Hadronic Contributions to the Muon $g - 2$ from Lattice QCD
Where do we stand?

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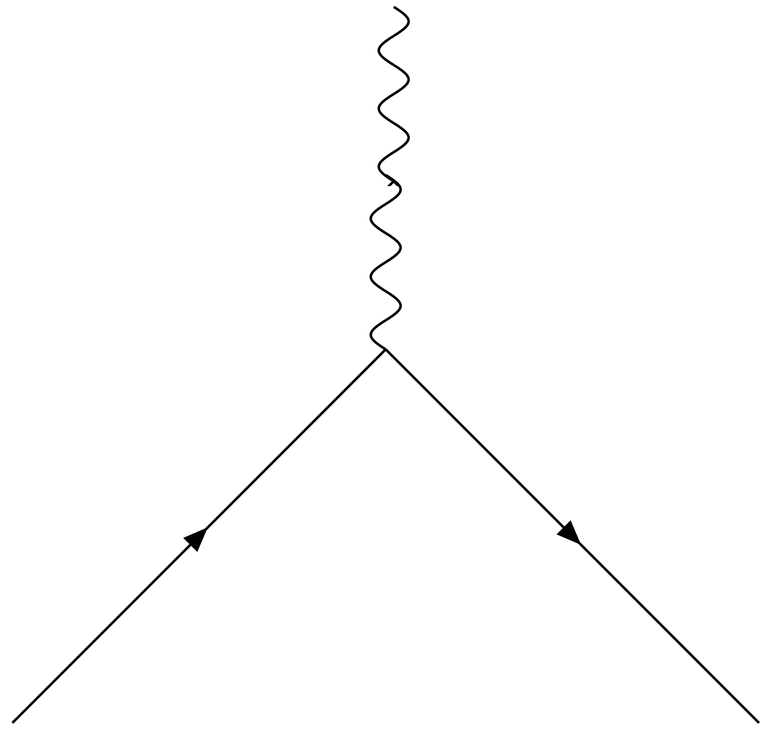
Anomalous magnetic moments

Particle with mass m and charge e :

$$\boldsymbol{\mu} = g\,\frac{e\hbar}{2m}\,\boldsymbol{S}, \qquad \boldsymbol{S} = \frac{\boldsymbol{\sigma}}{2}$$



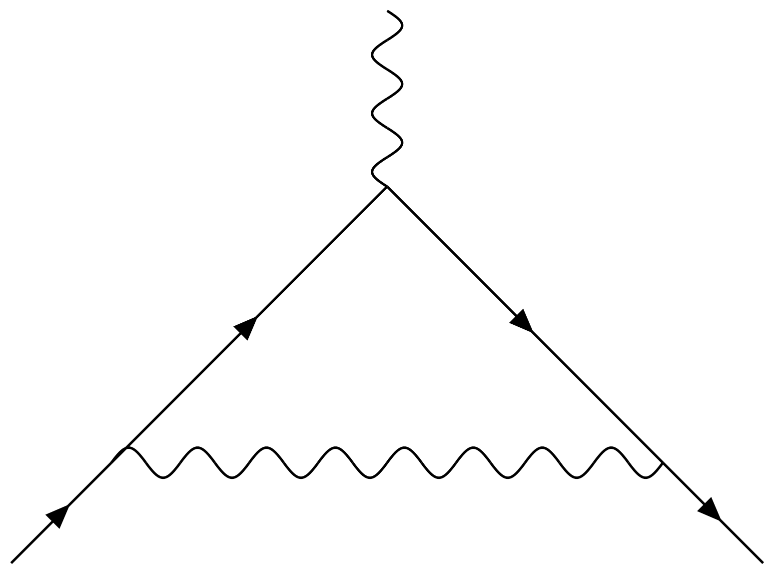
Pauli equation: $g = 2$

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = \left\{\frac{1}{2m}\left(\boldsymbol{p} - e\boldsymbol{A}\right)^2 + e\Phi - \frac{e\hbar}{2m}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}\right\}\psi(\boldsymbol{x},t)$$

Quantum corrections: $g = 2(1+a)$, a : anomalous magnetic moment

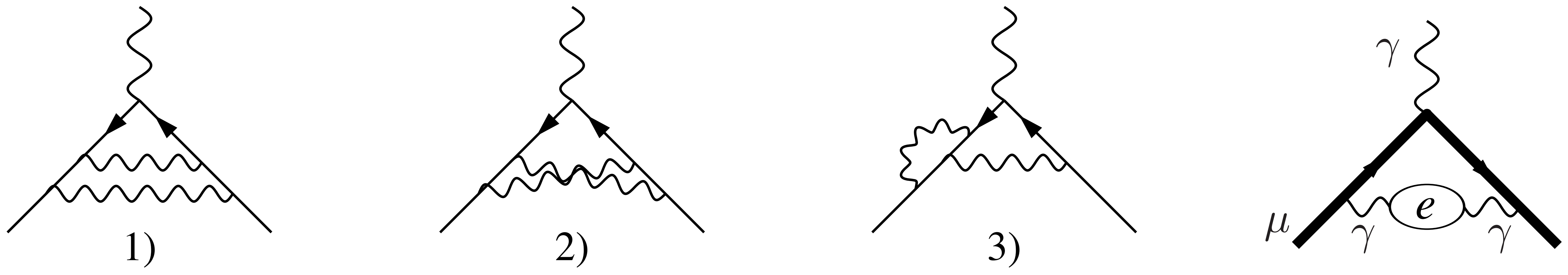
First-order QED correction calculated by Schwinger:

$$g = 2\left(1 + \frac{\alpha}{2\pi}\right)$$



Higher-order corrections

QED corrections:



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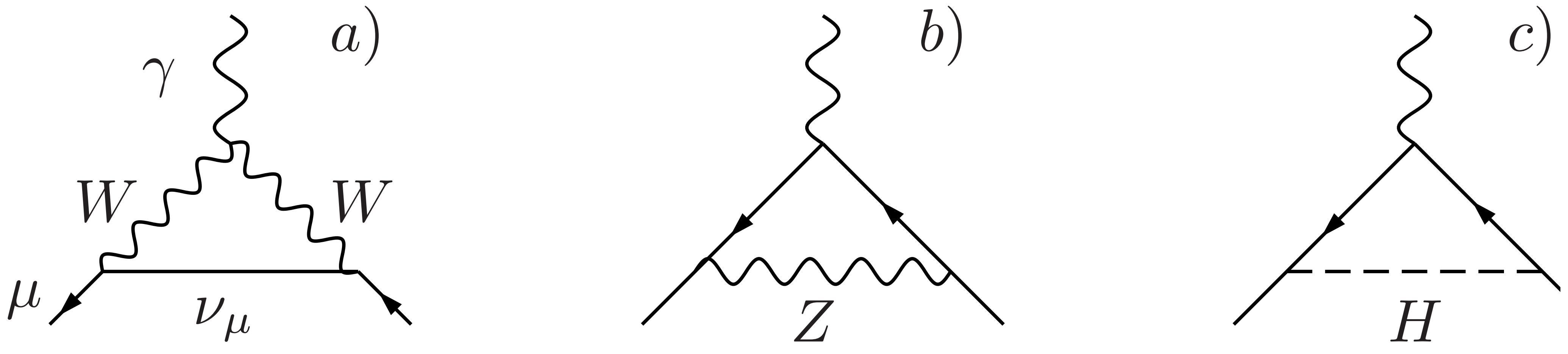
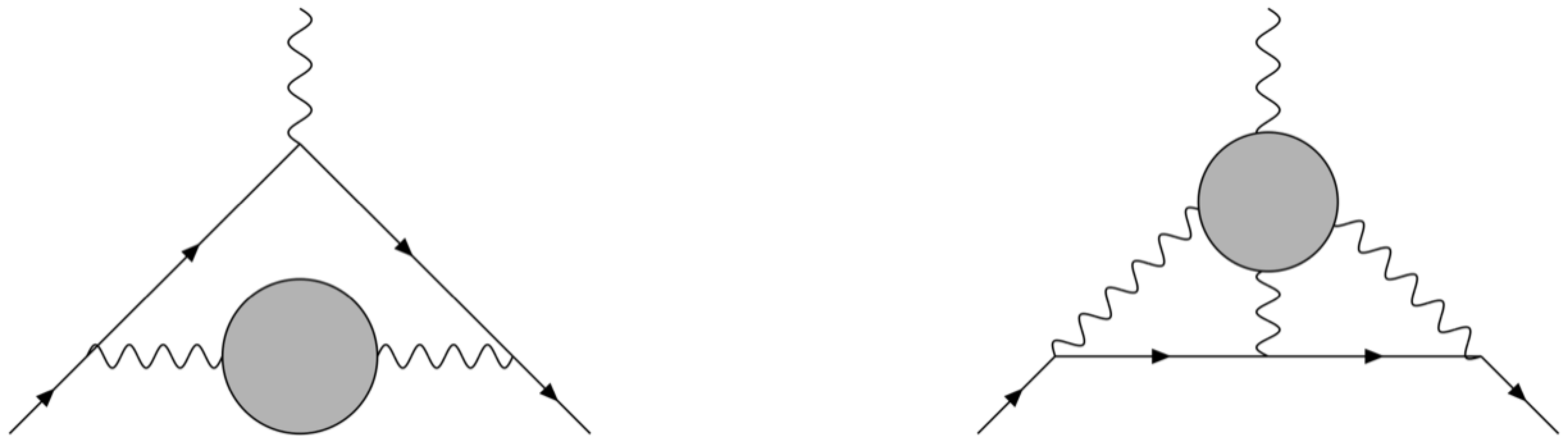
Weak corrections:



Hadronic corrections:



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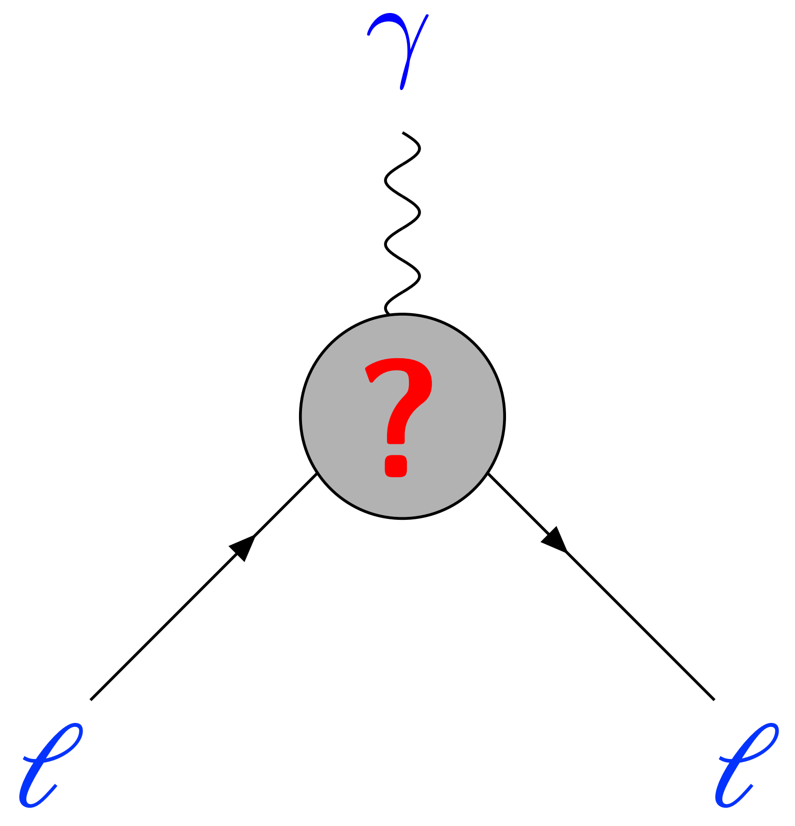
The muon anomalous magnetic moment as a probe for new physics

