The halo-independent approach as a problem of moments

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- The halo-independent approach
- The problem of moments
- The halo-independent approach as a problem of moments
- First application: the DAMA unmodulated signal
  with isotropic galactic velocity distributions

Recasting the halo-independent approach as a problem of moments can address questions beyond the comparison of experiments.

For example:

- maximum likelihood with an infinite number of nuisance parameters
- include direct- and indirect-detection data
- statistical tests of compatibility
- information on distribution function itself
- predictions for future experiments

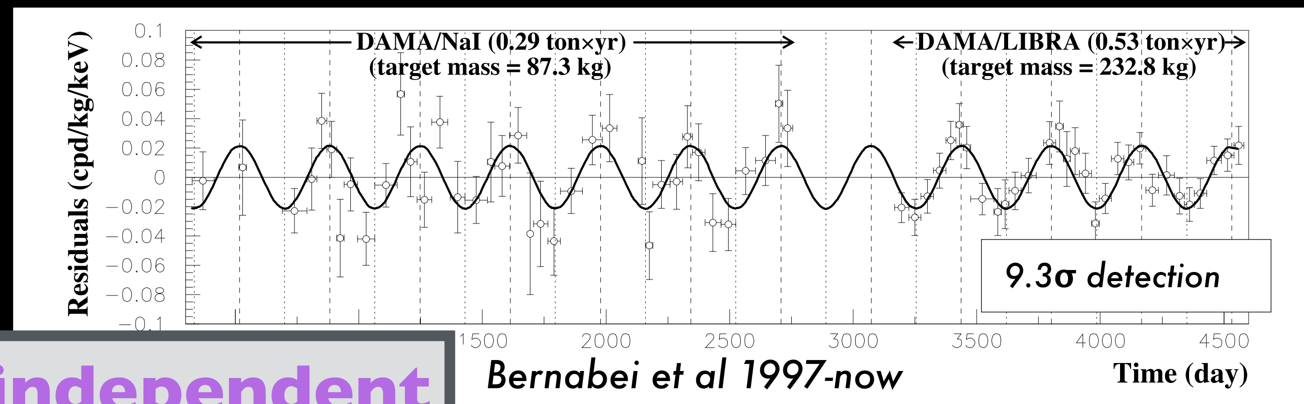
The halo-independent approach

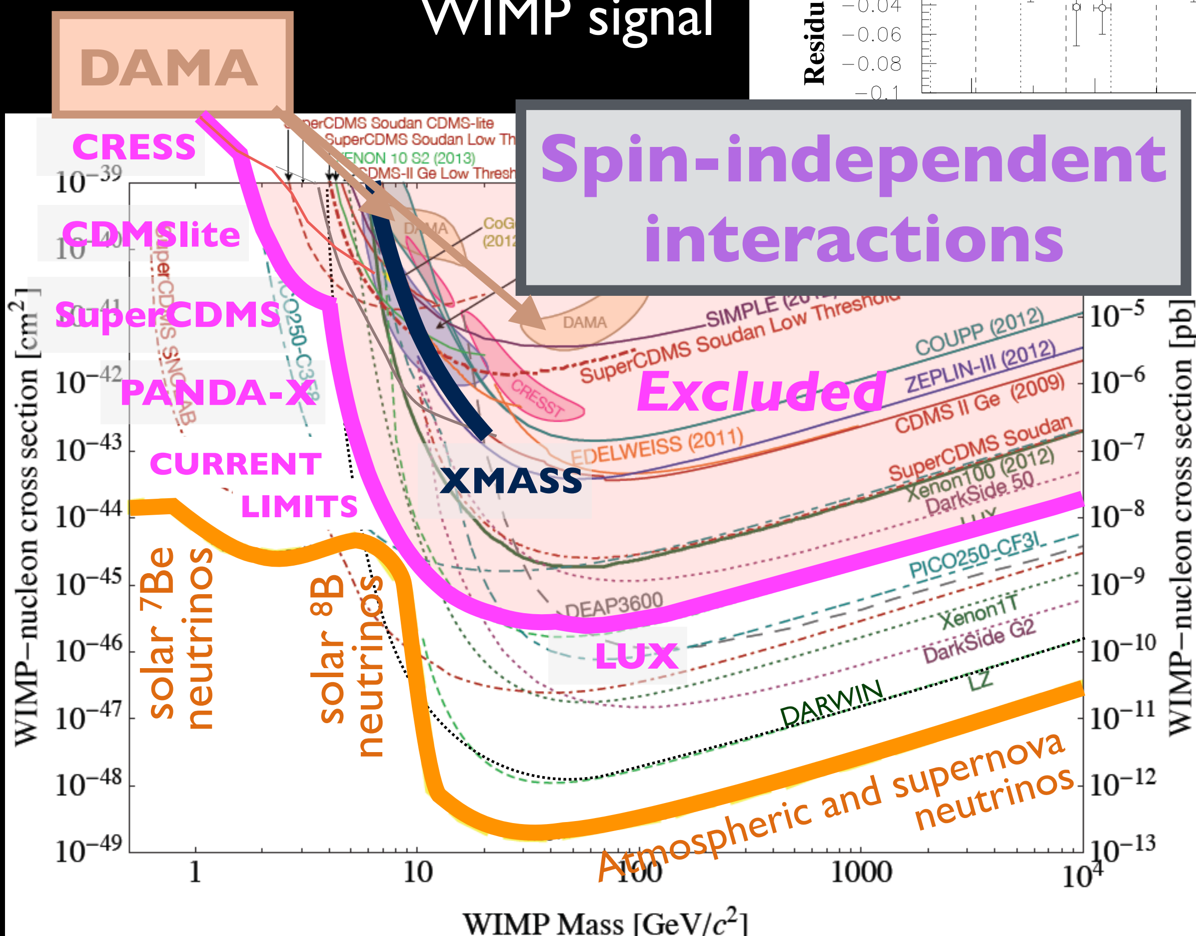
The DAMA modulation

"What does not kill me makes me stronger"

F. Nietsche, "Twilight of the Idols, or How to philosophize with a Hammer" (1888)

DAMA observes an annual modulation with the characteristics of a WIMP signal





The DAMA signal seems incompatible with other experiments

Spin-dependent, spin-independent. electron-WIMP, etc. interactions

Standard halo model (Maxwellian velocity distribution)

Halo-independent approach

Do not assume any particular
WIMP density or velocity distribution

One could put bounds separately for each assumed velocity distribution. But how does one put them together? Introduce the probability of a distribution? These questions are too hard. We follow an alternative route.

