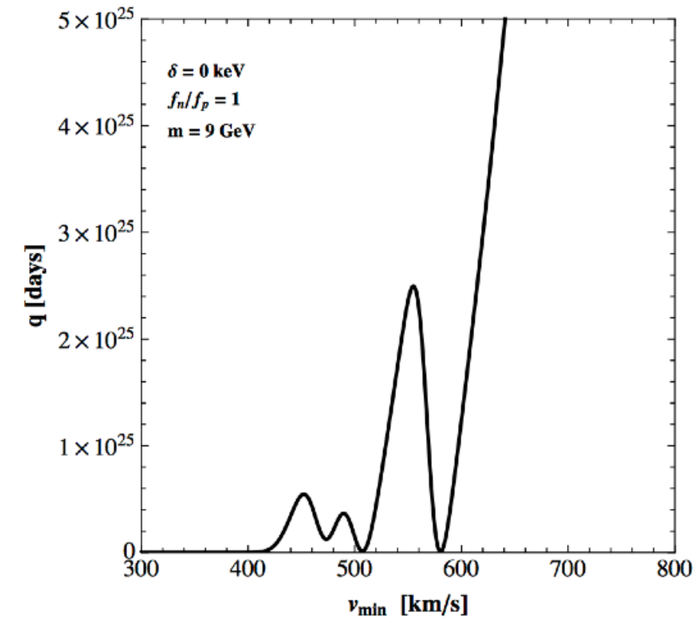
$$q(v_{\rm min})\lim_{\epsilon\to 0+}\frac{\tilde{\eta}(v_{\rm min}+\epsilon)-\tilde{\eta}(v_{\rm min})}{\epsilon} = 0$$

If q(v) only has isolated zeros… then halo function must be piecewise constant

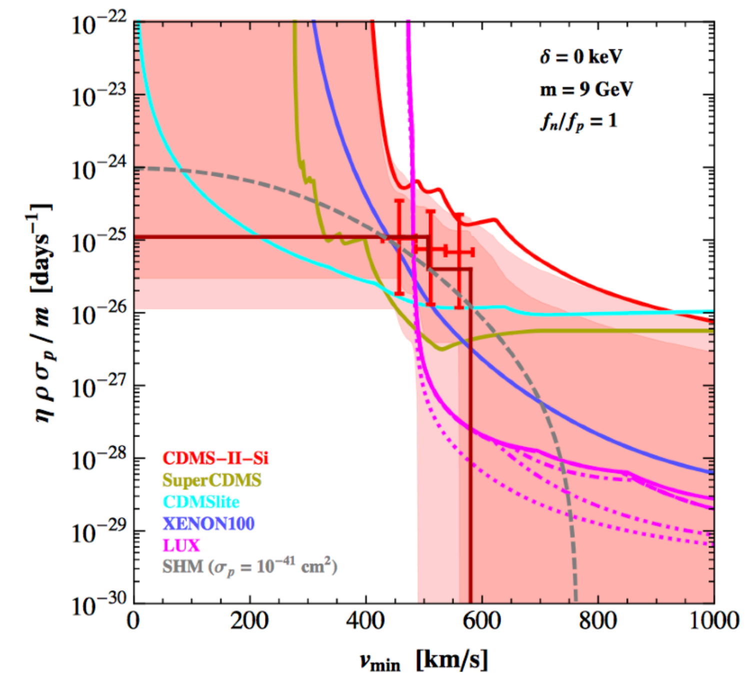
Prior Methods







Interpretation of crosses ambiguous



Minimize likelihood functional with respect to halo function
(enforcing monotonically decreasing requirement with KKT multipliers)

Karush-Kuhn-Tucker Conditions

$$L[\tilde{\eta}] \equiv -2 \ln \mathcal{L}[\tilde{\eta}]$$

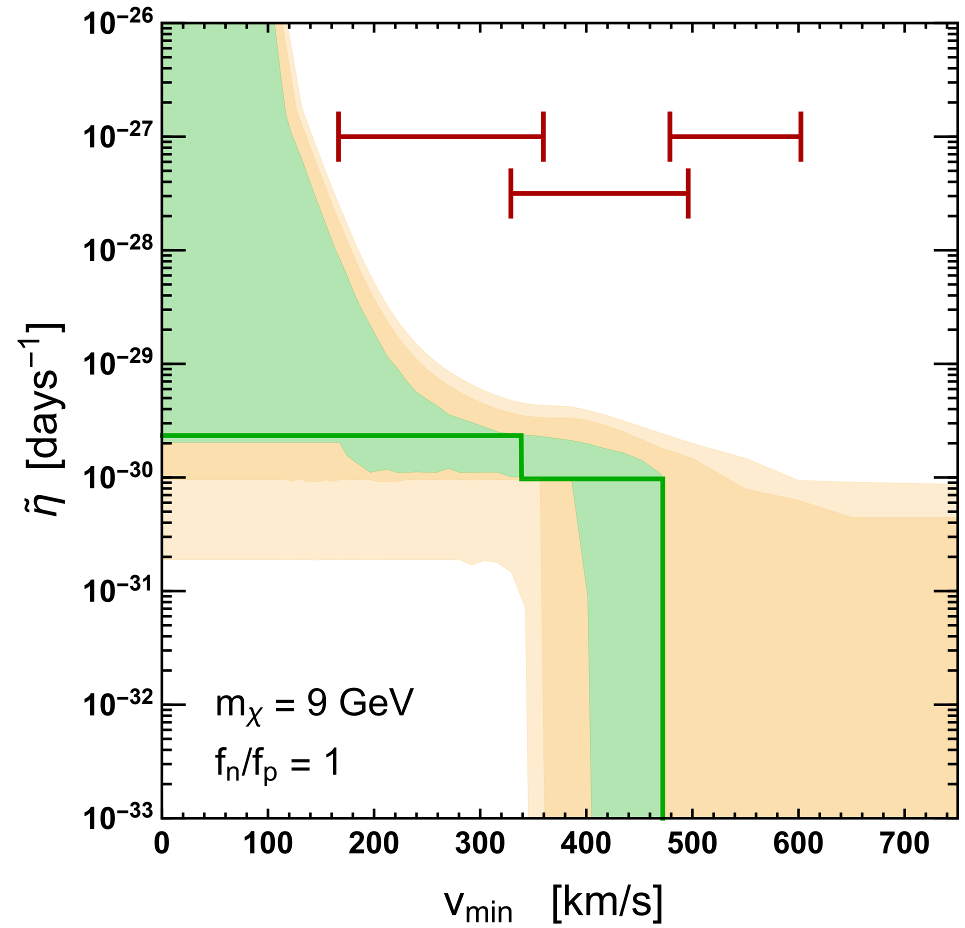
$$q(v_{\min}) = \int_{v_\delta}^{v_{\min}} dv\, \frac{\delta L}{\delta \tilde{\eta}(v)}$$