Lepton anomalous magnetic moments in the Standard Model:

$$a_\ell^{\mathrm{SM}} = a_\ell^{\mathrm{QED}} + a_\ell^{\mathrm{weak}} + a_\ell^{\mathrm{strong}}, \qquad \ell = e,\, \mu,\, \tau$$

BSM physics contribution:



$$a_\ell = a_\ell^{\mathrm{QED}} + a_\ell^{\mathrm{weak}} + a_\ell^{\mathrm{strong}} + a_\ell^{\mathrm{BSM}}$$

$$a_\ell^{\mathrm{BSM}} \propto m_\ell^2 / M_{\mathrm{BSM}}^2\,, \qquad \ell = e,\, \mu,\, \tau$$

→ sensitivity of a_μ enhanced by $(m_\mu/m_e)^2 \approx 4.3 \times 10^4$ relative to a_e

QED contributions to a_μ

QED contribution has been worked out to in perturbation theory to 5-loop order:

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Complete Tenth-Order QED Contribution to the Muon $g-2$

Tatsumi Aoyama,[1,2] Masashi Hayakawa,[3,2] Toichiro Kinoshita,[4,2] and Makiko Nio[2]

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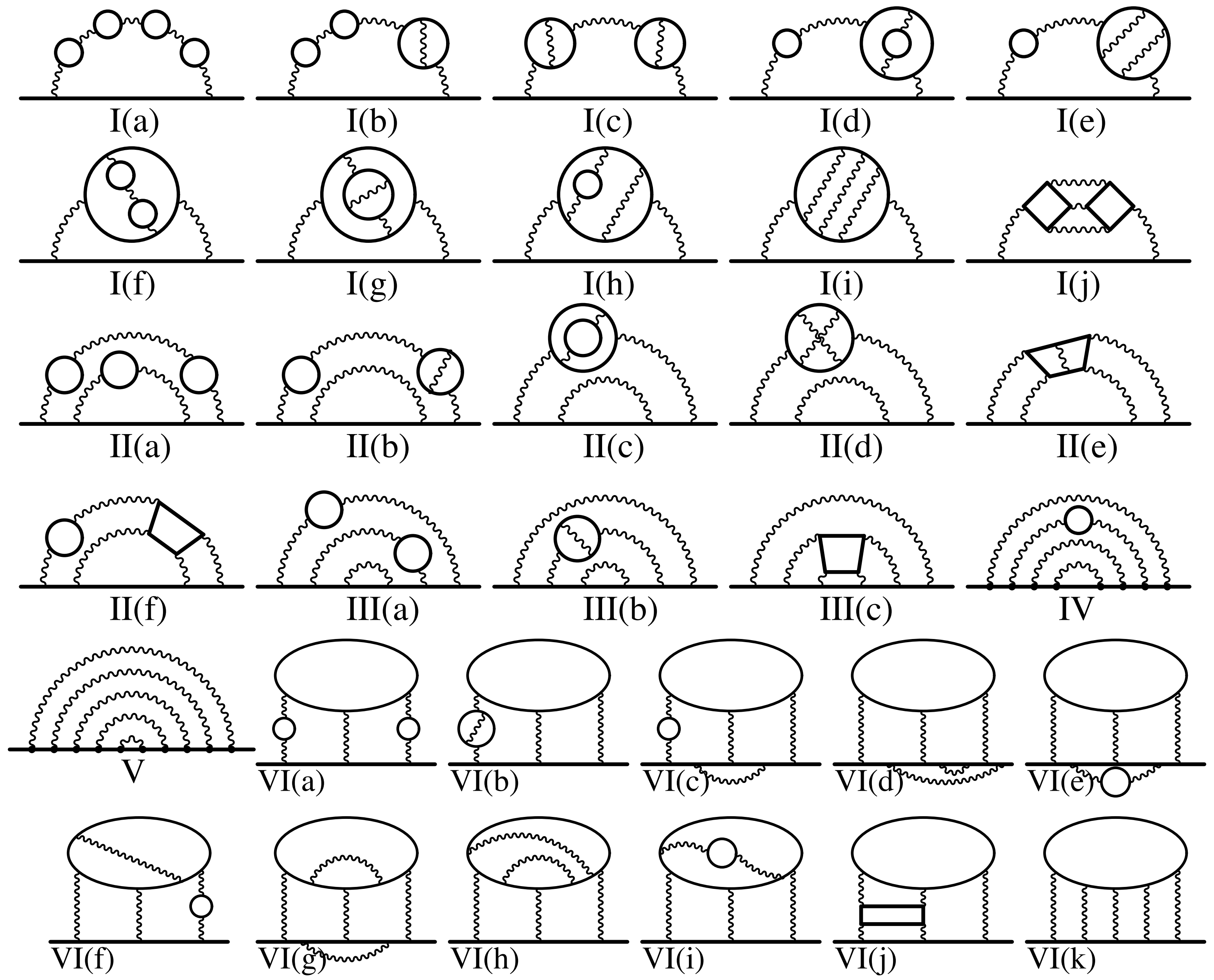
(Received 24 May 2012; published 13 September 2012)

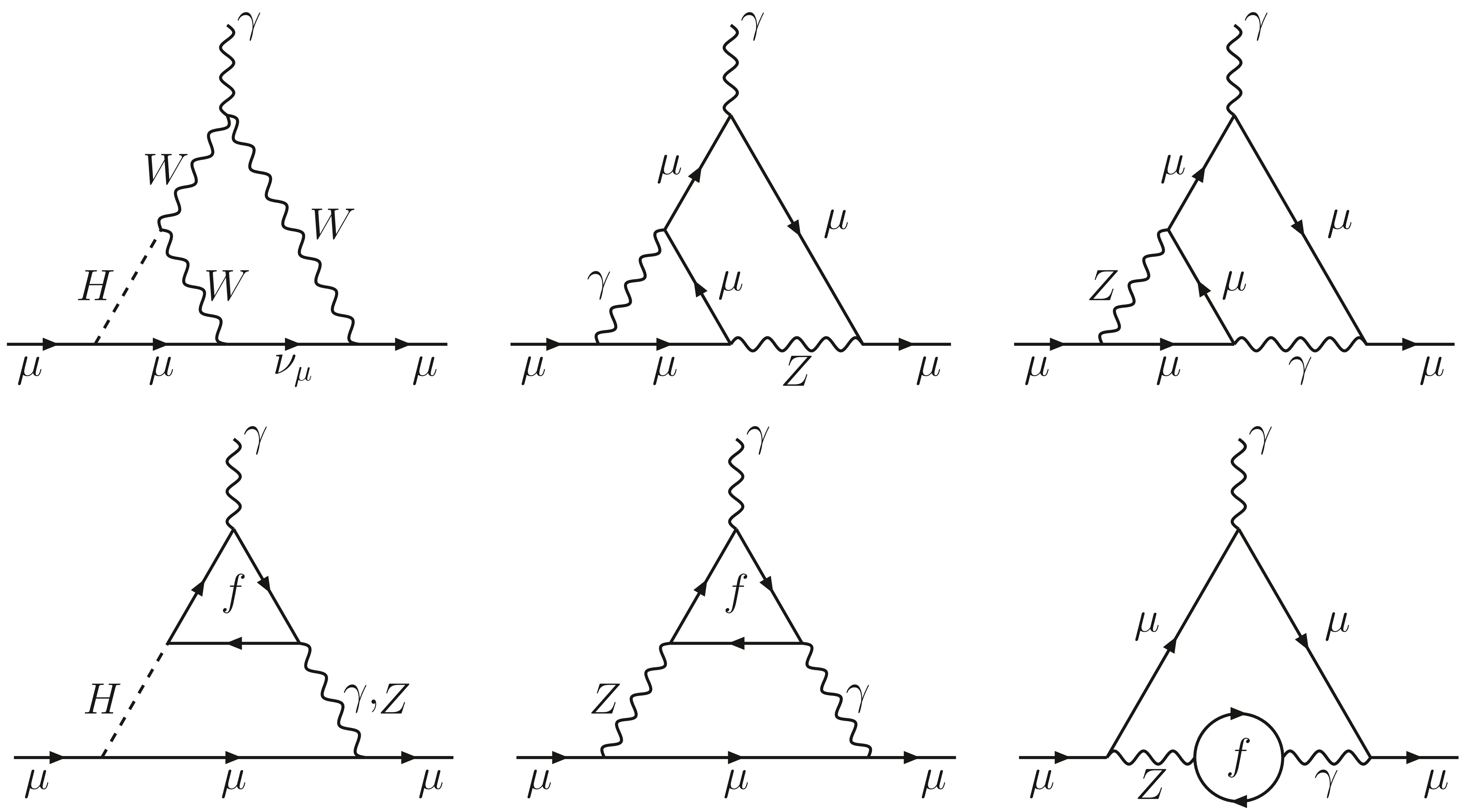
SM	116 591 810	100 %	#diagrams
QED(tot)	116 584 718.931	99,9939 %	
2	116 140 973.321	99,6133 %	1
4	413 217.626	0,3544 %	9
6	30 141.902	0,0259 %	72
8	381.004	0,0003 %	891
10	5.078	$4\cdot10^{-6}$ %	12672