Halo-independent approach

$$\begin{pmatrix}\text{event}\\ \text{rate}\end{pmatrix} = \begin{pmatrix}\text{detector}\\ \text{response}\end{pmatrix} \times \begin{pmatrix}\text{particle}\\ \text{physics}\end{pmatrix} \times (\text{astrophysics})$$

The scattering rate per unit target mass (recoil spectrum)

$$\frac{dR}{dE_R} = \frac{1}{m_T}\frac{\rho_\chi}{m_\chi}\int_{v>v_{\min}} v^2 \frac{d\sigma}{dE_R}\frac{f(\mathbf{v})}{v}\, d^3\mathbf{v}$$

The event rate per unit target mass (actually measured)

$$\frac{dR}{dE} = \int_0^\infty \mathcal{G}(E, E_R)\frac{dR}{dE_R}\, dE_R$$

Measured energy

Effective energy response function: probability of actually detecting an event that occurred

Recoil energy

Halo-independent approach

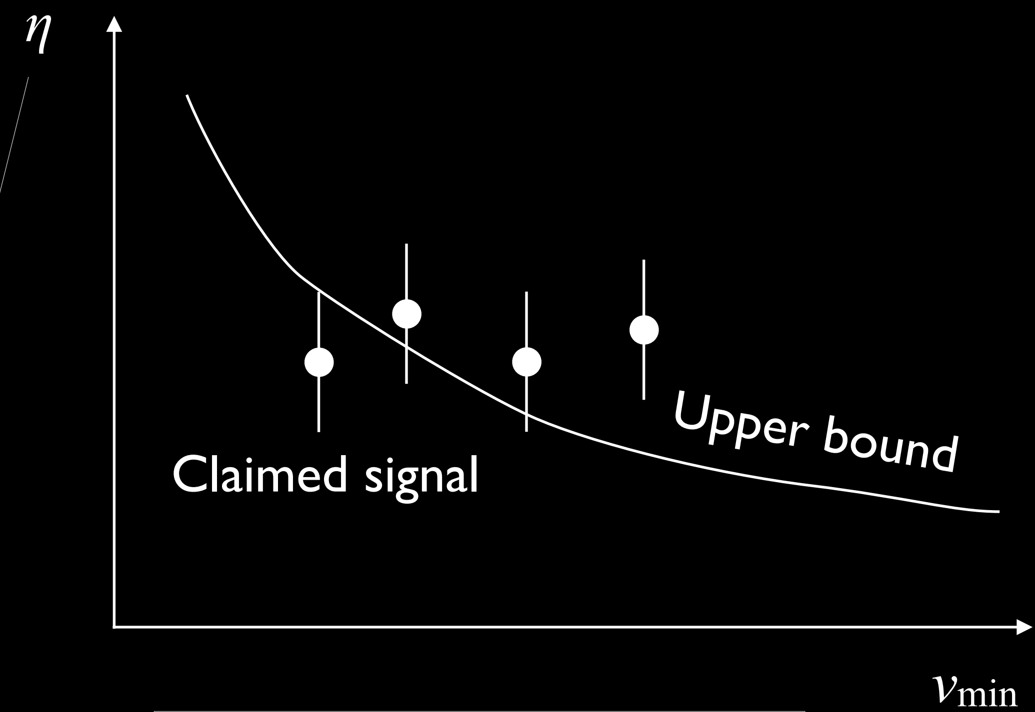
$$\begin{pmatrix}\text{event}\\ \text{rate}\end{pmatrix} = \begin{pmatrix}\text{detector}\\ \text{response}\end{pmatrix} \times \begin{pmatrix}\text{particle}\\ \text{physics}\end{pmatrix} \times (\text{astrophysics})$$

FIXED (particle physics) ARBITRARY (astrophysics)

Rescaled astrophysics factor common to all experiments

$$\eta(v_{\rm min}) = \frac{\rho_\chi}{m_\chi}\int_{v_{\rm min}}^{\infty} \frac{f(\mathbf{v})}{v}\, d^3v$$

"Velocity integral"
Proxy for dark matter flux



Minimum WIMP speed to impart recoil energy E_R

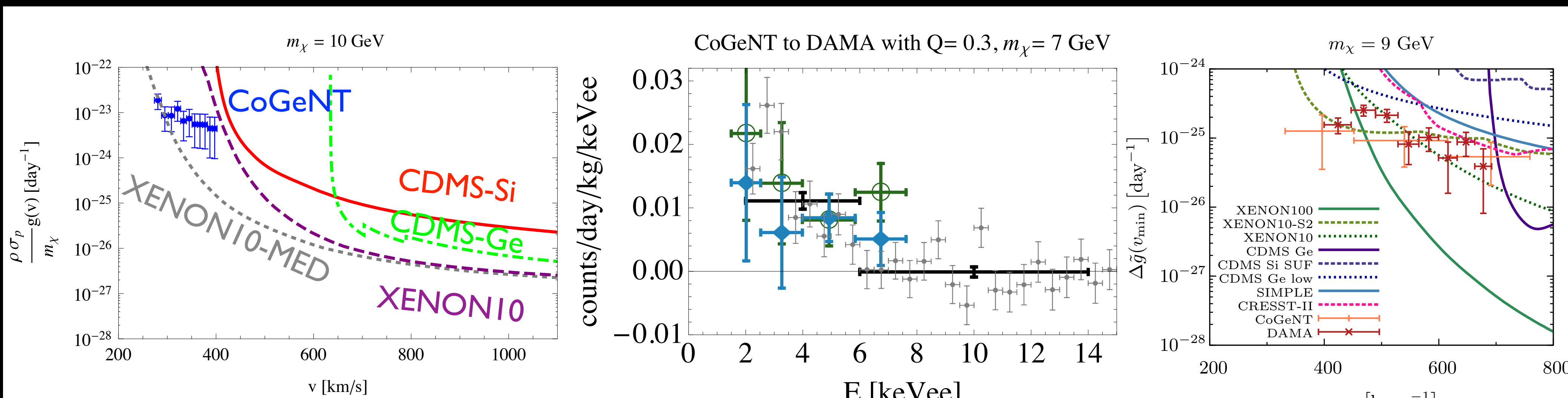
Fox, Liu, Wiener 2011; Gondolo, Gelmini 2012; Del Nobile, Gelmini, Gondolo, Huh 2013-14

Halo-independent approach

Find velocity integral from one experiment and use it for another.