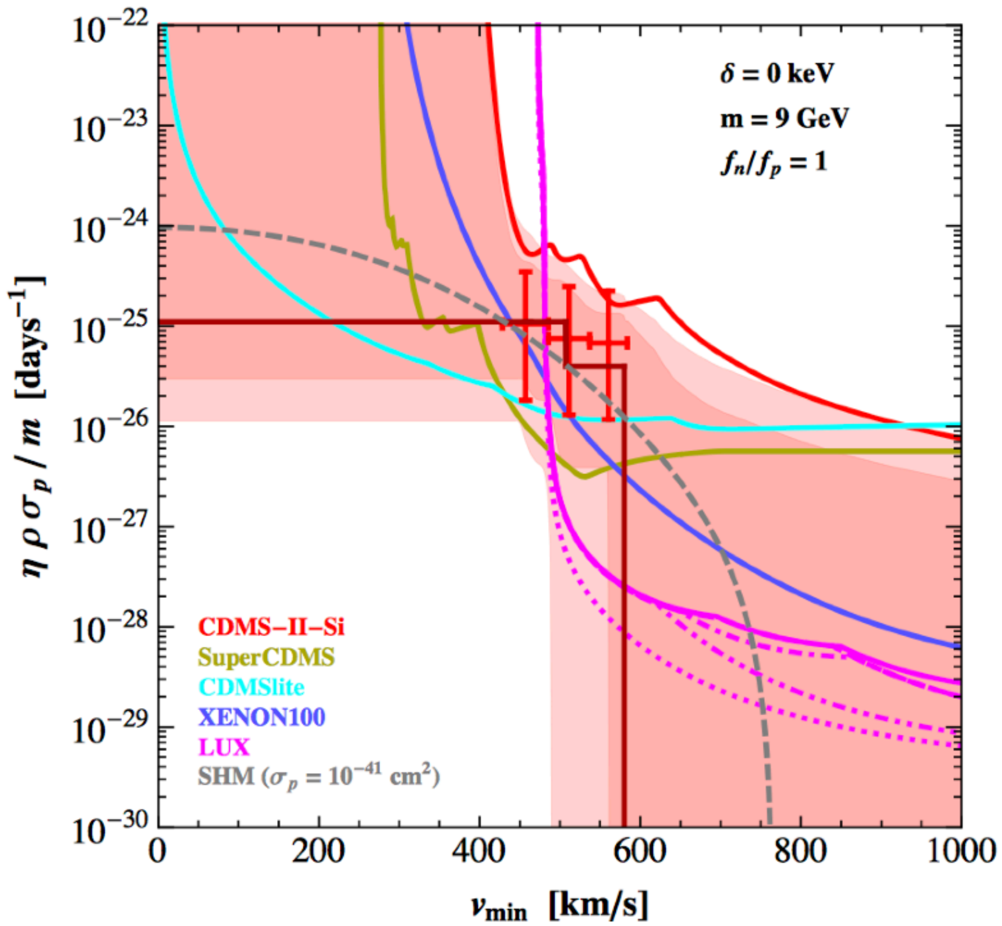
Defines KKT multiplier

$$q(v_{\min}) \lim_{\epsilon \to 0+} \frac{\tilde{\eta}(v_{\min}+\epsilon) - \tilde{\eta}(v_{\min})}{\epsilon} = 0$$