Electroweak contributions to a_μ

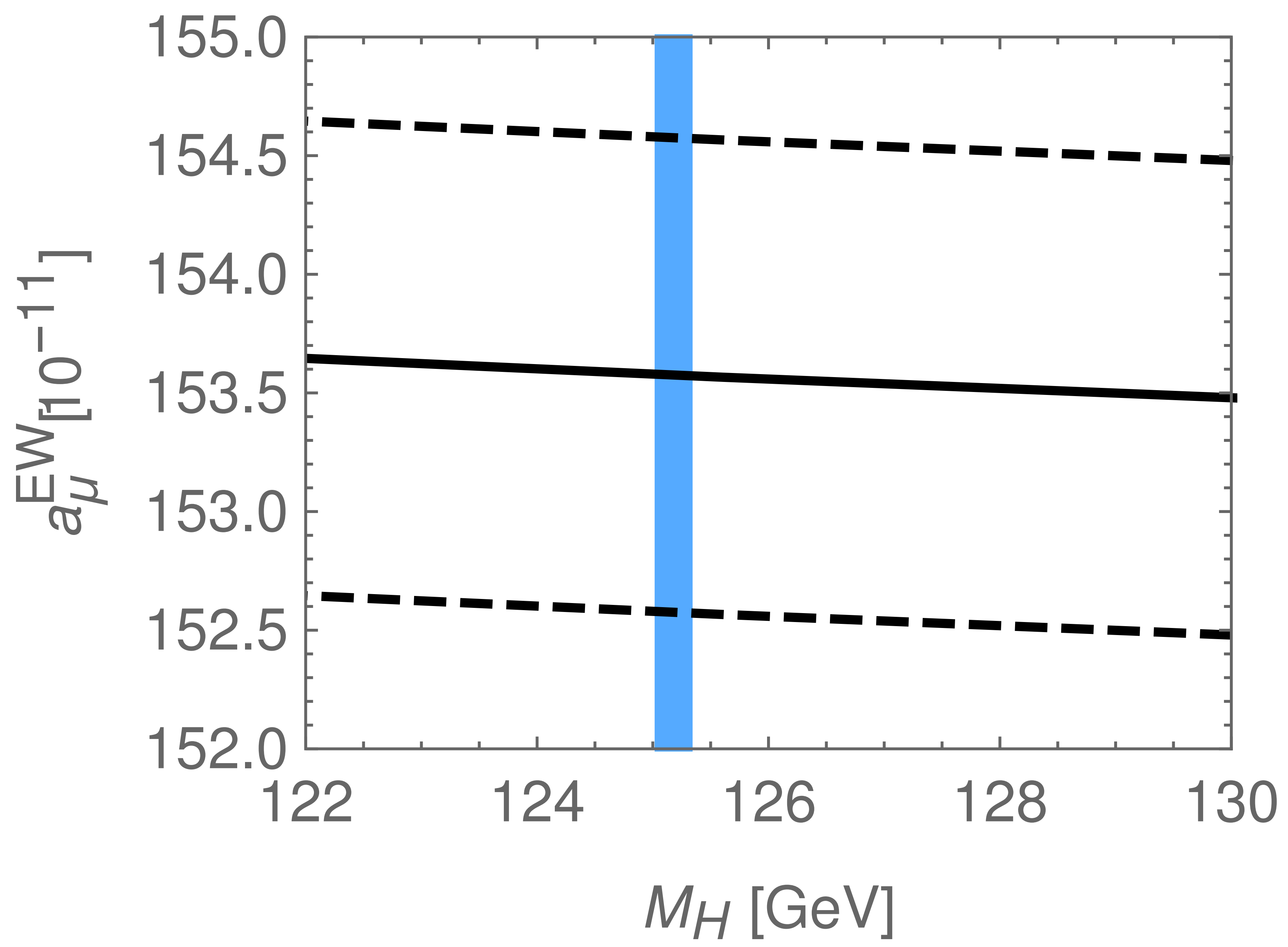
Weak contributions known to leading three-loop order

Sample two-loop diagrams:



$$a_\mu^{\text{weak}} = (153.6 \pm 1.0) \times 10^{-11}$$

Dependence on the Higgs mass



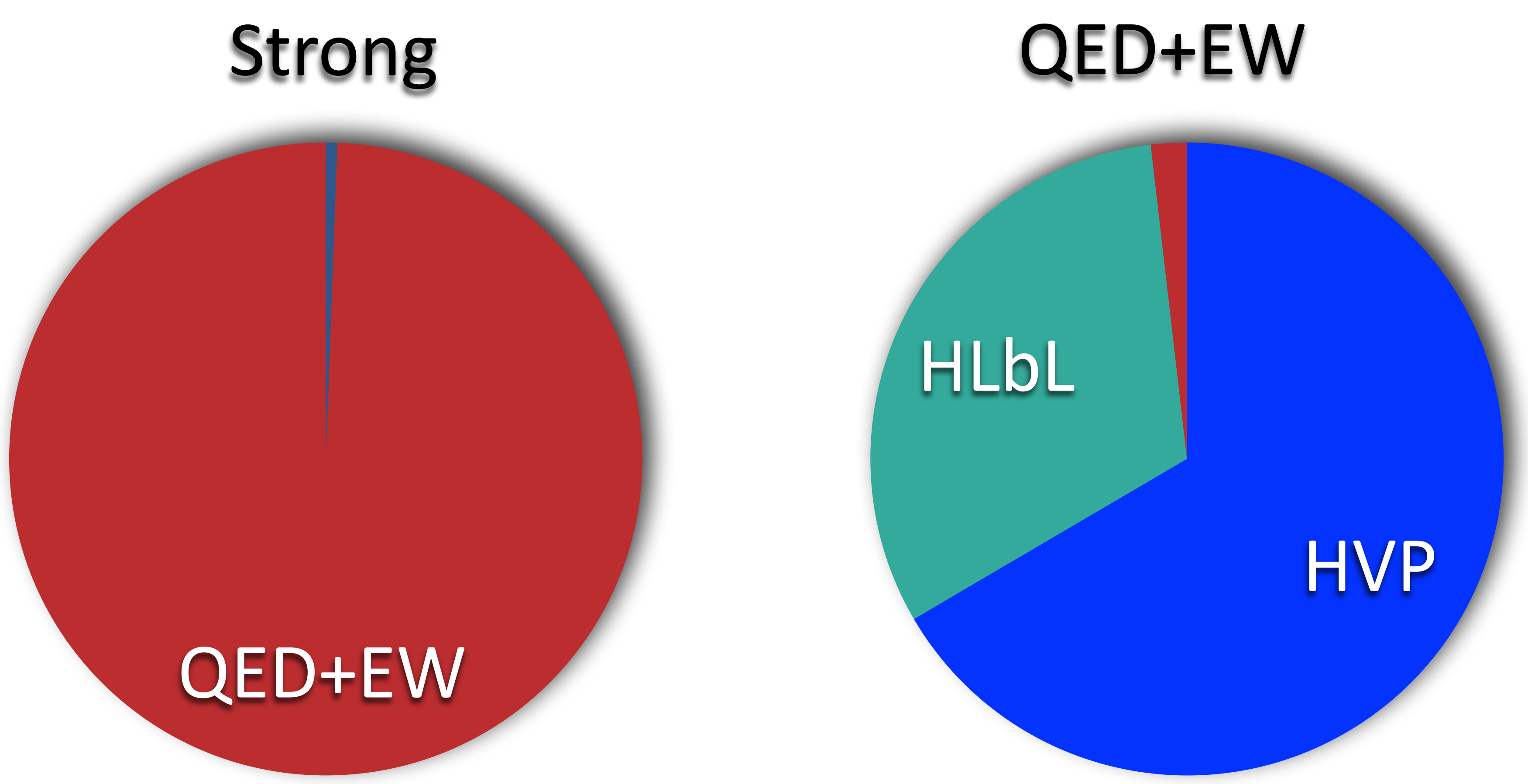
[Gnendiger et al., arXiv:1306.5546]

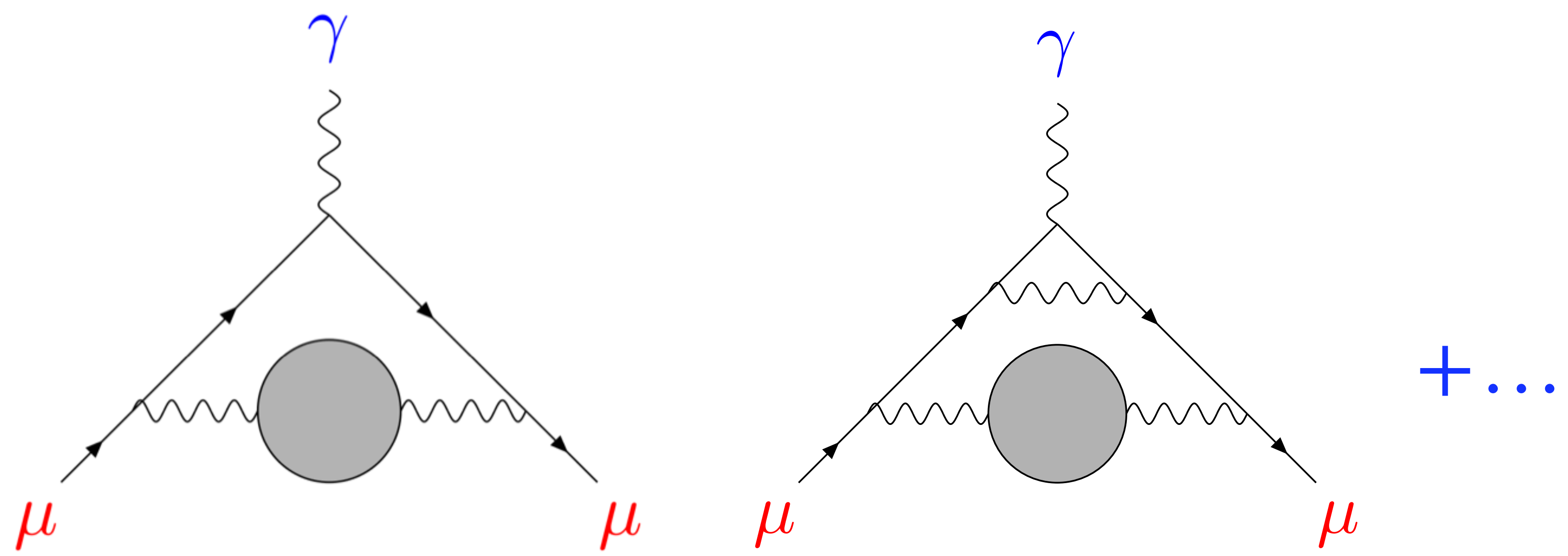
[Aoyama et al., Phys. Rep. 887 (2020) 1, arXiv:2006.04822]

Hadronic contributions to a_μ

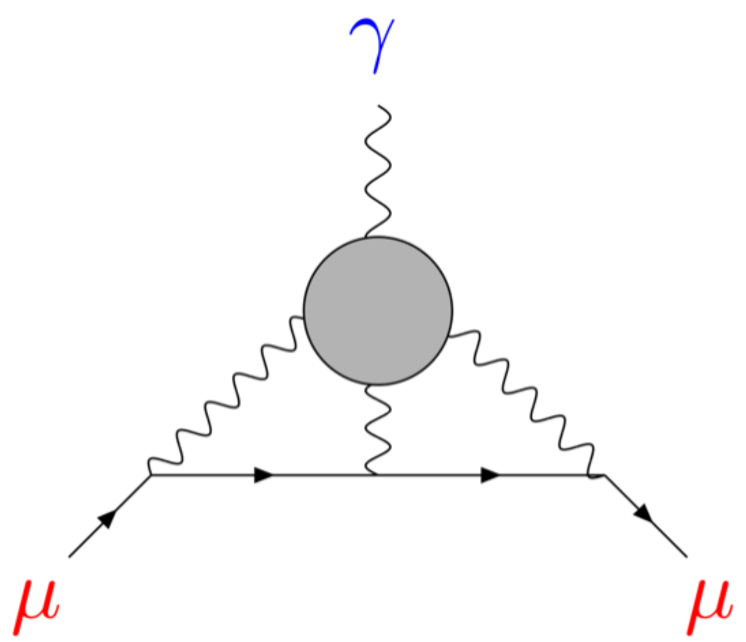
QED and electroweak contributions account for 99.994% of the SM prediction for a_μ

Error is dominated by strong interaction effects





Hadronic vacuum polarisation (HVP)



Hadronic light-by-light scattering (HLbL)

Two main approaches:

- Dispersion theory using experimentally determined cross sections ("data-driven")
- Lattice QCD calculations ("ab initio")