Fox, Liu, Weiner 2011

$$\frac{dR}{dE_R} = \frac{A^2 F^2(E_R)}{2\mu_{\chi p}^2}\,\tilde{\eta}(v_{\min}) \quad \Rightarrow \quad \tilde{\eta}(v_{\min}) = \frac{2\mu_{\chi p}^2}{A^2 F^2(E_R)}\,\frac{dR}{dE_R}$$



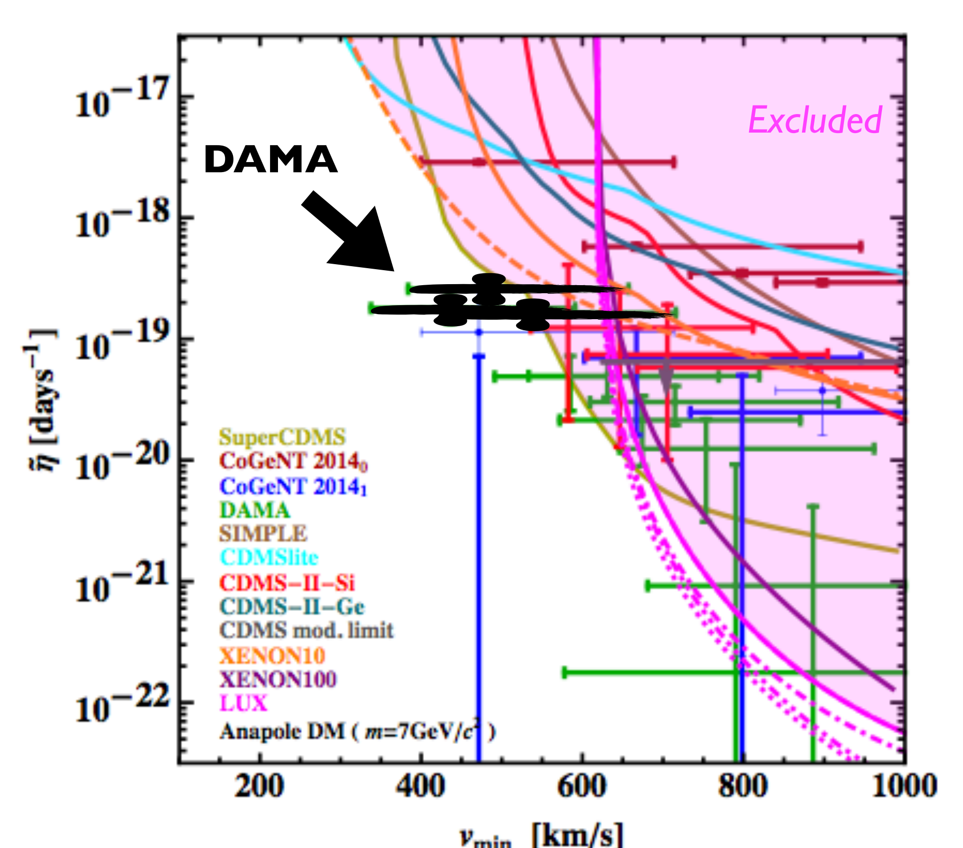
Fox, Liu, Weiner 2011 — *Fox, Kopp, Lisanti, Weiner 2011* — *Frandsen et al 2011*

Needs unique relation between measured energy and minimum WIMP speed, available for ***single-target detectors with excellent energy resolution***.
For composite targets, lucky event pattern in CRESST allowed inversion.

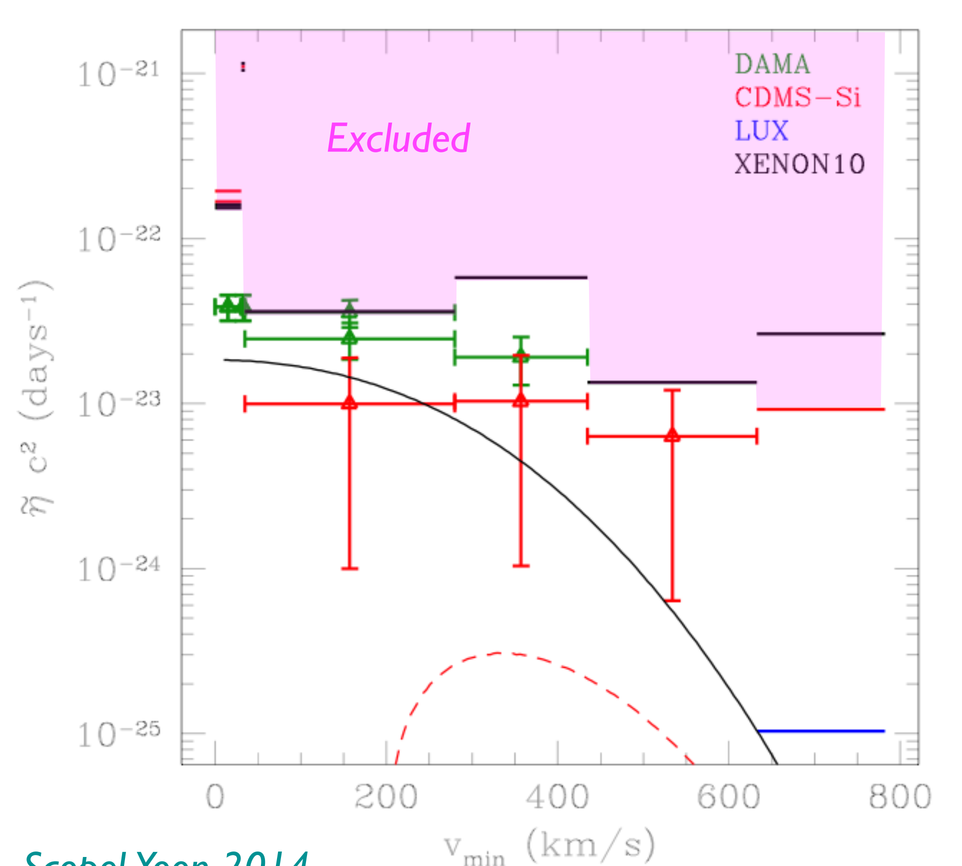
Halo-independent approach

In general, for ***composite targets and finite energy resolution***, one can still find ***weighted averages of the velocity integral***.