If q(v) only has isolated zeros… then halo function must be piecewise constant

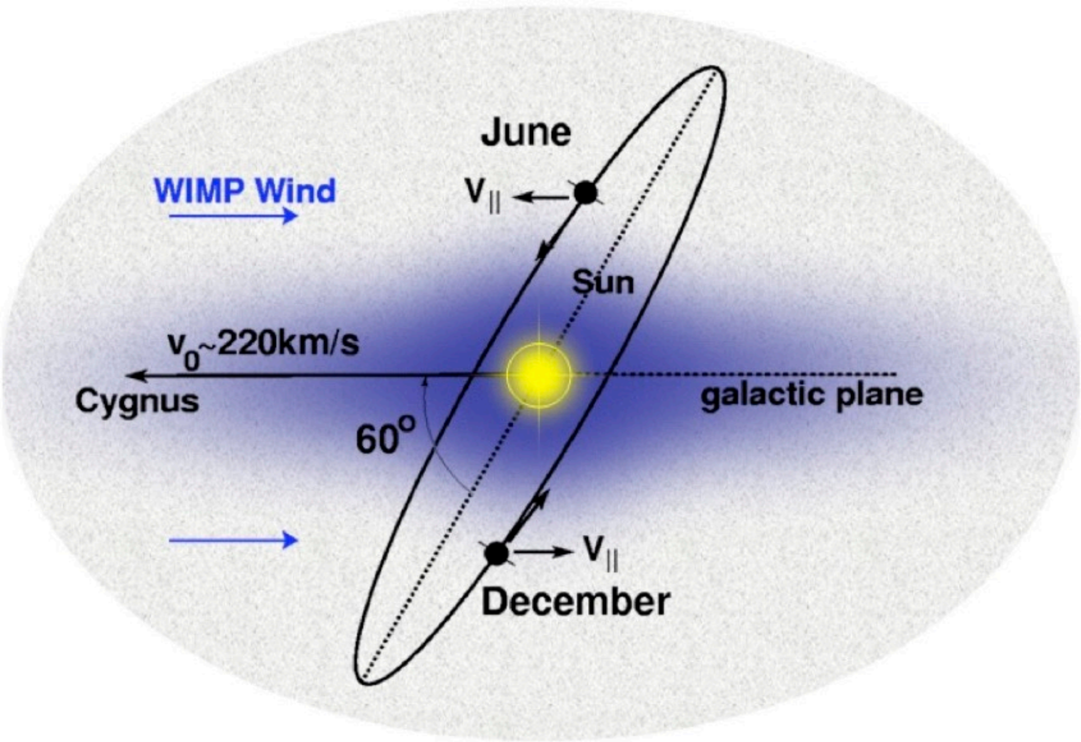
Quick Example





In the event of degenerate best-fit region one identify this as well

Annual Modulation



Earth's rotation about the Sun produces modulation in the scattering rate

Conventionally, assume form of f(v) in Galaxy, use Galilean transformation

$$\vec{u} = \vec{v}_\odot + \vec{v}_\oplus(t) + \vec{v}$$

Recall:

$$R_{\alpha i}(t) = \int d^3v\, \mathcal{C}\, \frac{\mathcal{H}_{\alpha i}(\vec{v})}{v}\, f(\vec{v}, t)$$

Let us now change variables to absorb time-dependence in H:

$$R_{\alpha i}(t) = \int d^3u\, \mathcal{C}\, \frac{\mathcal{H}^{\rm gal}_{\alpha i}(\vec{u}, t)}{|\vec{u} - \vec{v}_\odot - \vec{v}_\oplus(t)|}\, f_G(\vec{u})$$

Note we are now working with velocity, not speed, distribution

Annual Modulation

Time-averaged halo function:

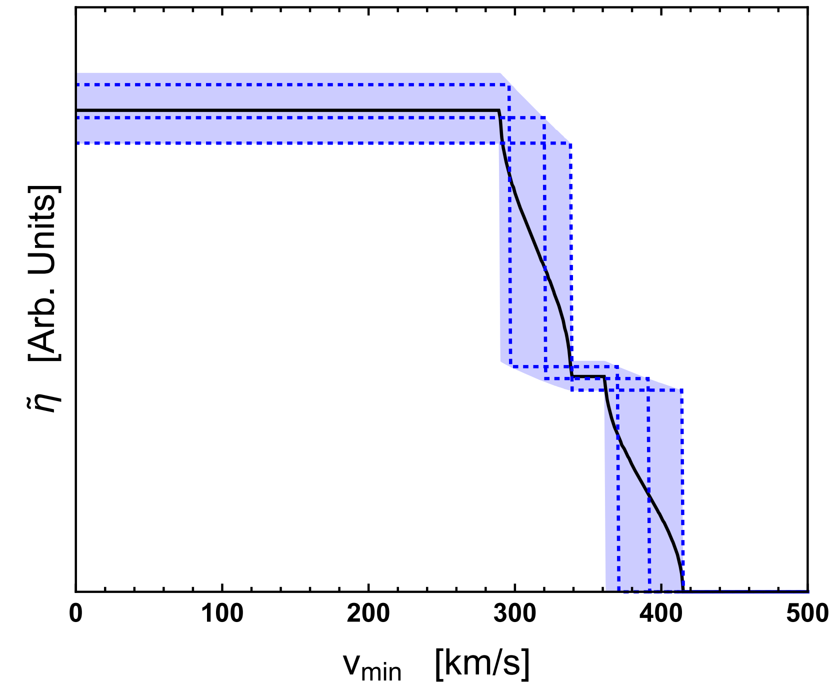
$$\tilde{\eta}^0_{BF}(v_{\min}) = \sum_{h=1}^{\mathcal{N}} \frac{\mathcal{C} f_h^{\rm gal}}{\bar{v}_h(v_{\min})} \qquad \frac{1}{\bar{v}_h(v_{\min})} \equiv \frac{1}{T}\int dt \, \frac{\Theta(|\vec{u}_h - \vec{v}_\odot - \vec{v}_\oplus(t)| - v_{\min})}{|\vec{u}_h - \vec{v}_\odot - \vec{v}_\oplus(t)|}$$

A few notes:

- Now working with 3D velocity distribution rather than speed
  - Minimization done done w.r.t. 4N parameters (quickly becomes numerically taxing)
- Best-fit halo function *only* piecewise constant at fixed times
- Require *at most* N streams, not (N - 1)