Standard Model prediction — White Paper estimate

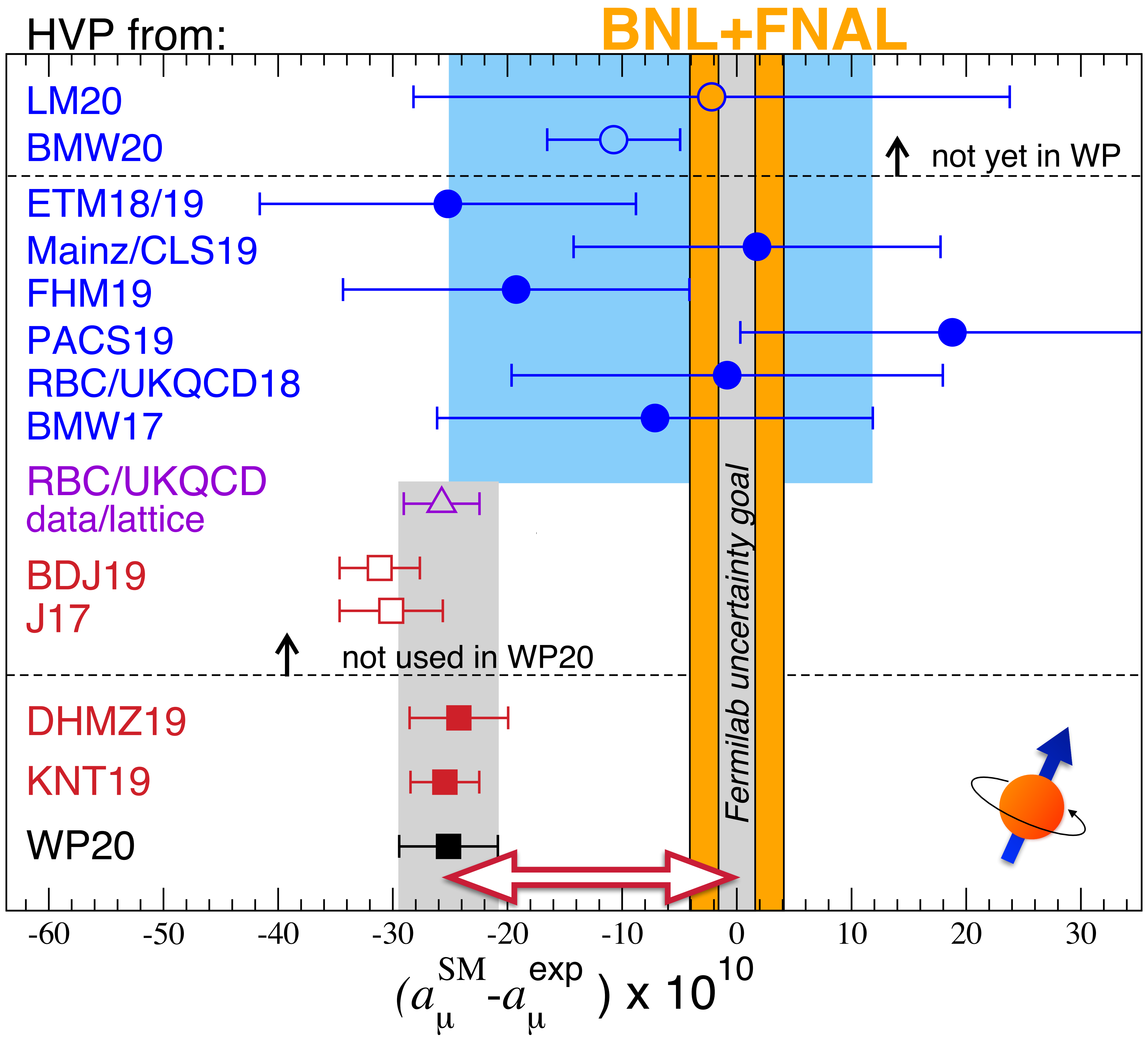
Contributions to the muon $g-2$ from electromagnetism, weak and strong interactions:

QED:	$116\,584\,718.9\,(1) \times 10^{-11}$	0.001 ppm	
Weak:	$153.6(1.0) \times 10^{-11}$	0.01 ppm	
Hadronic vacuum polarisation:	$6845(40) \times 10^{-11}$	0.34 ppm	[0.6%]
Hadronic light-by-light scattering:	$92(18) \times 10^{-11}$	0.15 ppm	[20%]

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm weak} + a_\mu^{\rm hvp} + a_\mu^{\rm hlbl} = 116\,591\,810(43) \times 10^{-11} \qquad 0.37\ \text{ppm}$$

[Aoyama et al., Phys. Rep. 887 (2020) 1, arXiv:2006.04822]

Standard Model prediction versus experiment



SM prediction (White Paper):

$$a_\mu^{\mathrm{SM}} = 116\,591\,810(43) \times 10^{-11}$$

FNAL E989 (2021):

$$a_\mu^{\mathrm{E989}} = 116\,592\,040(54) \times 10^{-11}$$

Combined with BNL E821 (2004):

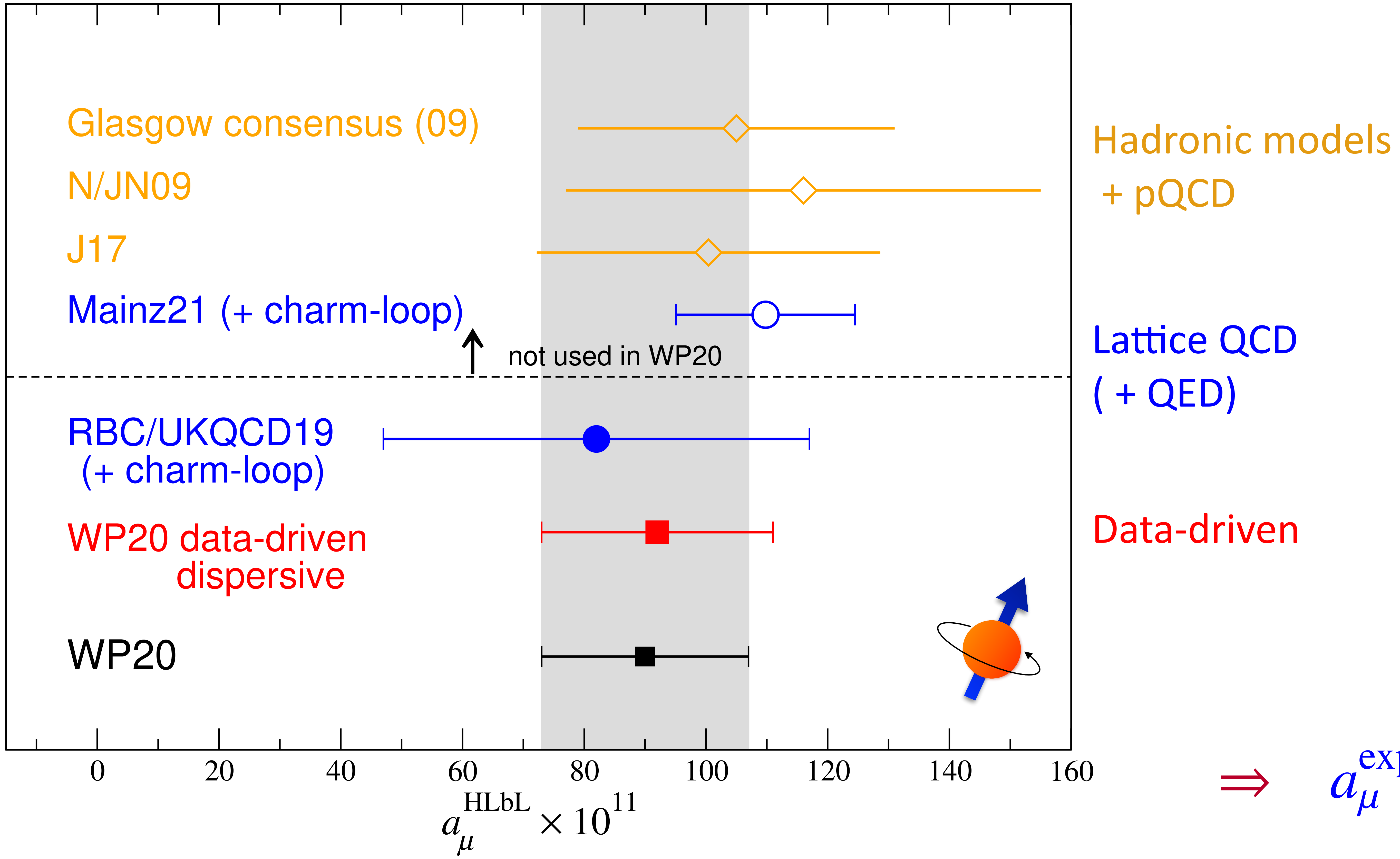
$$a_\mu^{\mathrm{exp}} = 116\,592\,061(41) \times 10^{-11}$$

$$\Rightarrow \quad a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = 251(59) \times 10^{-11} \qquad (4.2\,\sigma)$$

[Aoyama et al., Phys. Rep. 887 (2020) 1; Colangelo et al., arXiv:2203.15810]

Hadronic light-by-light scattering

[Aoyama et al., Phys. Rep. 887 (2020) 1; Colangelo et al., arXiv:2203.15810]



White Paper:

$$a_\mu^{\rm hlbl} = (92 \pm 18) \cdot 10^{-11}$$

Direct lattice calculations (Mainz):

$$a_\mu^{\rm hlbl} = (106.8 \pm 14.7) \cdot 10^{-11}$$

(excluding charm loop)

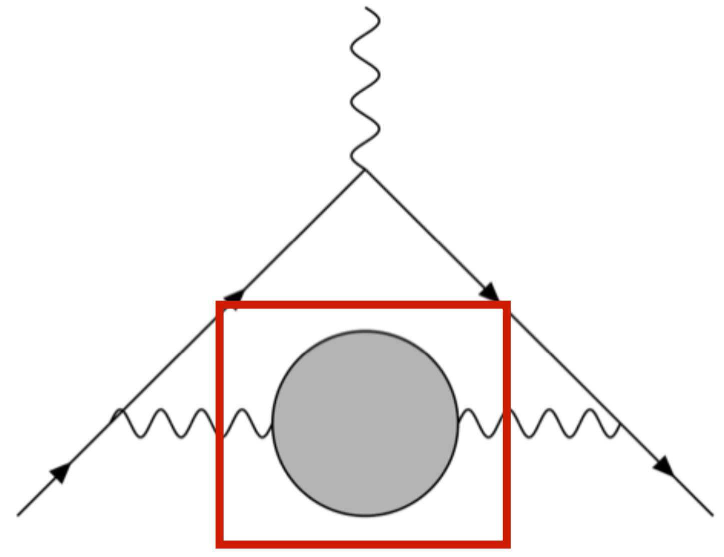
$$a_\mu^{\rm hlbl,\,c} = (2.8 \pm 0.5) \cdot 10^{-11}$$

[Chao et al., Eur. Phys. J. C81 (2021) 7, 651; arXiv:2204.08844]

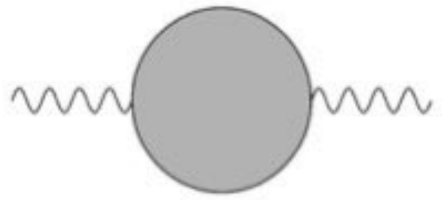
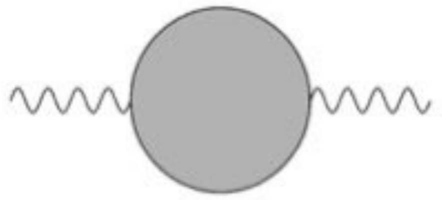
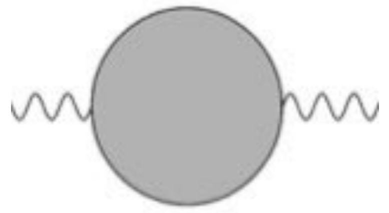
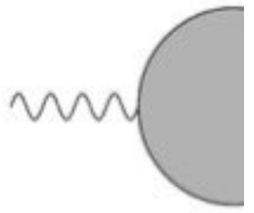
$$\Rightarrow \quad a_\mu^{\rm exp} - a_\mu^{\rm SM}\Big|_{\rm Mainz}^{\rm hlbl} = 234(59) \times 10^{-11} \qquad (4.0\,\sigma)$$

Hadronic light-by-light scattering not the dominant source of uncertainty!