DAMA may be compatible with null searches for anapole and exothermic dark matter.



Del Nobile, Gelmini, Gondolo, Huh 2014



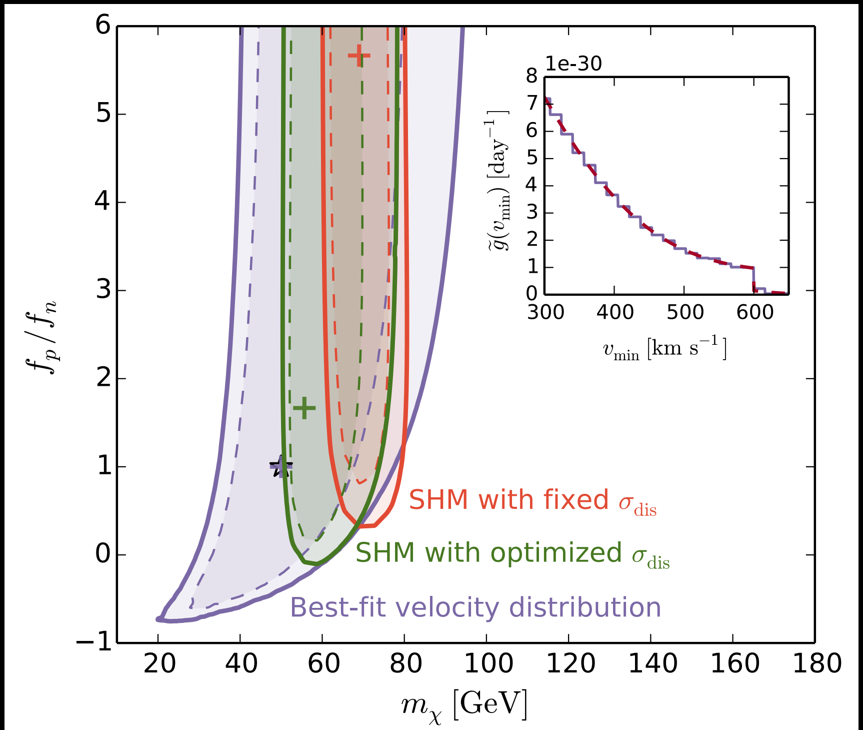
Scopel, Yoon 2014

Allows for ***any velocity and energy dependent cross section***, and indirect searches through ***neutrinos from the Sun/Earth***.

Halo-independent approach

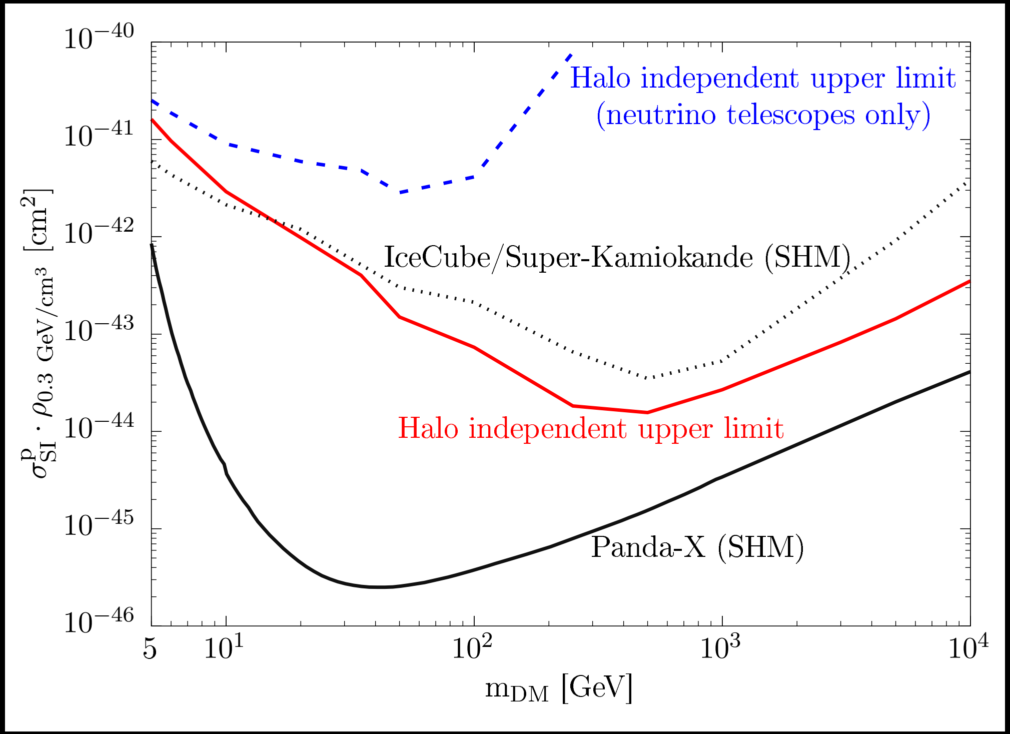
Alternatively, one has sampled ***discretized velocity distributions*** to find bounds from direct and indirect experiments (neutrinos from the Sun).