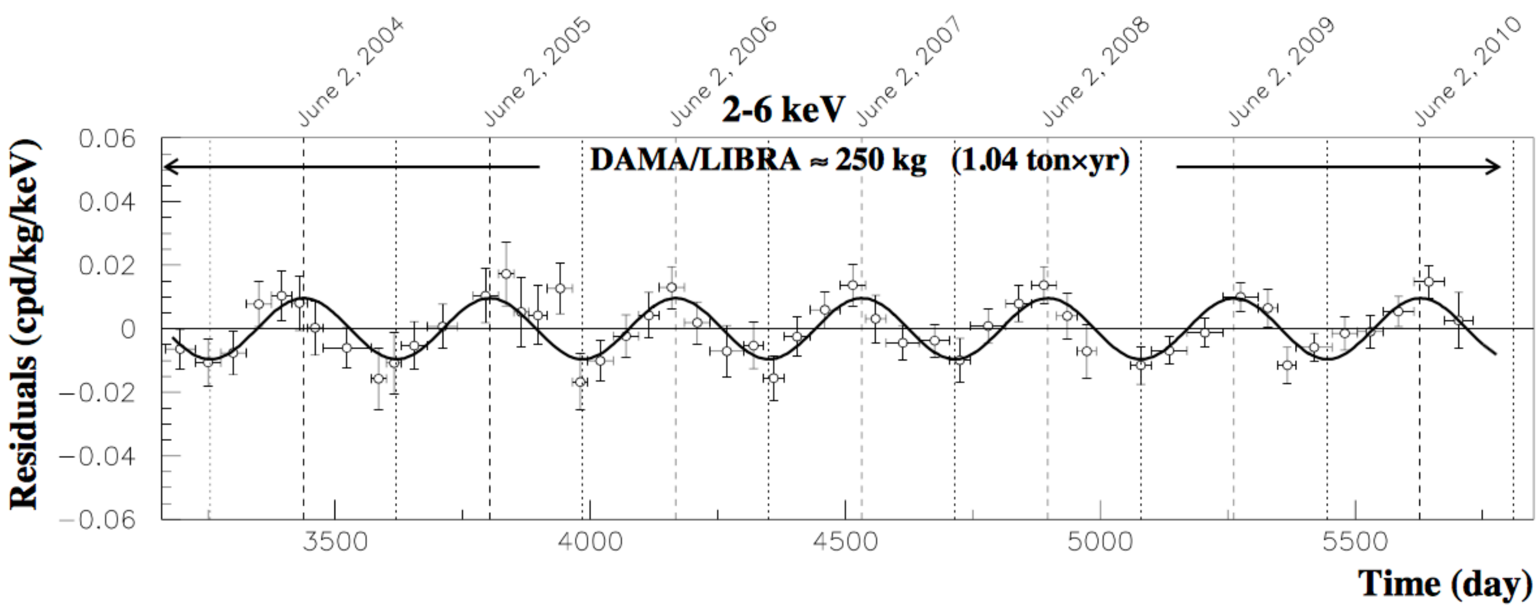
Constrained Analysis:

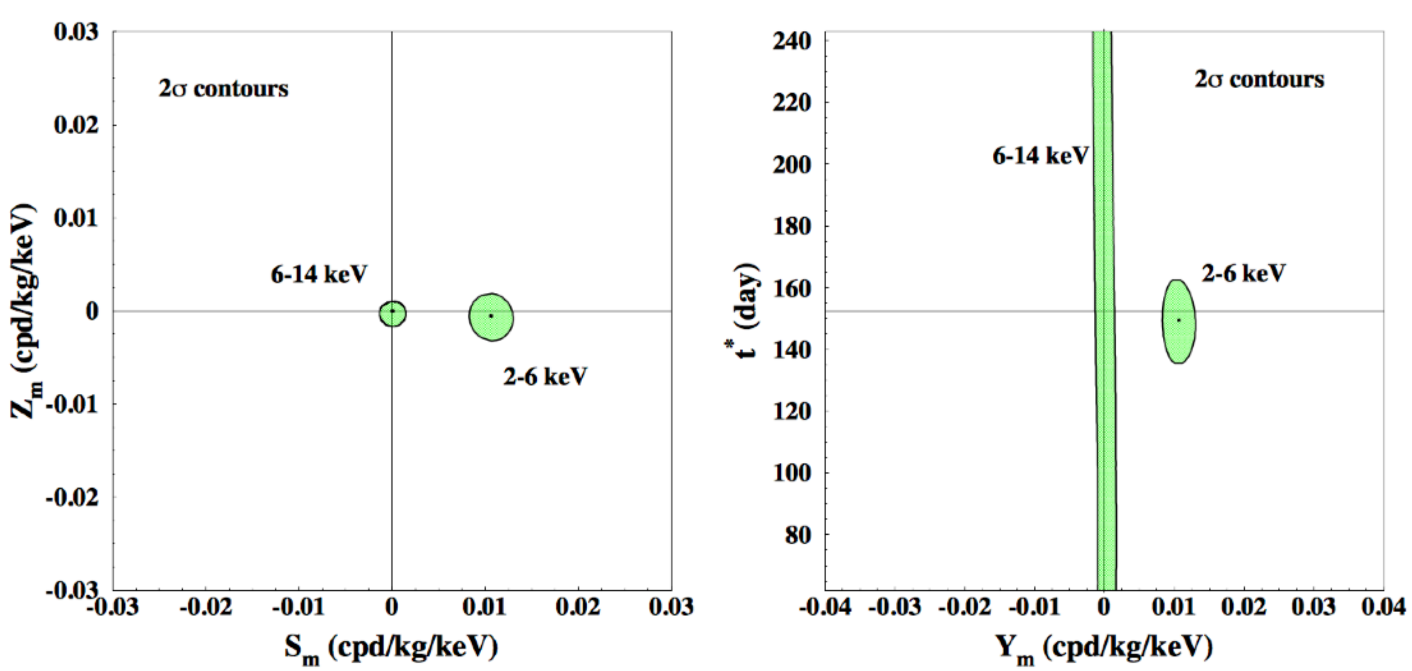
$$\tilde{\eta}^* = \mathcal{C} \sum_{h=1}^{\mathcal{N}+1} f_h^{\rm gal} \frac{1}{T} \int dt \, \frac{\Theta(|\vec{u}_h - \vec{v}_\odot - \vec{v}_\oplus(t)| - v^*)}{|\vec{u}_h - \vec{v}_\odot - \vec{v}_\oplus(t)|}$$