Hadronic vacuum polarisation from dispersion theory



Analyticity, unitarity & optical theorem imply:

 $= \int \frac{ds}{\pi(s-q^2)}\,\mathrm{Im}$  $2\,\mathrm{Im}$  $= \sum_{\mathrm{had}} \int d\Phi \,\Big|$  $\Big|^2$

$\propto \sigma(e^+e^- \to \mathrm{hadrons})$

$$a_\mu^{\mathrm{hvp}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R_{\mathrm{had}}(s)\hat{K}(s)}{s^2}, \qquad R_{\mathrm{had}}(s) = \frac{3s}{4\pi\,\alpha(s)}\,\sigma(e^+e^- \to \mathrm{hadrons})$$

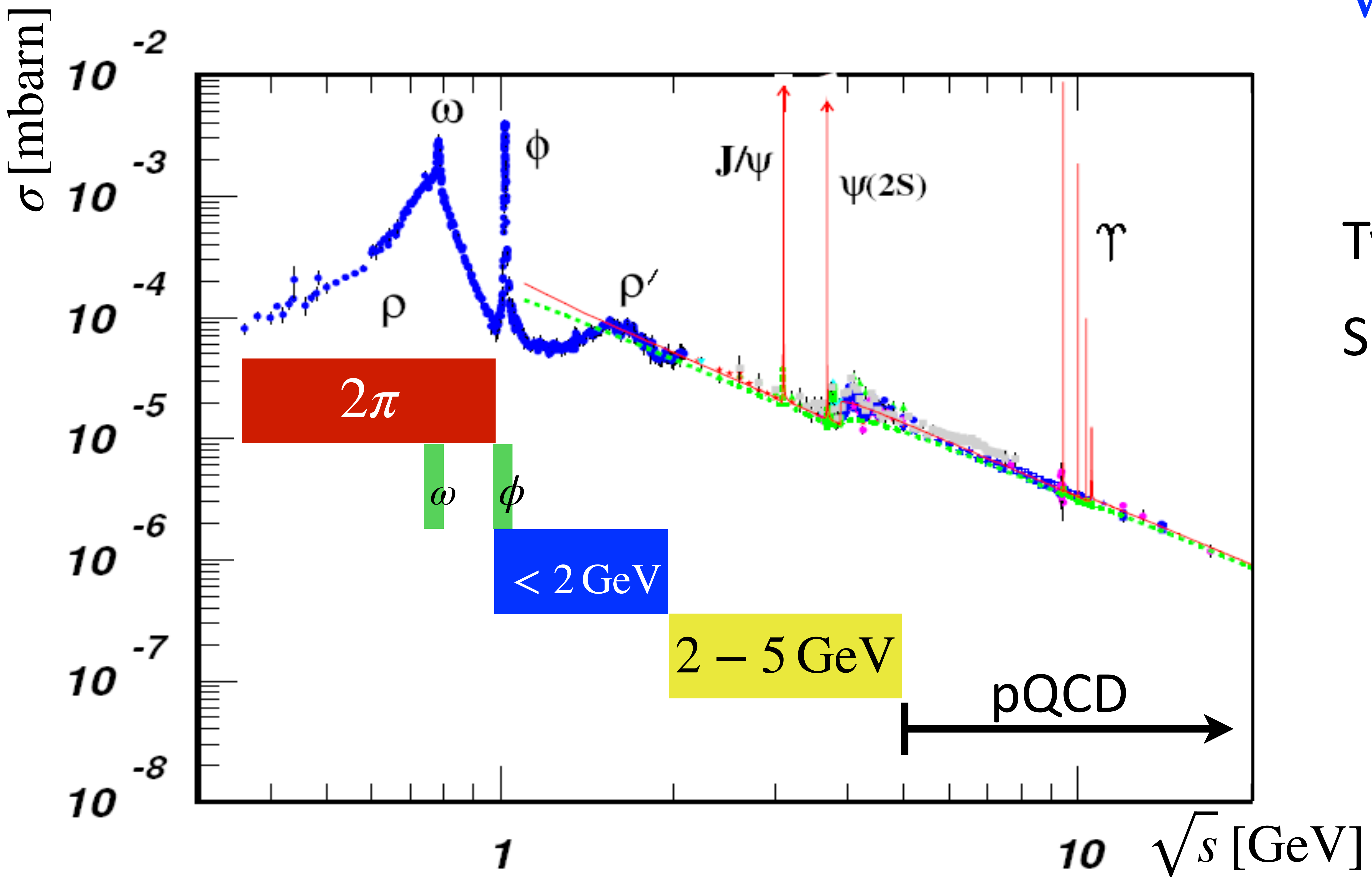
"R-ratio"

Hadronic effects cannot be treated in perturbation theory

- Use experimental data for $R_{\mathrm{had}}(s)$ in the low-energy regime ("data-driven approach")
- Standard Model prediction is subject to experimental uncertainties

Data-driven approach: Hadronic cross sections

$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R_{\rm had}(s)\hat{K}(s)}{s^2}$$



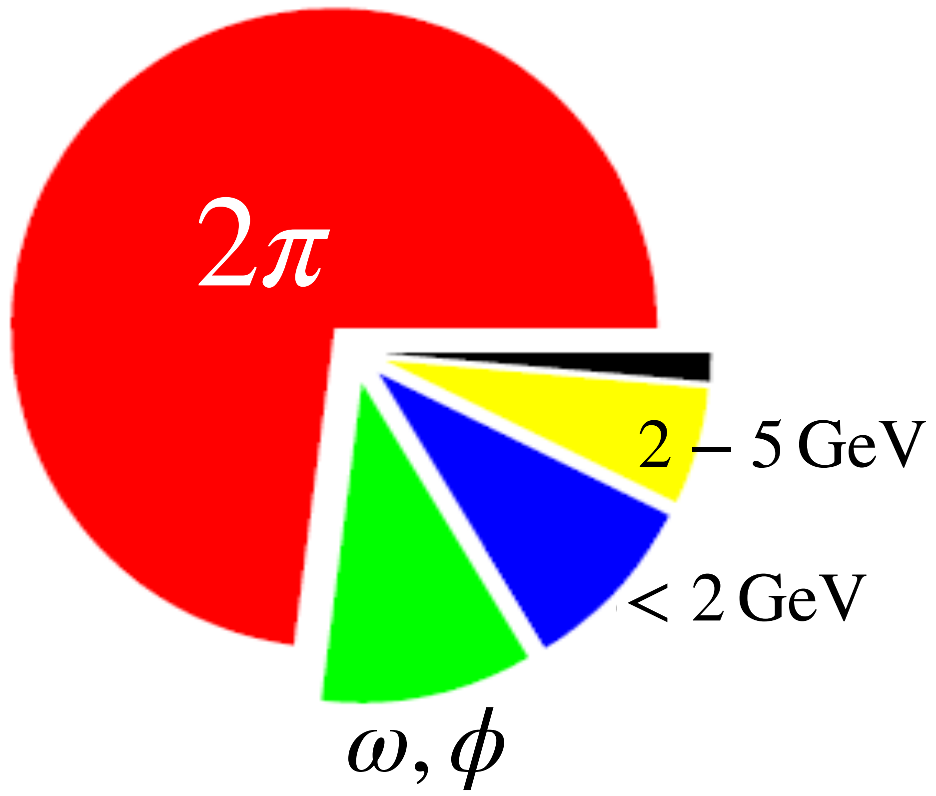
Decade-long effort to measure e^+e^- cross sections

$\sqrt{s} \lesssim 2\,\mathrm{GeV}$: sum of exclusive channels

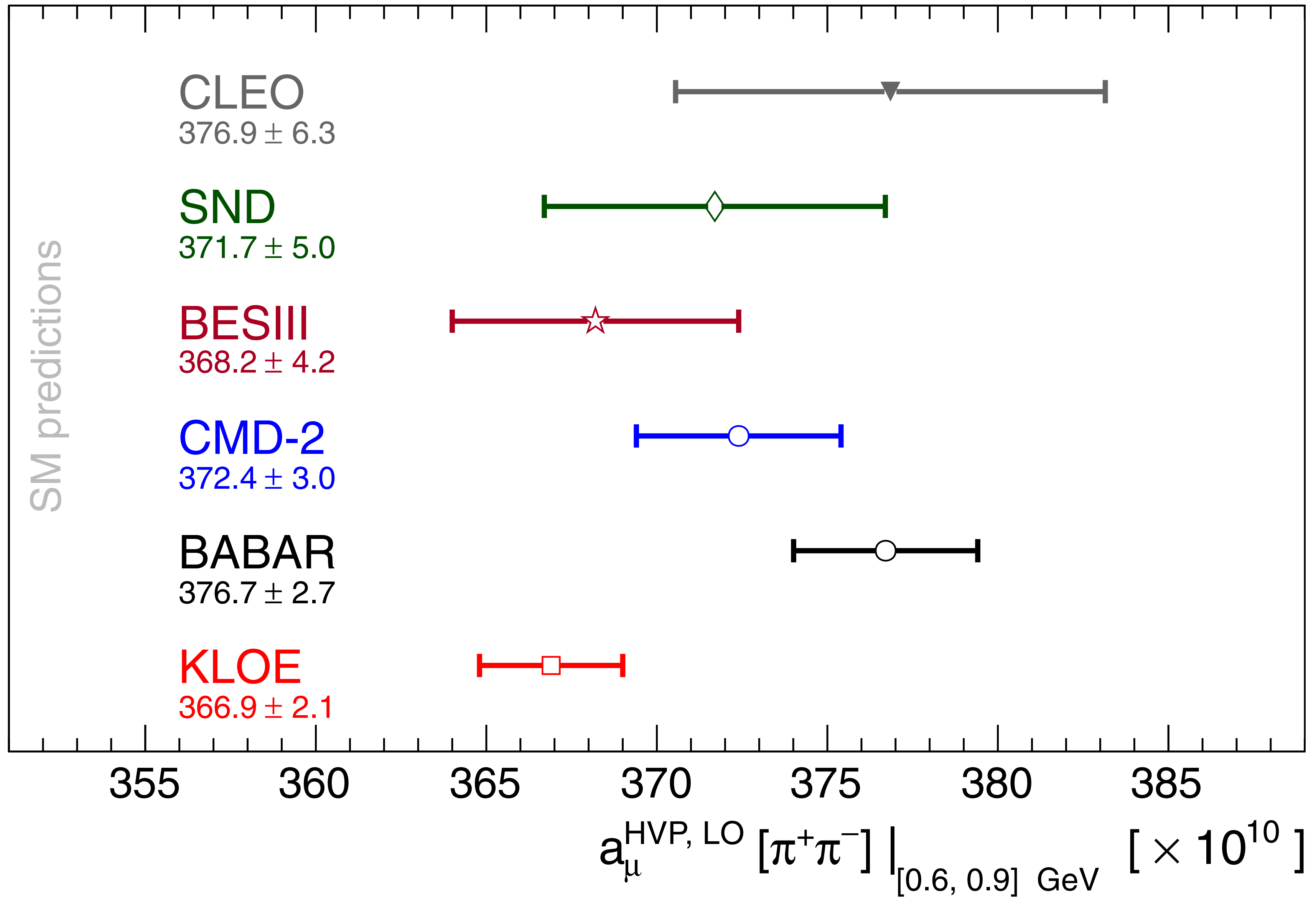
$\sqrt{s} > 2\,\mathrm{GeV}$: inclusive channels, narrow resonances, perturbative QCD