Likelihood for particle-physics parameters (mock data)



Feldstein, Kahlhoefer 2014

Bounds on cross section from direct detection and neutrinos from the Sun



Ibarra, Rappelt 2017

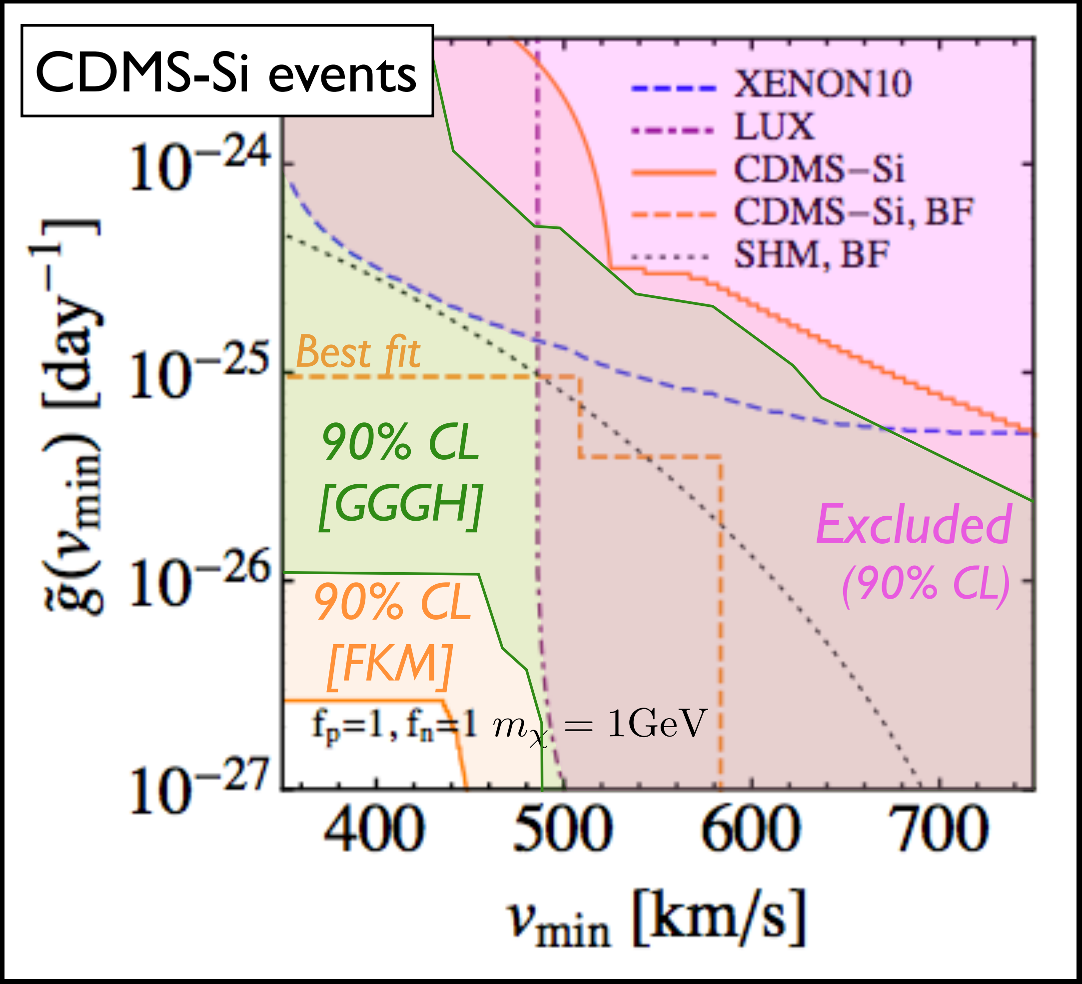
Halo-independent approach

Open questions include the statistical significance of the bounds obtained and of the comparison of experiments.

Unbinned likelihood analysis

$$\mathcal{L} = \frac{e^{-\int_{E_{\rm min}}^{E_{\rm max}} \frac{dR}{dE}\, dE}}{N!} \prod_{i=1}^{N} \frac{dR}{dE}\Big|_{E=E_i}$$

The extent of the 90% CL region is still unclear



Fox, Kahn, McCullough 2015

Gelmini, Georgescu, Gondolo, Huh 2015

Halo-independent approach

Observables are integrals of the velocity distribution.

For example,

Event rate $$\frac{dR}{dE} = \int \mathcal{H}(\mathbf{v})\, f(\mathbf{v})\, d^3v$$

where $$\mathcal{H}(\mathbf{v}) = \int dE_R \; \mathcal{G}(E, E_R) \, \frac{v\rho_\chi}{m_T m_\chi} \, \frac{d\sigma}{dE_R}$$

is the event rate for a monochromatic WIMP beam of velocity $\mathbf{v}$.

Question: if we know some observables, can we estimate others?

The problem of moments

Chebyshev's problem of moments

What can be said about a probability distribution if its first *N* moments are known?

$$\int f(x)\,dx = 1$$ (normalization)

$$\int x\, f(x)\,dx = \mu_1$$ (mean)

$$\int x^2\, f(x)\,dx = \mu_2$$

$$\cdots\cdots\cdots$$

$$\int x^N\, f(x)\,dx = \mu_N$$

Here the μ_i are given numbers.

Markov's problem of moments

What can be said about a probability distribution if N of its generalized moments are known?

$$\int f(x)\,dx = 1 \qquad \text{(normalization)}$$

$$\int h_1(x)\,f(x)\,dx = y_1$$

$$\int h_2(x)\,f(x)\,dx = y_2$$

$$\cdots\cdots\cdots$$

$$\int h_N(x)\,f(x)\,dx = y_N$$

Here the $h_i(x)$ are given integrable functions, and the y_i are given numbers.

Bounds on integrals of f(x)

Bienaymé 1853, Chebyshev 1867, Markov 1884, Stieltjes 1884

Markov's inequality

For any probability distribution $f(x)$ defined on $x > 0$ with mean μ, and $a > 0$,

$$\int_a^\infty f(x)\,dx \le \frac{\mu}{a}$$

Chebyshev's inequality

For any probability distribution $f(x)$ with mean μ and dispersion σ, and $a > 0$,

$$\int_{\mu-a\sigma}^{\mu+a\sigma} f(x)\,dx \le \frac{1}{a^2}$$

Usually not very powerful but there are many ways to sharpen them.