DAMA/LIBRA

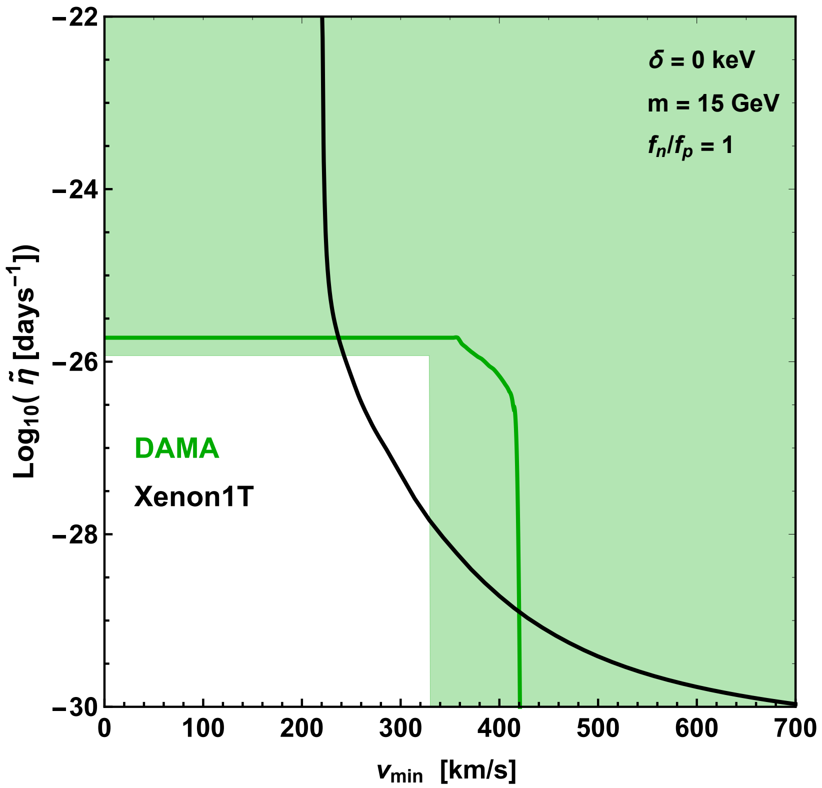
Infamous DAMA modulation at > 9 sigma

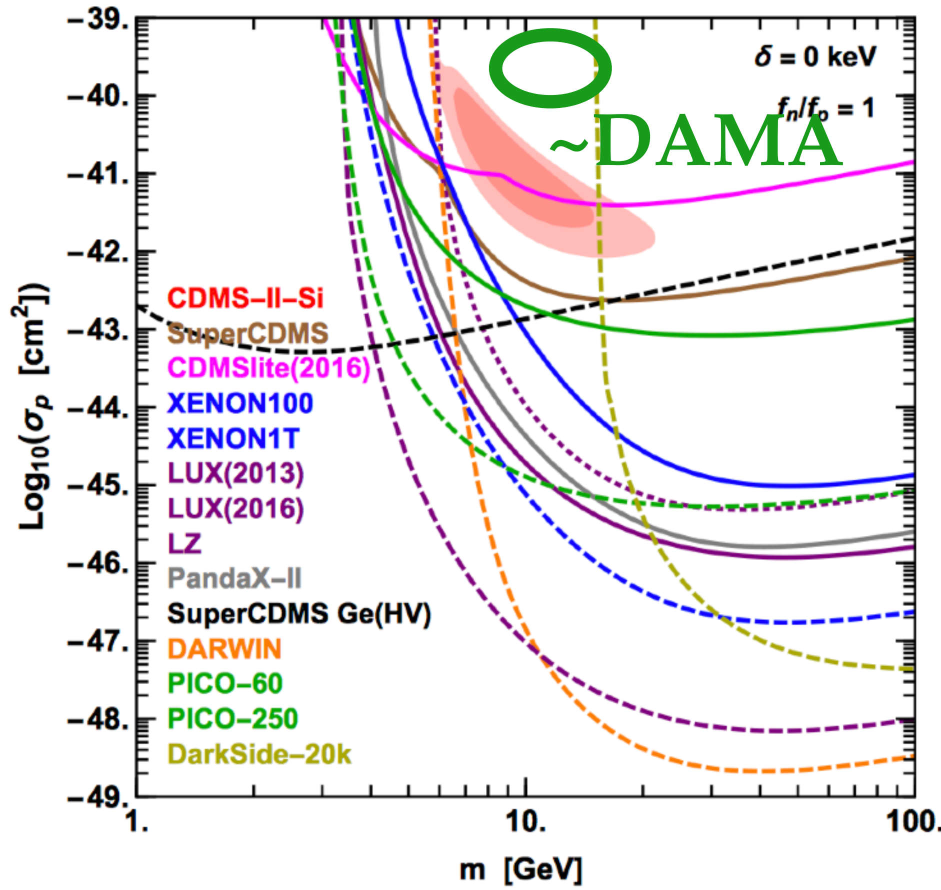




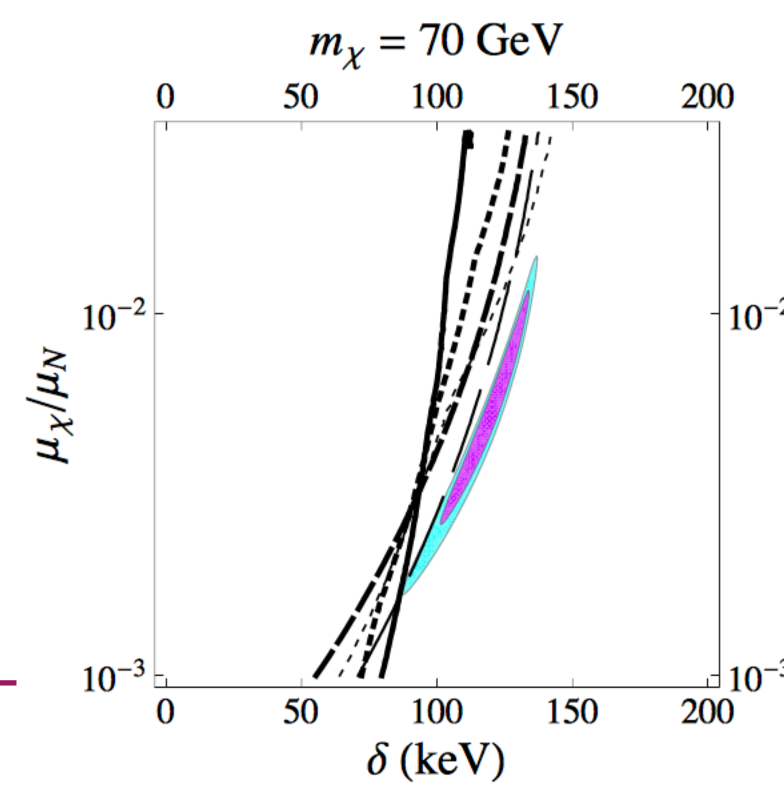
We can now infer preferred galactic velocity distributions, use these to calculate time-averaged rates and make apples-to-apples comparison with e.g. Xenon1T