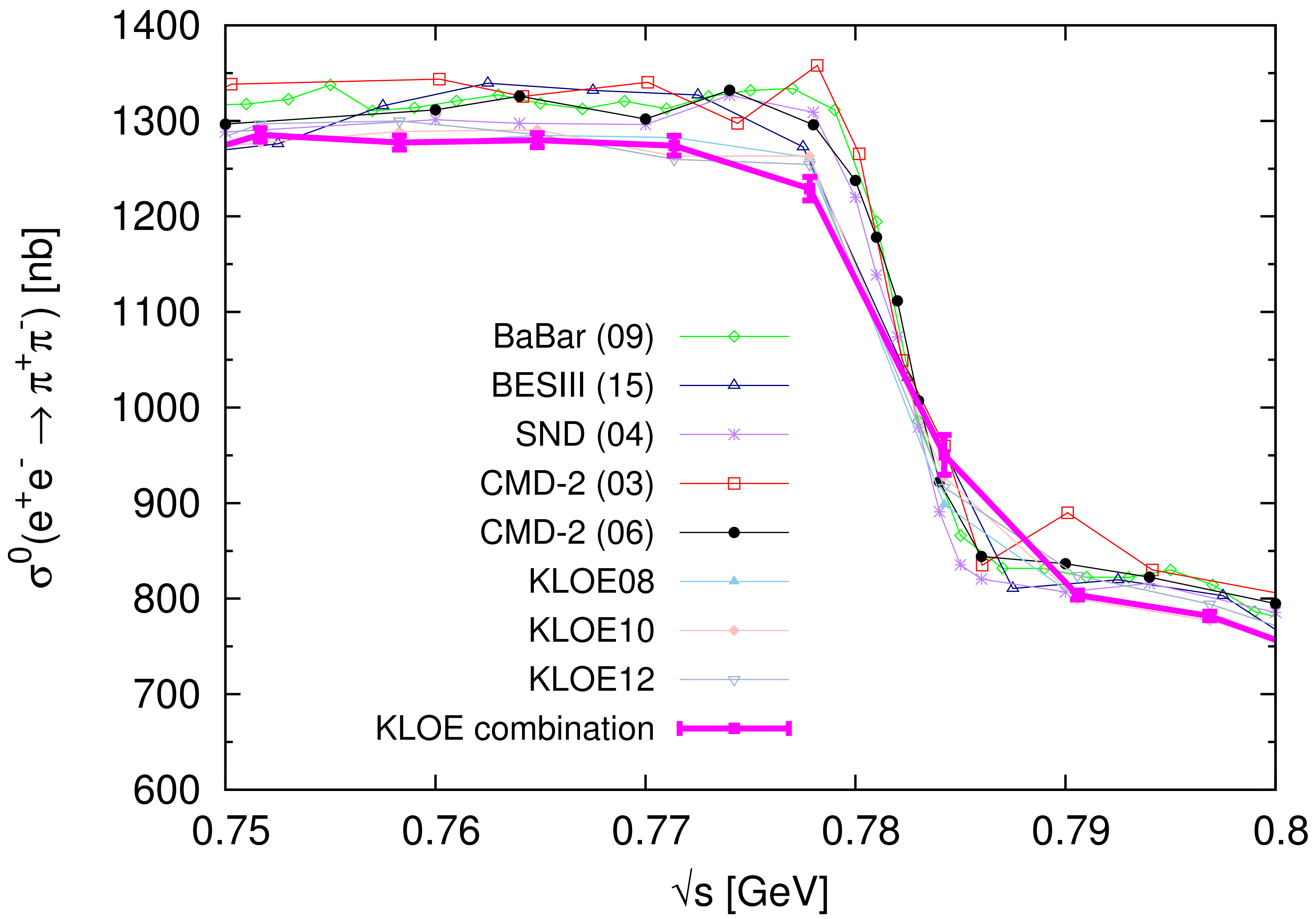
Two-pion channel accounts for $\approx 70\,\%$ of LO-HVP

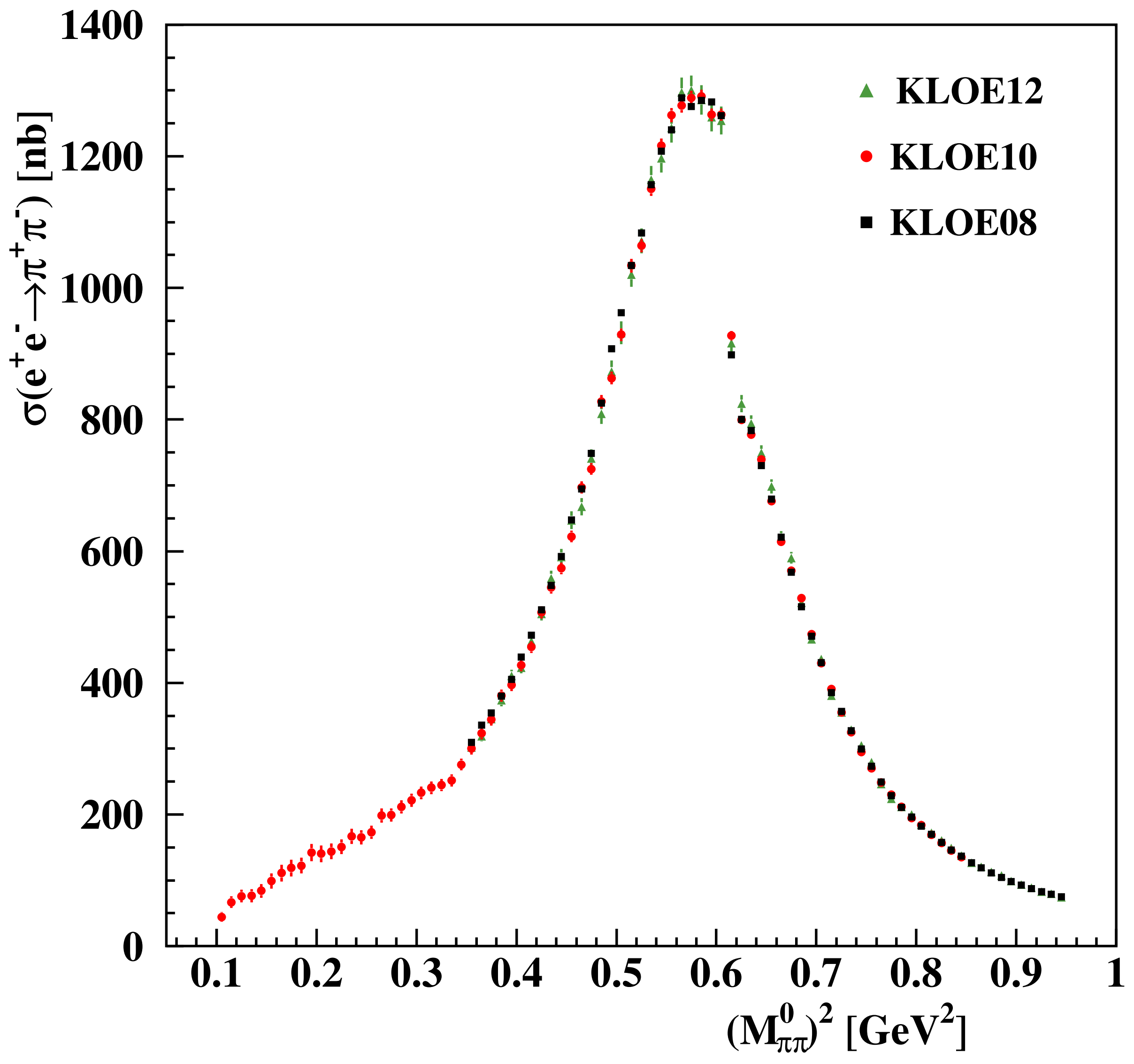
Subleading channels: ω, ϕ decays, final states with 3 pions, 2 kaons, 4 pions,…



Two-pion channel







- Tension in the data for $e^+e^- \to \pi^+\pi^-$ between BaBar and KLOE
- Extended (re-)analysis of ISR data: BaBar (in progress) and KLOE (planned)
- New data: SND-3 (published), CMD-3 (expected) and BESIII
- Future prospects at Belle II

Evaluation of the dispersion integral

Many different groups and analyses (DHMZ, KNT, FJ, CHHKS, BHLS,…)

Disagreement for some exclusive channels

	DHMZ19	KNT19	Difference
$\pi^+\pi^-$	507.85(0.83)(3.23)(0.55)	504.23(1.90)	3.62
$\pi^+\pi^-\pi^0$	46.21(0.40)(1.10)(0.86)	46.63(94)	−0.42
$\pi^+\pi^-\pi^+\pi^-$	13.68(0.03)(0.27)(0.14)	13.99(19)	−0.31
$\pi^+\pi^-\pi^0\pi^0$	18.03(0.06)(0.48)(0.26)	18.15(74)	−0.12
K^+K^-	23.08(0.20)(0.33)(0.21)	23.00(22)	0.08
K_SK_L	12.82(0.06)(0.18)(0.15)	13.04(19)	−0.22
$\pi^0\gamma$	4.41(0.06)(0.04)(0.07)	4.58(10)	−0.17
Sum of the above	626.08(0.95)(3.48)(1.47)	623.62(2.27)	2.46
[1.8, 3.7] GeV (without $c\bar{c}$)	33.45(71)	34.45(56)	−1.00
J/ψ, $\psi(2S)$	7.76(12)	7.84(19)	−0.08
$[3.7, \infty)$ GeV	17.15(31)	16.95(19)	0.20
Total $a_\mu^{\text{HVP, LO}}$	$694.0(1.0)(3.5)(1.6)(0.1)_\psi(0.7)_{\text{DV+QCD}}$	692.8(2.4)	1.2

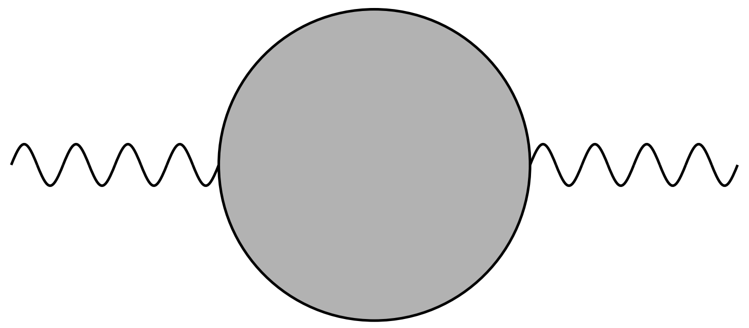
Merging procedure: average of individual results + theoretical constraints + conservative error estimate (reflecting tensions in the data, differences in procedures)

$$a_\mu^{\text{hvp, LO}} = 693.1(2.8)_{\text{exp}}(2.8)_{\text{syst}}(0.7)_{\text{DV+QCD}} \times 10^{-10} = 693.1(4.0) \times 10^{-10} \qquad [0.6\%]$$