Bounds on integrals of f(x)

The fundamental theorem (generalized Chebyshev inequalities)

Hoeffding 1955, Richter 1957, Mulholland&Rogers 1958, Isii 1960, Winkler 1988, Pinelis 2016

For probability distributions $f(x)$ that satisfy the *N*+1 moment conditions

$$\int h_i(x)\, f(x)\, dx = y_i \qquad (i = 0, 1, \ldots, N; h_0(x) = 1; y_0 = 1)$$

one has

$$\inf\left[\int g(x)\, f_{\rm e}(x)\, dx\right] \le \int g(x)\, f(x)\, dx \le \sup\left[\int g(x)\, f_{\rm e}(x)\, dx\right]$$

where the inf and the sup are over "extreme distributions" (positive sums of Dirac delta functions)

$$f_{\rm e}(x) = \sum_{j=0}^{N} \lambda_j\, \delta(x - x_j), \quad \lambda_j \ge 0\,, \quad \sum_{j=0}^{N} \lambda_j h_i(x_j) = y_i\,, \quad \left|h_i(x_j)\right| \ne 0\,.$$

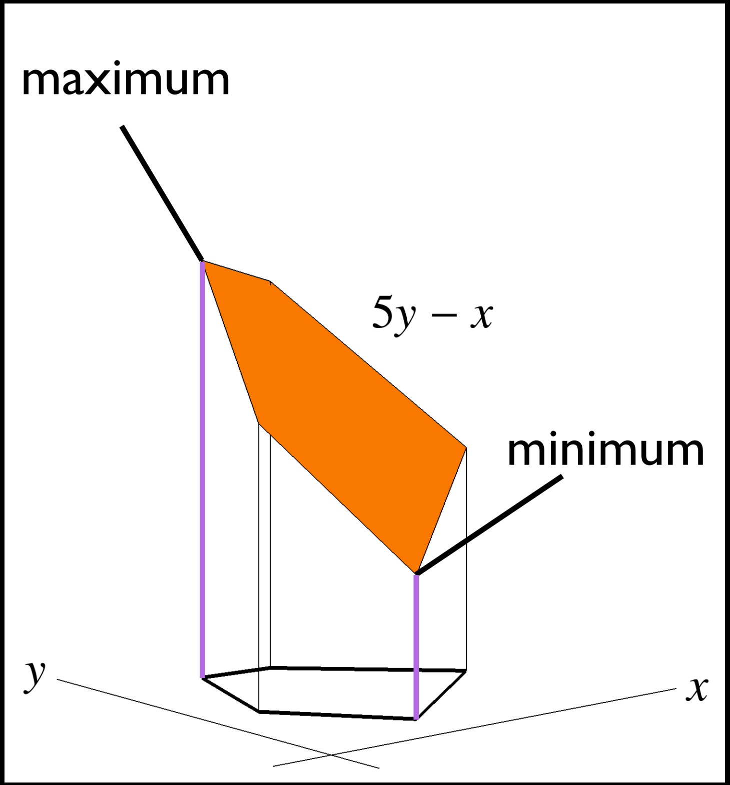
These inequalities are strict. They also apply for values of y_i in a region.

Bounds on integrals of f(x)

Finite-dimensional analog

Linear optimization

To find the maximum and minimum of a linear function of x, y, z, … defined on a convex region it is enough to compute the function at the vertices of the region.



Example: for a polygonal region, the maximum and minimum values are achieved at one of the vertices.

"Extreme distributions" are analogous to vertices. The fundamental theorem states that the maximum and minimum values of the linear functional $\int g(x)\, f(x)\, dx$ occur at an extreme distribution (vertex).

The halo-independent approach as a problem of moments

The halo-independent approach as a problem of moments

Observables are generalized moments of the velocity distribution.

For example, event rate $\frac{dR}{dE} = \int \mathcal{H}(\mathbf{v})\, f(\mathbf{v})\, d^3v$

We can access all the power of generalized Chebyshev inequalities and linear optimization in the infinite-dimensional space of distributions.

The fundamental theorem (generalized Chebyshev inequalities)
Given (ranges for) *N* measured observables, strict upper and lower bounds on any other observable can be found using at most *N*+1 streams.

First application:
estimating the DAMA unmodulated signal

Signals as integrals of f(v)

Gondolo, Scopel 2017

Write modulated and unmodulated signals as integrals over the same velocity distribution. For this purpose, use velocity distribution in galactic rest frame.

$$f_{\rm lab}(\mathbf{v},t) = f_{\rm gal}(\mathbf{u}) \qquad \mathbf{u} = \mathbf{v} + \mathbf{v}_\odot + \mathbf{v}_\oplus(t)$$

$$S_i(t) = \int \mathcal{H}_i(\mathbf{v})\, f_{\rm lab}(\mathbf{v},t)\, d^3v = \int \mathcal{H}_i^{\rm gal}(\mathbf{u},t)\, f_{\rm gal}(\mathbf{u})\, d^3u$$

Unmodulated signals in each energy bin (constant Fourier coefficient)

$$S_{0,i} = \int \mathcal{H}_{0,i}^{\rm gal}(\mathbf{u})\, f_{\rm gal}(\mathbf{u})\, d^3u$$

$$\mathcal{H}_{0,i}^{\rm gal}(\mathbf{u}) = \frac{2}{T} \int_0^T dt\, \mathcal{H}_i\big(\mathbf{u} - \mathbf{v}_\odot - \mathbf{v}_\oplus(t)\big)$$

Modulation amplitudes in each energy bin (cosine Fourier coefficient)

$$S_{m,i} = \int \mathcal{H}_{m,i}^{\rm gal}(\mathbf{u})\, f_{\rm gal}(\mathbf{u})\, d^3u$$

$$\mathcal{H}_{m,i}^{\rm gal}(\mathbf{u}) = \frac{1}{T} \int_0^T dt\, \cos[\omega(t-t_0)]\, \mathcal{H}_i\big(\mathbf{u} - \mathbf{v}_\odot - \mathbf{v}_\oplus(t)\big)$$

Profile likelihood

Gondolo, Scopel 2017

Likelihood of DAMA modulation amplitudes

$$-2 \ln \mathcal{L}(\underset{\sim}{S_m}) = \sum_{j=1}^{N} \left(\frac{S_{m,j} - S_{m,j}^{\exp}}{\Delta S_j^{\exp}} \right)^2$$

Profile the likelihood over all velocity distributions that satisfy the given data (infinitely-many nuisance parameters)