DAMA/LIBRA



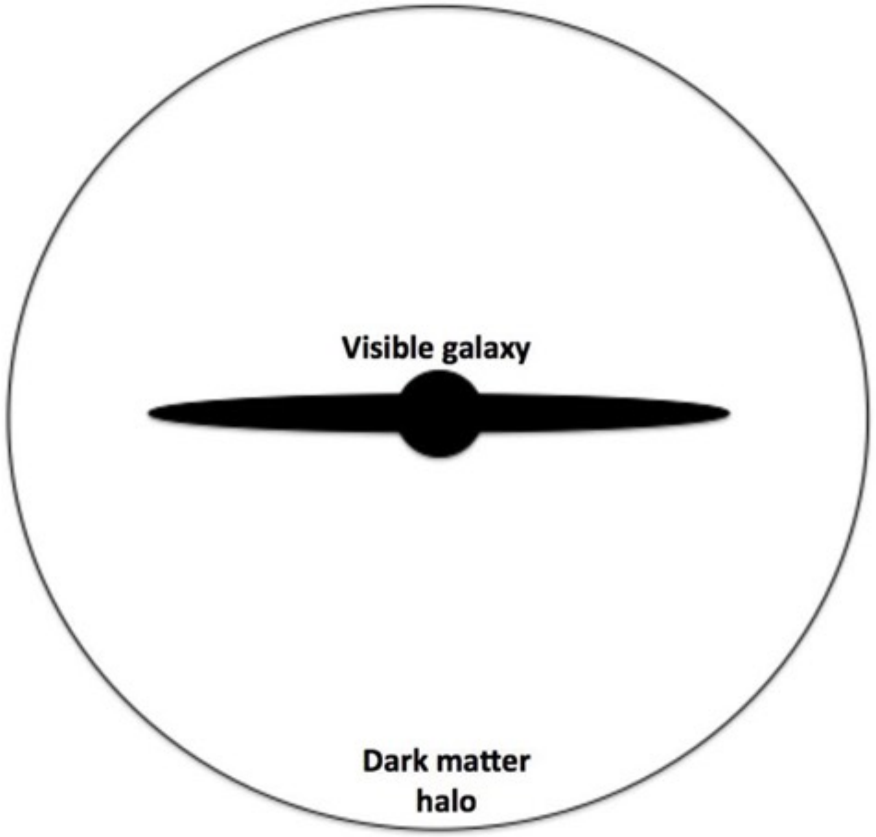


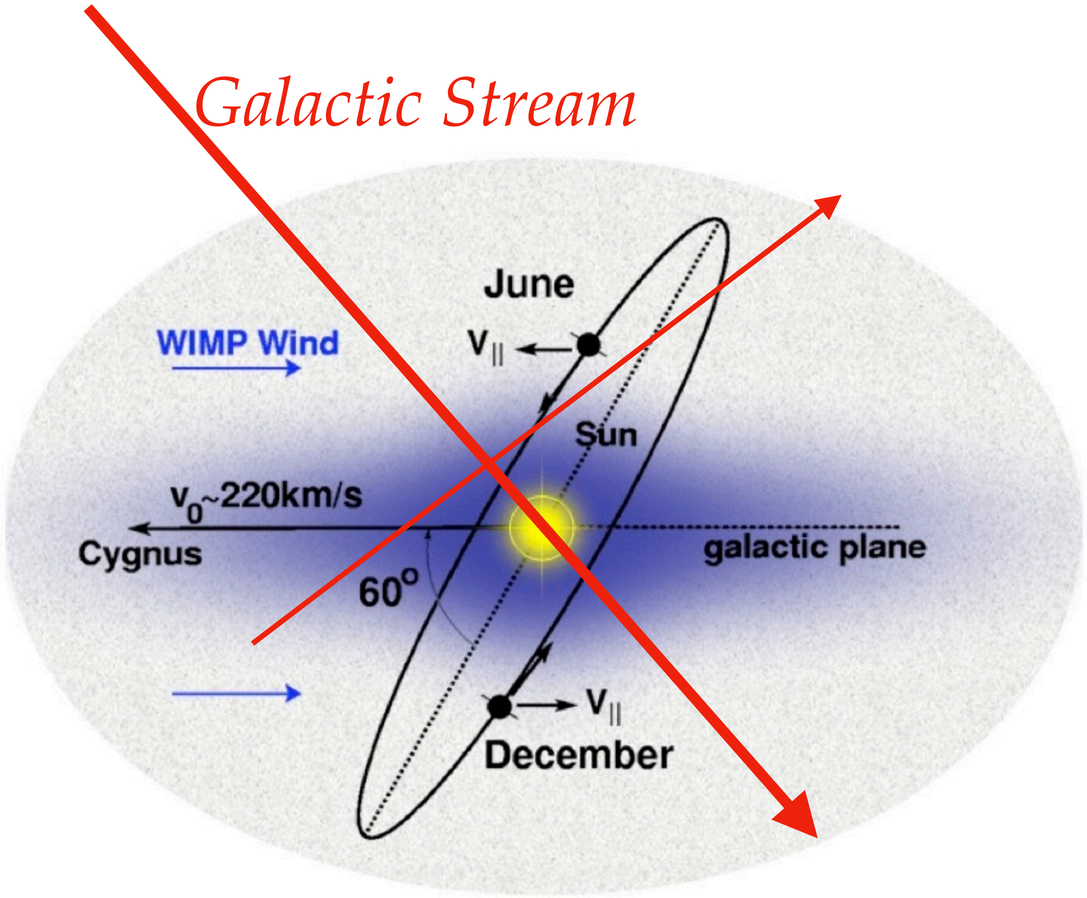
1703.06892 (SJW, Gelmini)