Hadronic vacuum polarisation from Lattice QCD

Lattice QCD does NOT determine the R-ratio from first principles

Primary observable: $G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} G_{kk}(\vec{x},t), \quad G_{\mu\nu}(x) = \left\langle j_\mu^{\rm em}(x) j_\nu^{\rm em}(0)\right\rangle$



Electromagnetic current: $j_\mu^{\rm em}(x) = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \ldots$

Vacuum polarisation function for Euclidean momenta $-Q^2 < 0$:

$$\underbrace{4\pi^2\left\{\Pi(-q^2) - \Pi(0)\right\}}_{\hat{\Pi}(Q^2)} = \frac{Q^2}{3}\int_0^\infty ds\,\frac{R(s)}{s(s+Q^2)} = \frac{1}{Q^2}\int_0^\infty dt\,G(t)\left[Q^2t^2 - 4\sin^2\left(\tfrac{1}{2}Qt\right)\right]$$

t : Euclidean time [Bernecker & Meyer 2011]

$$a_\mu^{\rm hvp,LO} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dQ^2\,f(Q^2)\,\hat{\Pi}(Q^2) = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dt\,G(t)\underbrace{\int_0^\infty dQ^2\,f(Q^2)\left[t^2 - \frac{4}{Q^2}\sin^2\left(\tfrac{1}{2}Qt\right)\right]}_{\tilde{K}(t)}$$

[Lautrup, Peterman & de Rafael 1972, Blum 2002]

Hadronic vacuum polarisation from Lattice QCD

Time-momentum representation (TMR) *[Bernecker & Meyer 2011]*

$$a_\mu^{\mathrm{hvp,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t), \qquad G(t) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} \left\langle j_k^{\mathrm{em}}(\vec{x},t)\, j_k^{\mathrm{em}}(0) \right\rangle$$

$\tilde{K}(t)$: analytically known kernel function

Why computing the HVP contribution is a challenge

$$a_\mu^{\mathrm{hvp,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t), \qquad G(t) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} \left\langle j_k^{\mathrm{em}}(\vec{x},t)\, j_k^{\mathrm{em}}(0) \right\rangle$$

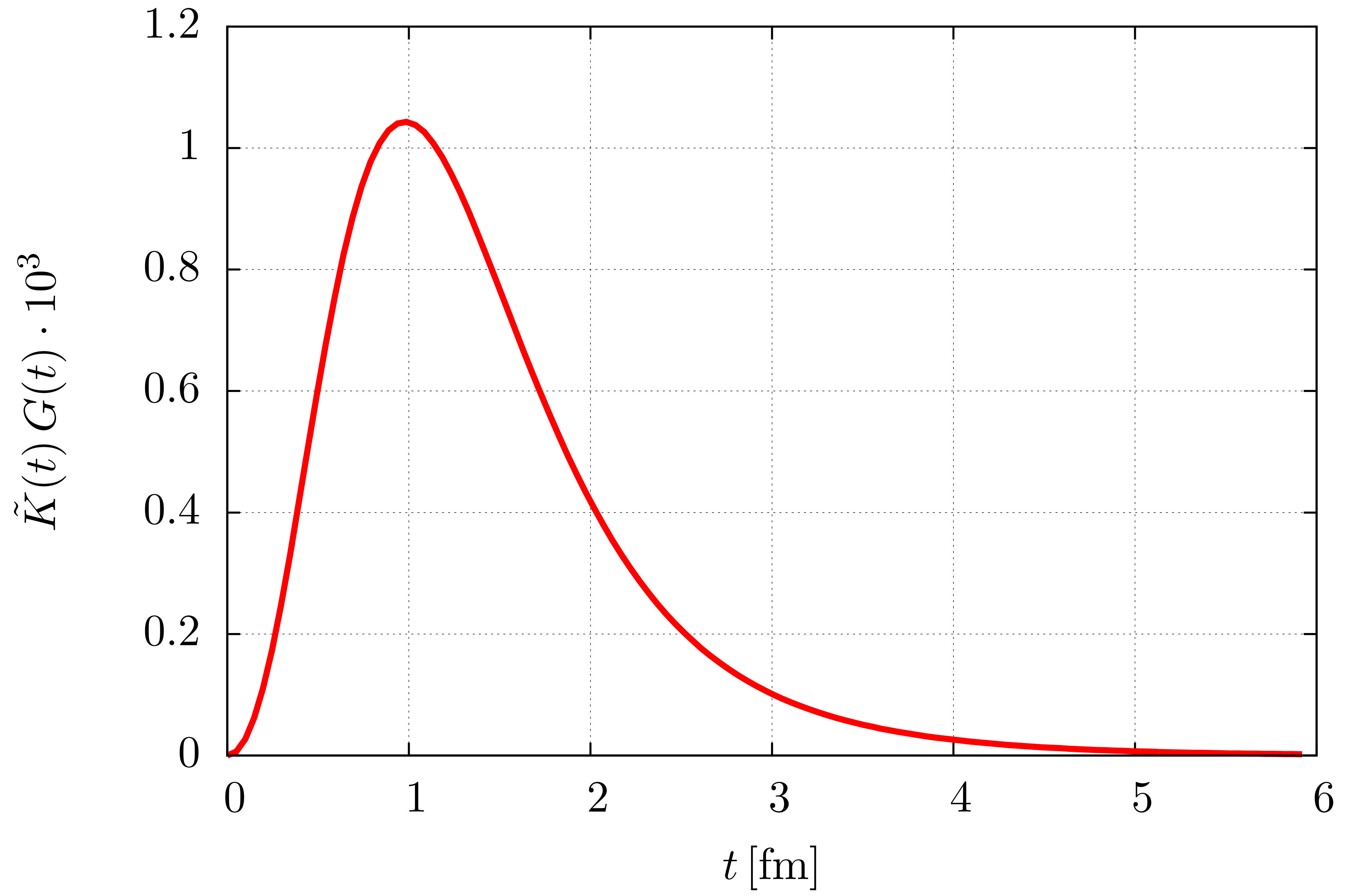
Sub-percent statistical precision;
exponentially growing signal-to-noise in $G(t)$ as $t \to \infty$

Correct for finite-volume effects

Control discretisation effects

Quark-disconnected diagrams:
control statistical & stochastic noise

Isospin breaking: $m_u \neq m_d$ and QED



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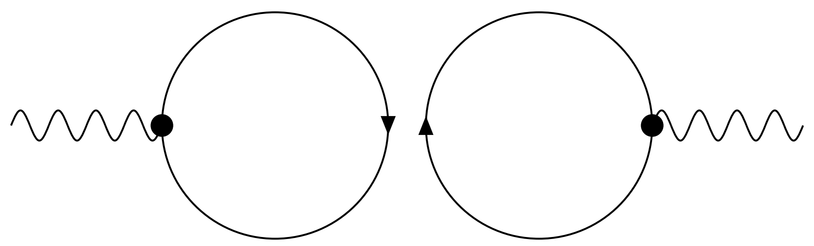
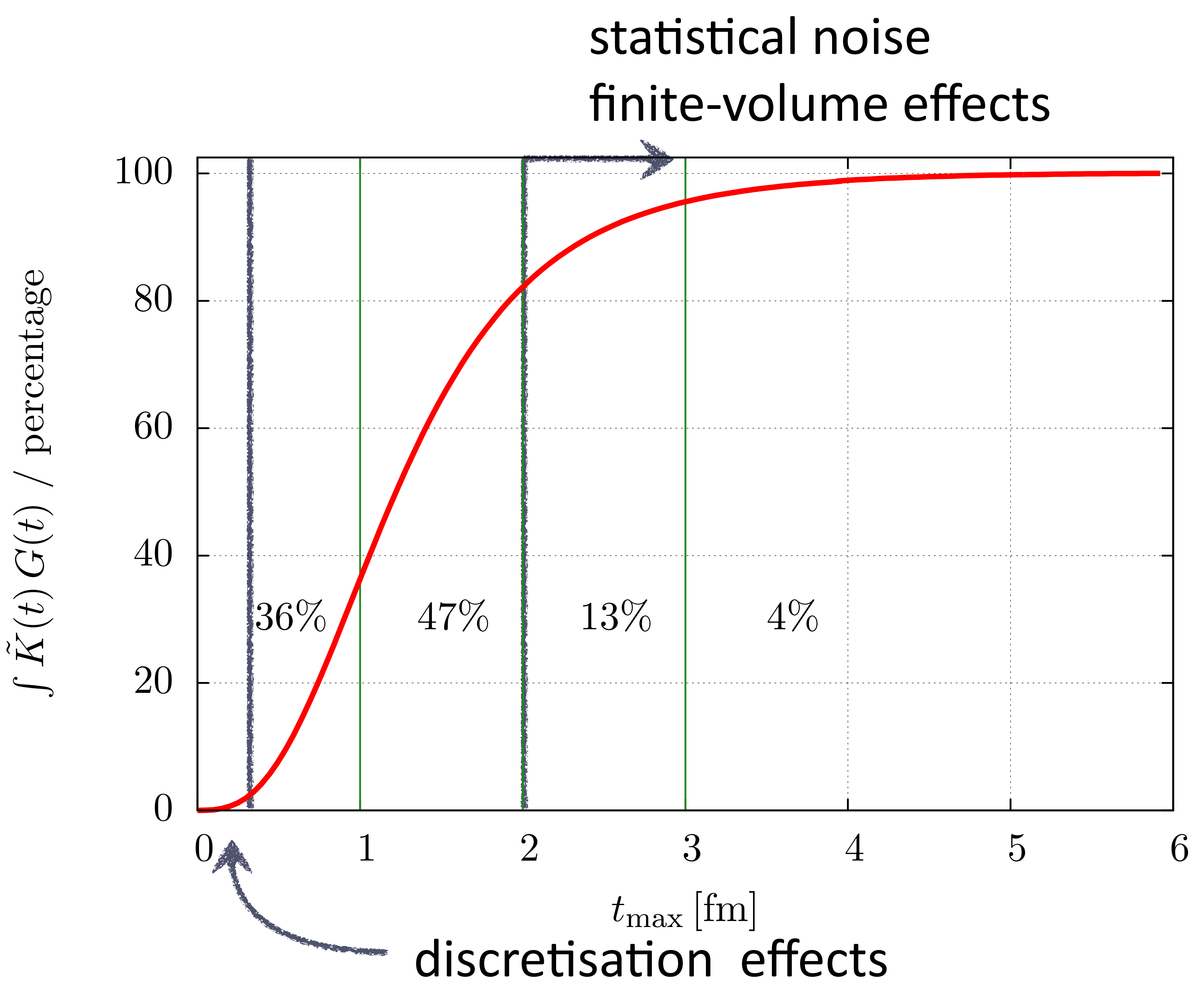


Isospin breaking: $m_u \neq m_d$ and QED



Why computing the HVP contribution is a challenge

$$a_\mu^{\rm hvp,\,LO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t), \qquad G(t) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} \left\langle j_k^{\rm em}(\vec{x},t)\, j_k^{\rm em}(0) \right\rangle$$