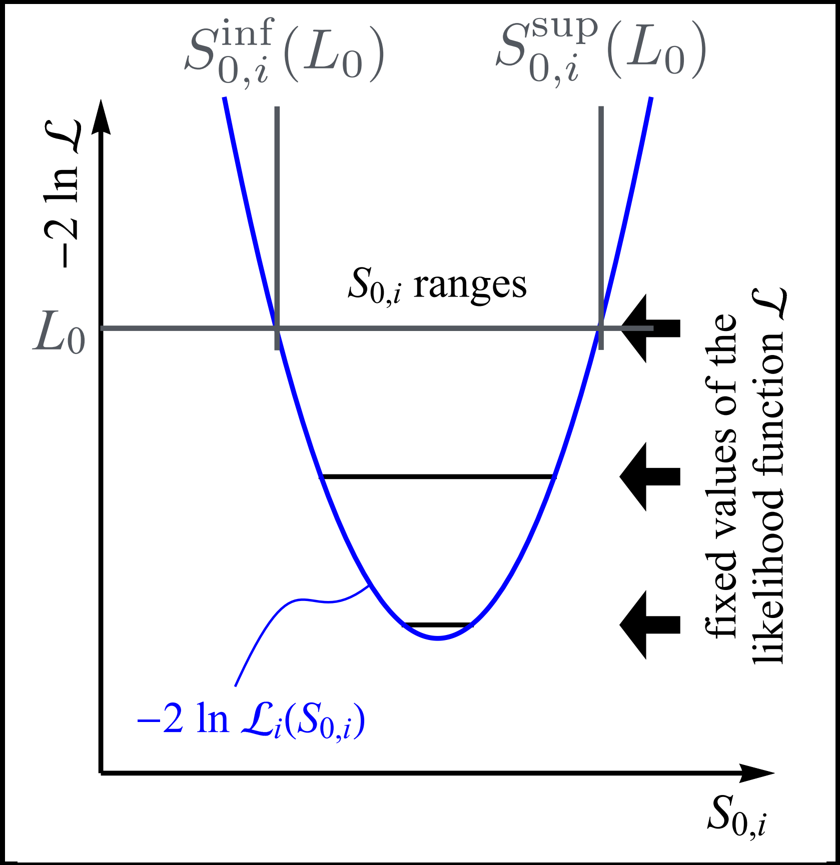
$$\mathcal{L}_i(S_{0,i}) = \sup_{f_{\rm gal} \in \mathcal{A}(S_{0,i})} \mathcal{L}(\underset{\sim}{S_m})$$

where $\mathcal{A}(S_{0,i})$ is the set of distributions that satisfy the moment constraints

$$S_{0,i} = \int \mathcal{H}_{0,i}^{\rm gal}(\mathbf{u}) \, f_{\rm gal}(\mathbf{u}) \, d^3u$$

Profile likelihood *(continued)*

Gondolo, Scopel 2017



Compute the profile likelihood as an extremization problem for S_{0i} at fixed likelihood L_0.

$$\text{Extremize } S_{0,i} = \int \mathcal{H}_{0,i}^{\text{gal}}(\mathbf{u})\, f_{\text{gal}}(\mathbf{u})\, d^3u$$

$$\text{subject to } \int f_{\text{gal}}(\mathbf{u})\, d^3u = 1$$

$$\int \mathcal{H}_{m,j}^{\text{gal}}(\mathbf{u})\, f_{\text{gal}}(\mathbf{u})\, d^3u = S_{m,j} \quad (j = 1, \ldots, N)$$

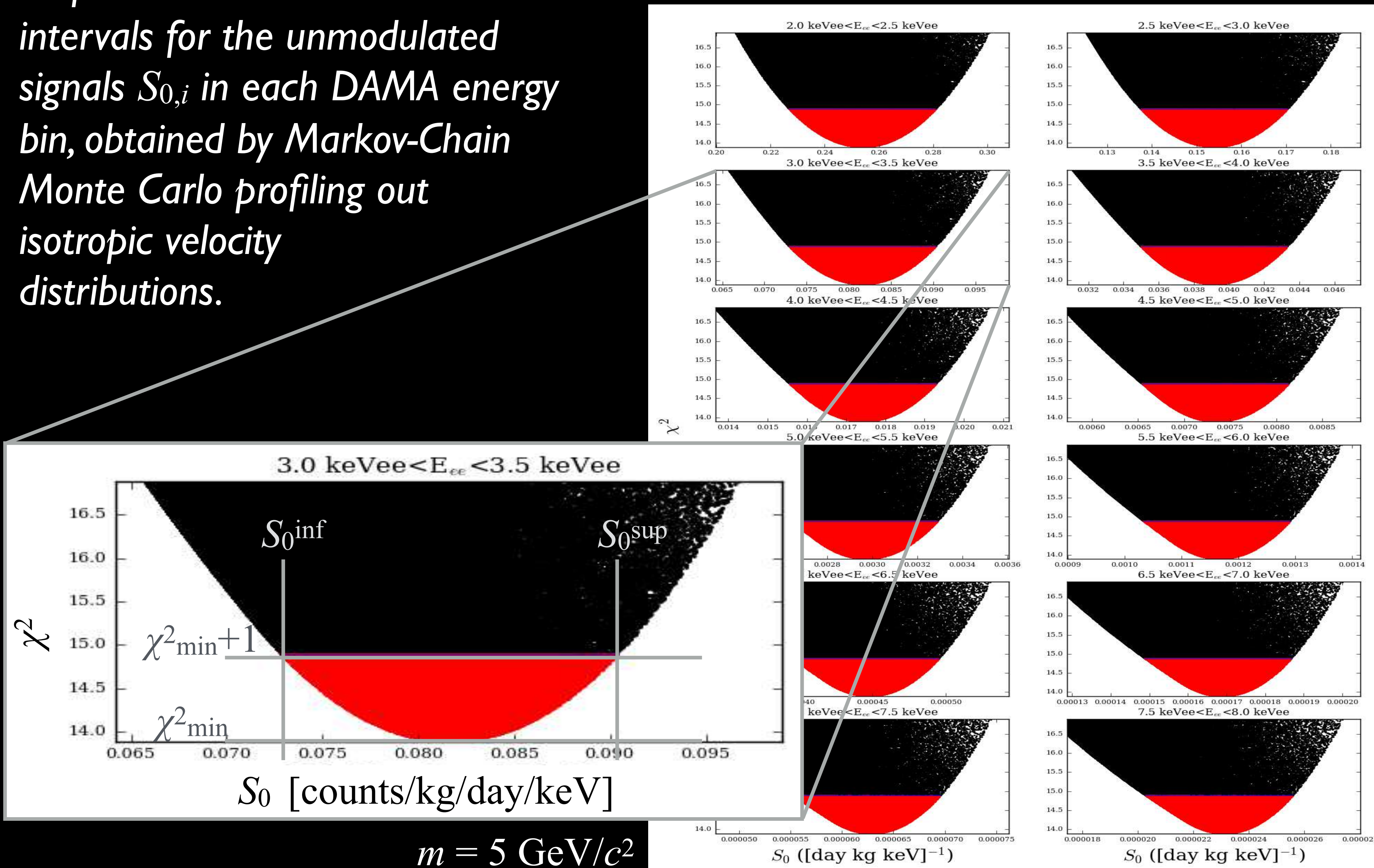
$$\mathcal{L}(\underset{\sim\sim\sim}{S_{m,j}}) \geq L_0$$