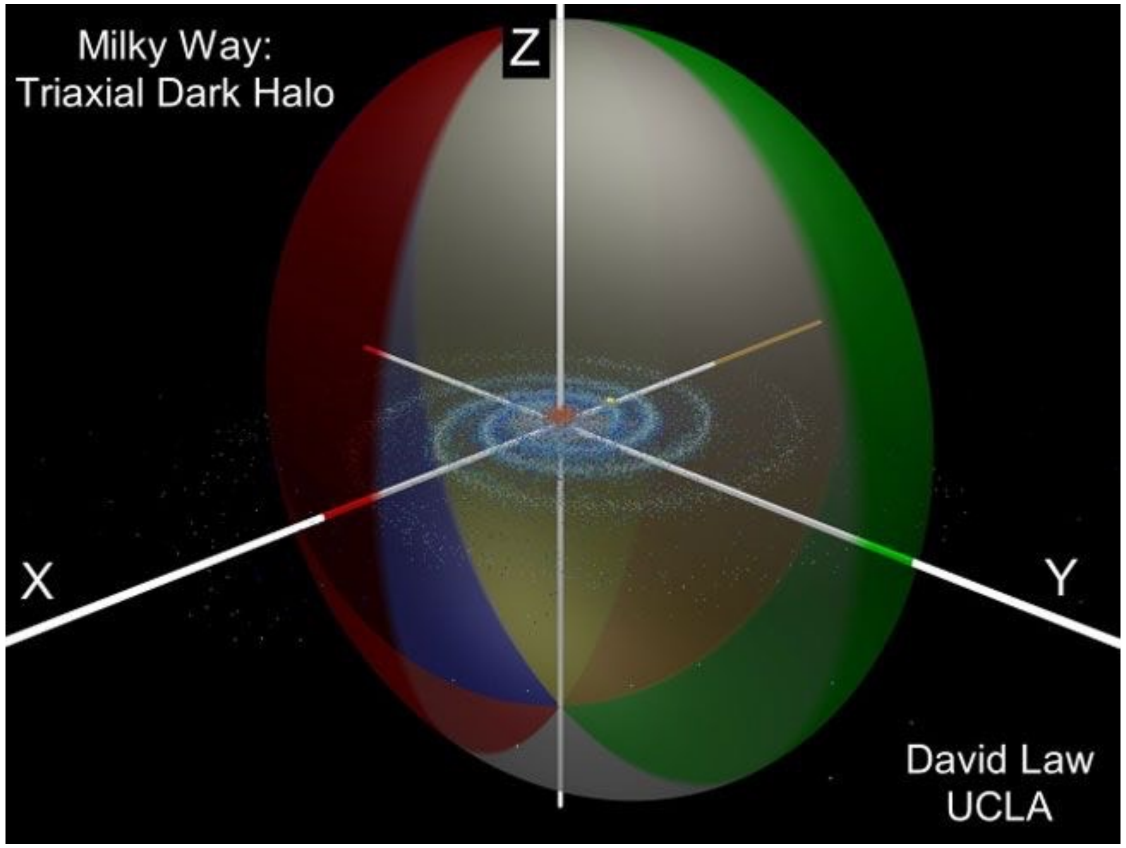


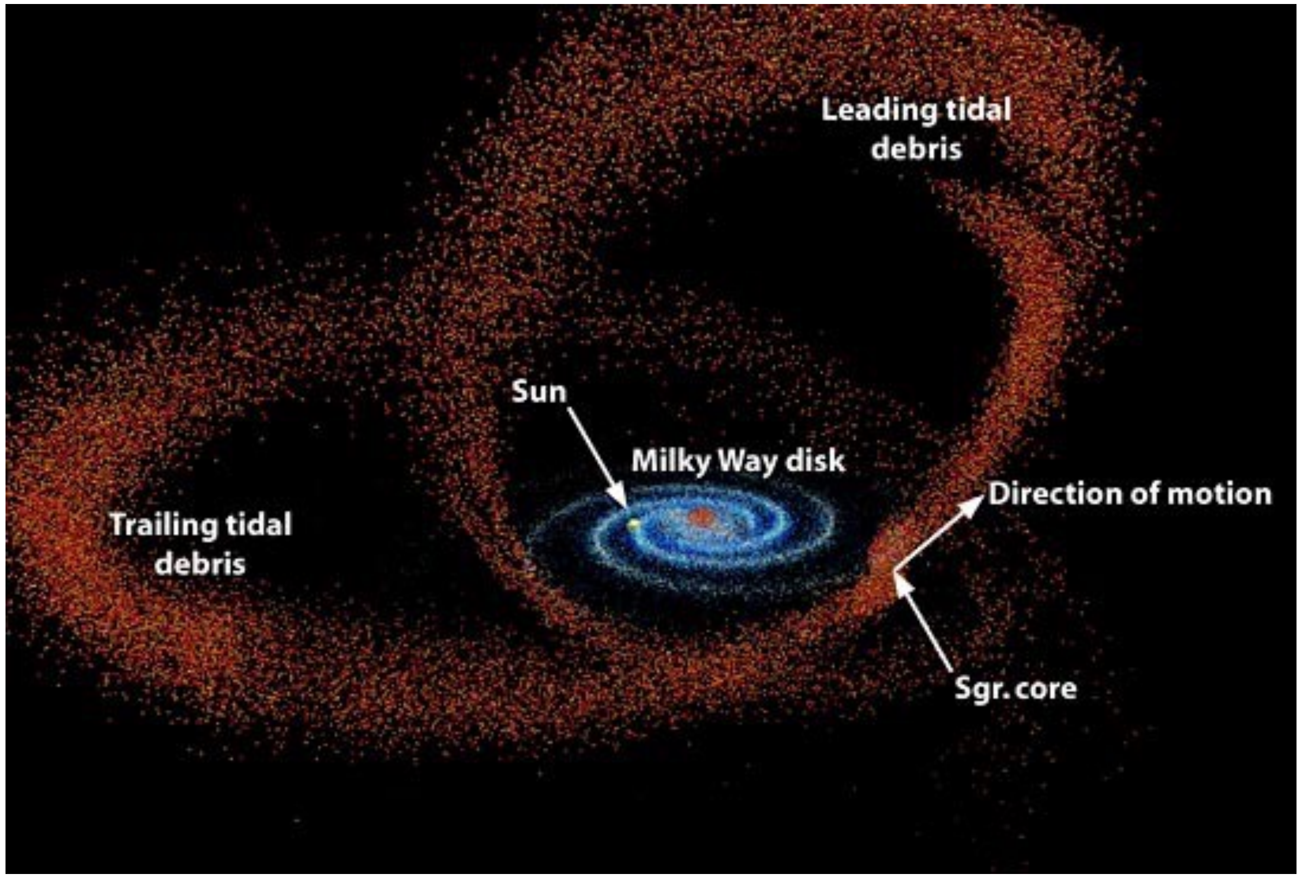
There exist more interesting considerations:

- Alternative Models
  - E.g. Magnetic inelastic dark matter (Chang, Weiner, Yavin — 1007.4200)
- Comparison with isotropic analysis (discussion to follow)


Visible galaxy
Dark matter
halo


Galactic Stream
June
WIMP Wind
$V_{||}$
Sun
v_0~220km/s
Cygnus
60°
galactic plane
$V_{||}$
December


Milky Way:
Triaxial Dark Halo
Z
X
Y
David Law
UCLA


Leading tidal
debris
Sun
Milky Way disk
Trailing tidal
debris
Direction of motion
Sgr. core

Isotropy

Enforcing isotropy makes velocity distribution more realistic and eases computation

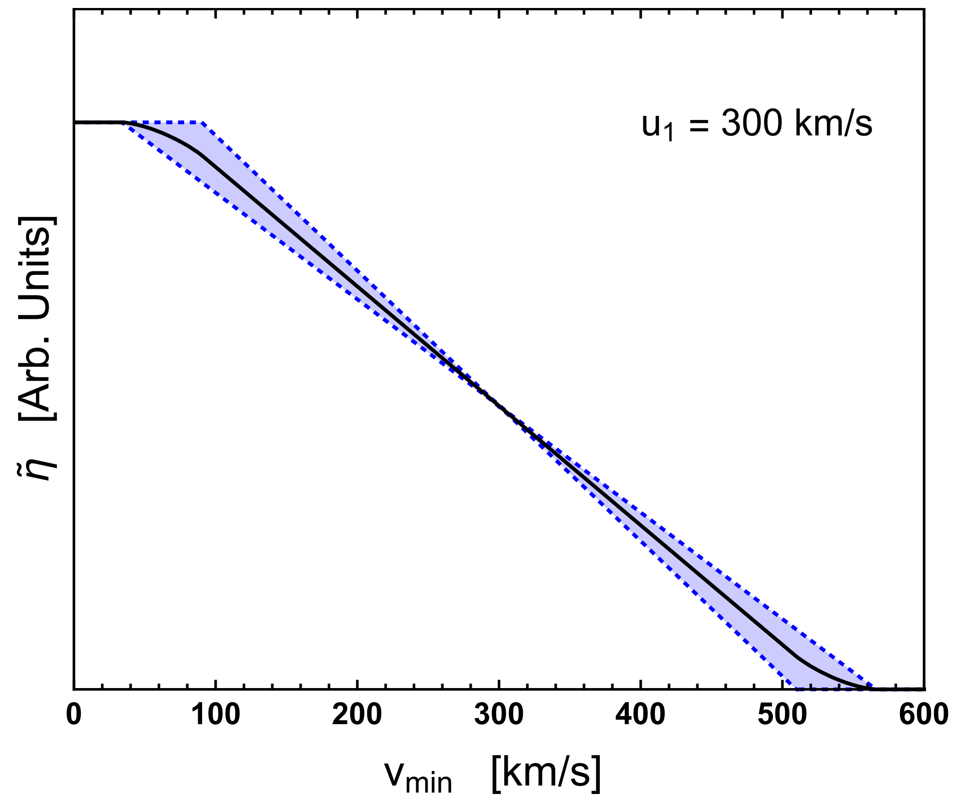
- Numerical simulations expect (more or less) isotropic distributions

$$f_G(\vec{u}) = f_G(|\vec{u}|)$$

$$R_{\alpha i}(t) = \int du\, \bar{\mathcal{H}}^{\text{gal}}_{\alpha I}(u,t)\, F^{\text{gal}}(u)$$

$$\bar{\mathcal{H}}^{\text{gal}}_{\alpha i}(u,t) \equiv \frac{1}{4\pi}\int d\Omega_u \mathcal{H}^{\text{gal}}_{\alpha i}(\vec{u},t)$$

$$F^{\text{gal}}(u) \equiv 4\pi u^2 f^{\text{gal}}(u)$$



$$\tilde{\eta}_{\text{BF}}(v_{\min},t) = \sum_{h=1}^{\mathcal{N}} \mathcal{C}F_h \times \begin{cases} \dfrac{1}{u_h} & v_{\min} \le u_h - u_\oplus(t) \\ \dfrac{u_\oplus(t) + u_h - v_{\min}}{2u_\oplus(t)u_h} & u_h - u_\oplus(t) < v_{\min} < u_h + u_\oplus(t) \end{cases}$$

Where:

$$u_\oplus(t) = |\vec{v}_\odot + \vec{v}_\oplus(t)|$$

Connections with Indirect Detection

Capture rate in Sun depends on same distribution

$$C = \sum_i \int_0^{R_\odot} 4\pi r^2 dr\, \eta_i(r) \frac{\rho_{\rm loc}}{m_\chi} \int_{v \le v^{\rm SUN}_{max,i}(r)} d^3v\, \frac{f(\vec{v})}{v} (v^2 + v_{\rm esc}(r)^2) \int_{m_\chi v^2/2}^{2\mu^2(v^2+v_{\rm esc}(r)^2)/m_A} dE_R\, \frac{d\sigma_i}{dE_R}$$

Can insert halo-independent DD results into indirect detection calculations, or perform joint analysis

Ibarra and Rappelt 2017