Restrict to velocity distributions that are isotropic in galactic frame (for faster computation)

Monte-Carlo

Gondolo, Scopel 2017

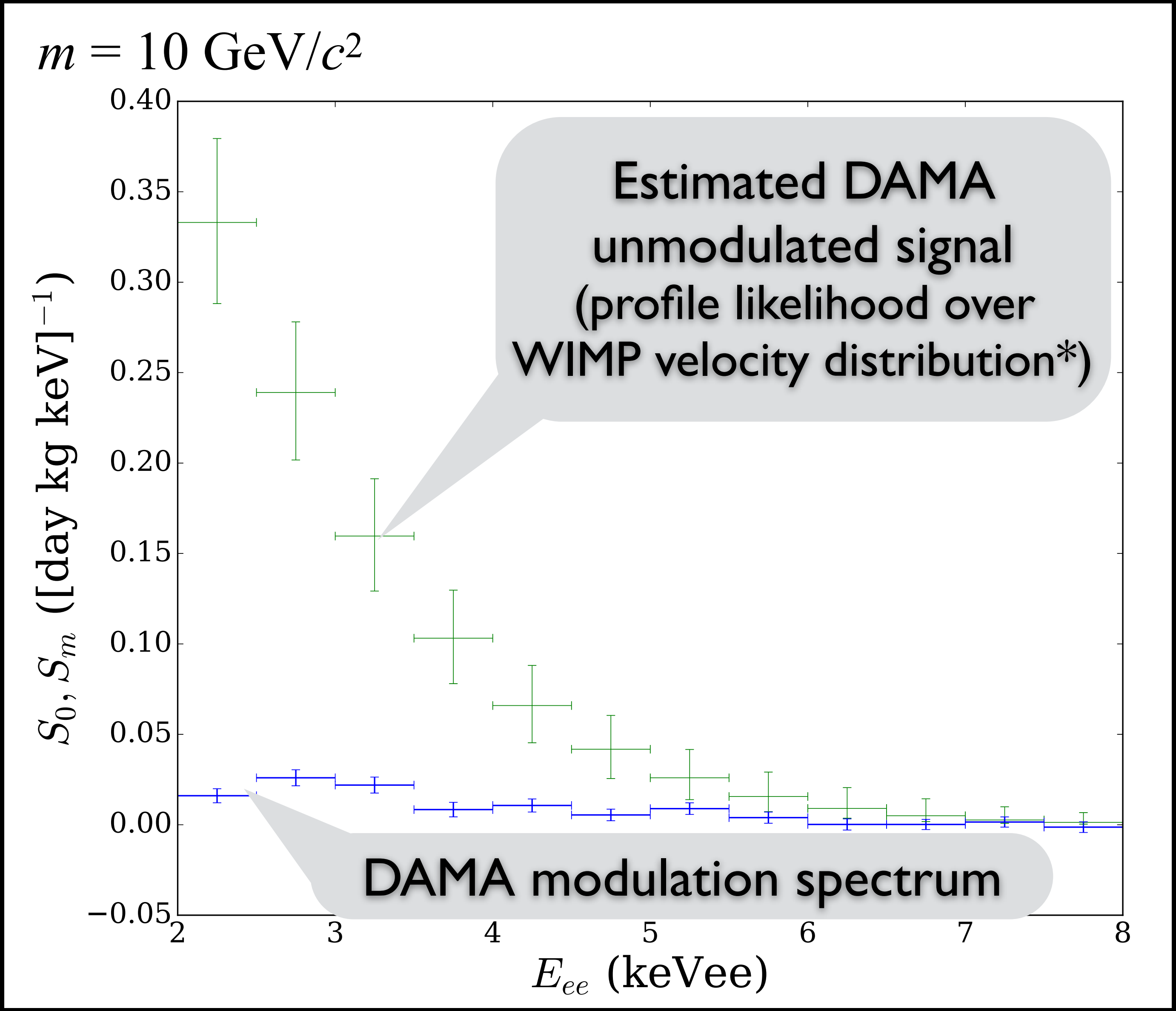
Profile likelihood and likelihood intervals for the unmodulated signals $S_{0,i}$ in each DAMA energy bin, obtained by Markov-Chain Monte Carlo profiling out isotropic velocity distributions.



Results

Gondolo, Scopel 2017

Halo-independent estimate of the DAMA unmodulated signal



$S_{0,i}$	$S_{0,i} + B_i$
$0.33^{+0.05}_{-0.05}$	1.029
$0.24^{+0.04}_{-0.04}$	1.228
$0.16^{+0.03}_{-0.03}$	1.294
$0.10^{+0.02}_{-0.03}$	1.140
$0.066^{+0.02}_{-0.02}$	0.956

The unmodulated signal is compatible with background+signal.

(*isotropic in galactic rest frame)

Summary

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- First application: the DAMA unmodulated signal with isotropic galactic velocity distributions

Recasting the halo-independent approach as a problem of moments can address questions beyond the comparison of experiments.

Work continues to understand the full power of this method and to bring it to complete fruition (e.g., include all data, statistical tests of compatibility, information on distribution function itself, etc.).