Sub-percent statistical precision;

exponentially growing signal-to-noise in $G(t)$ as $t \to \infty$

Correct for finite-volume effects

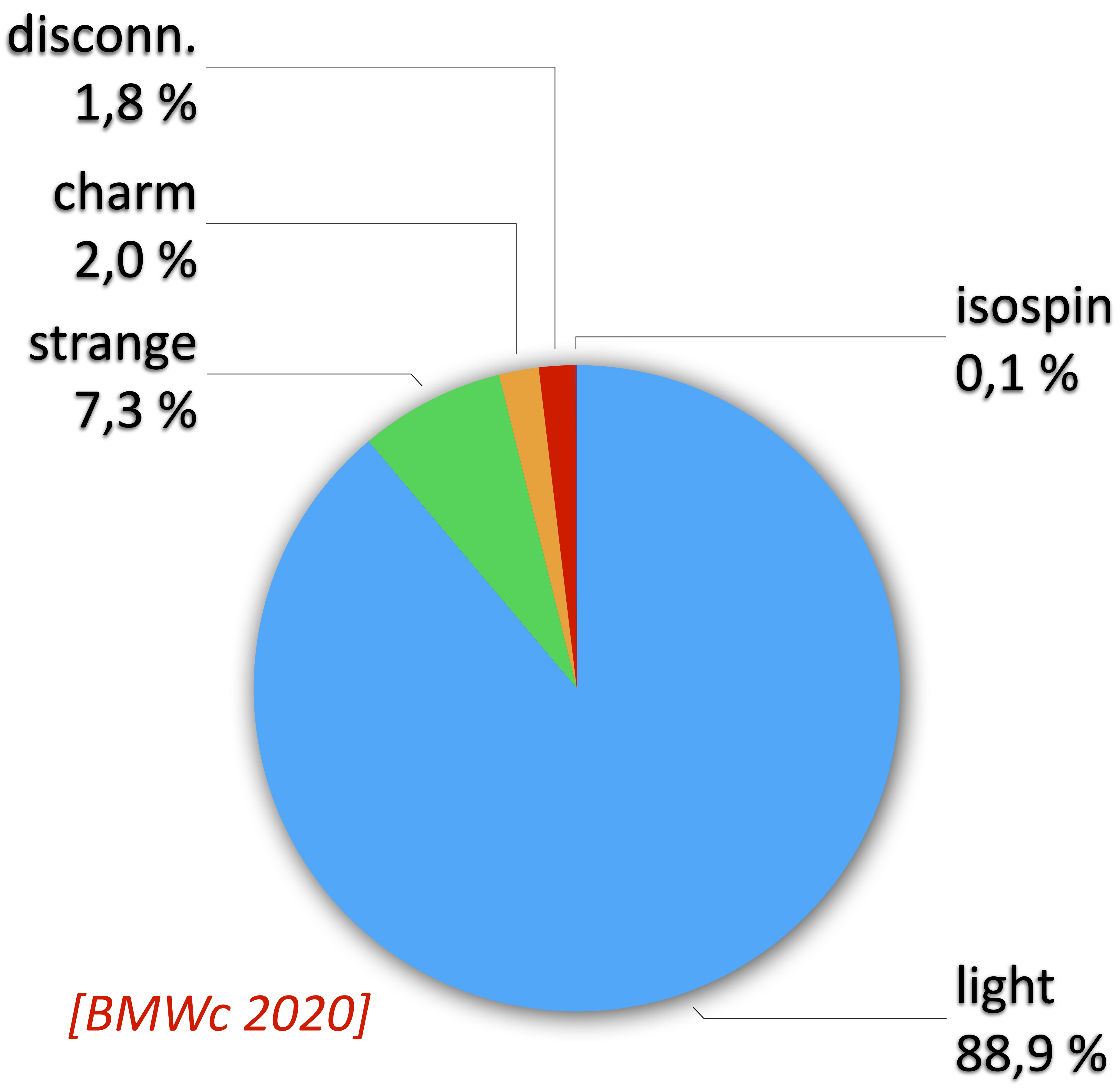
Control discretisation effects

Quark-disconnected diagrams:

control statistical & stochastic noise

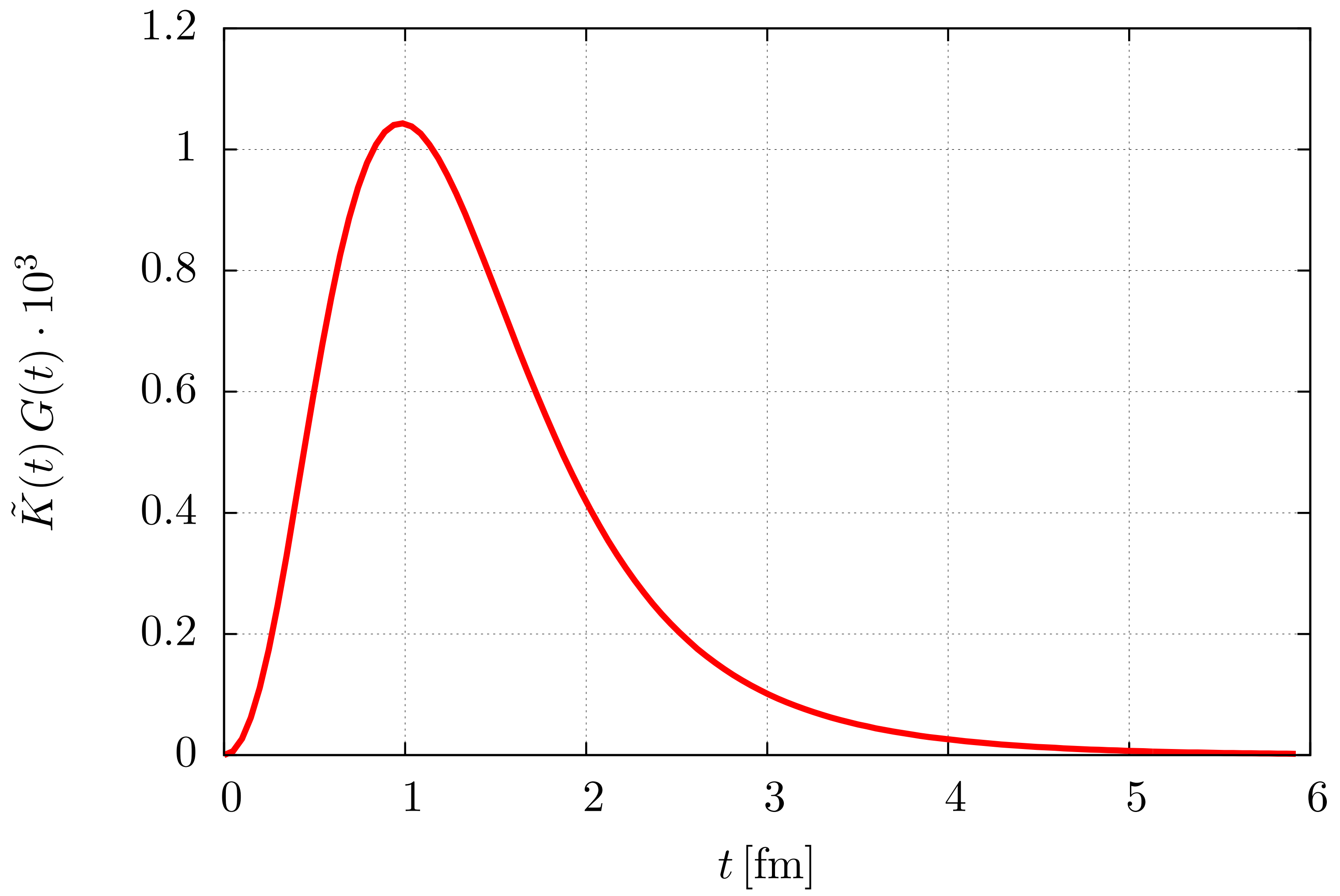
Isospin breaking: $m_u \neq m_d$ and QED

Light-quark connected contribution dominates



Controlling the long-distance tail of $G(t)$

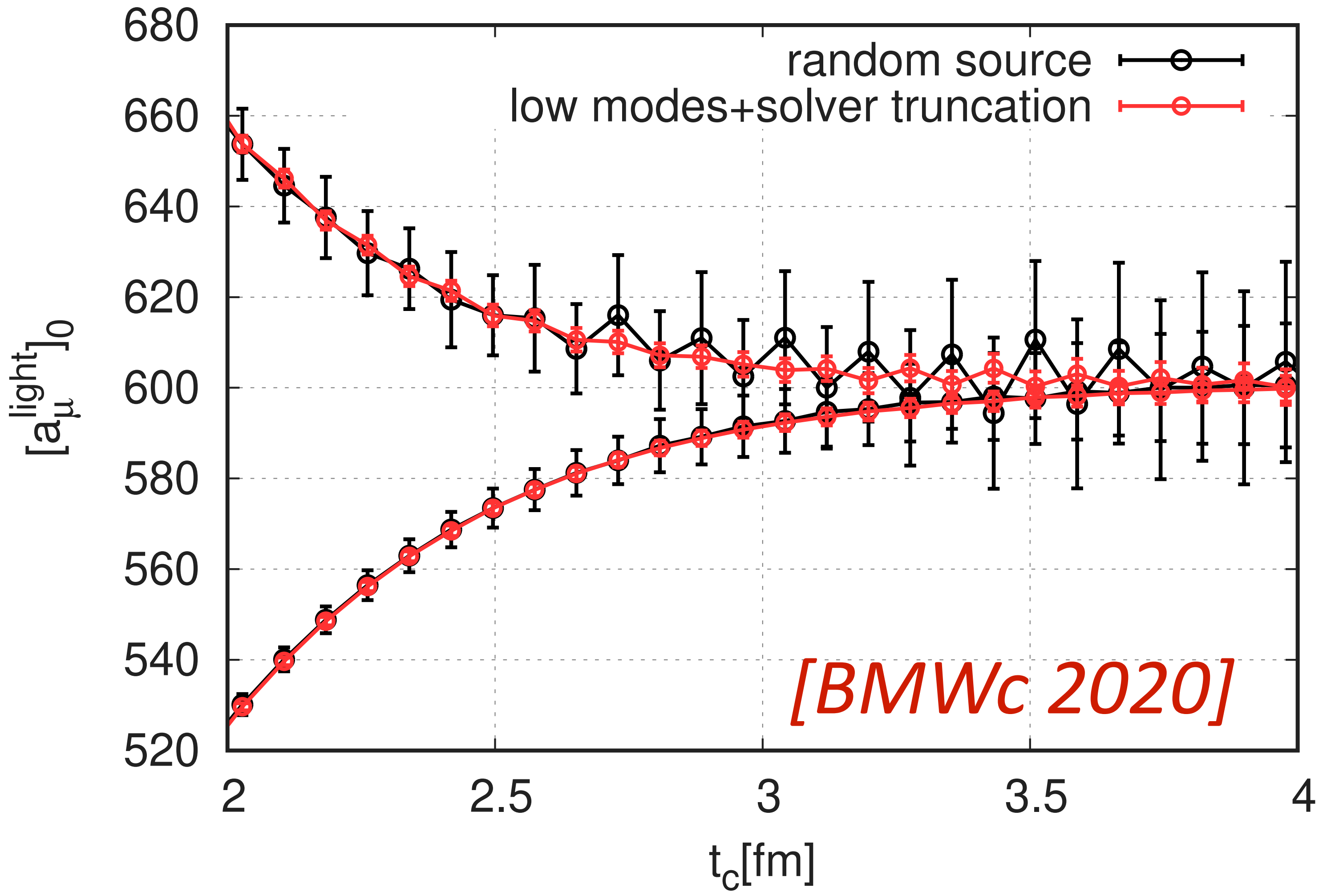
- Long-distance tail of the light quark contribution to $G(t)$: limiting factor for overall statistical precision
- Correlator dominated by isovector two-pion contribution

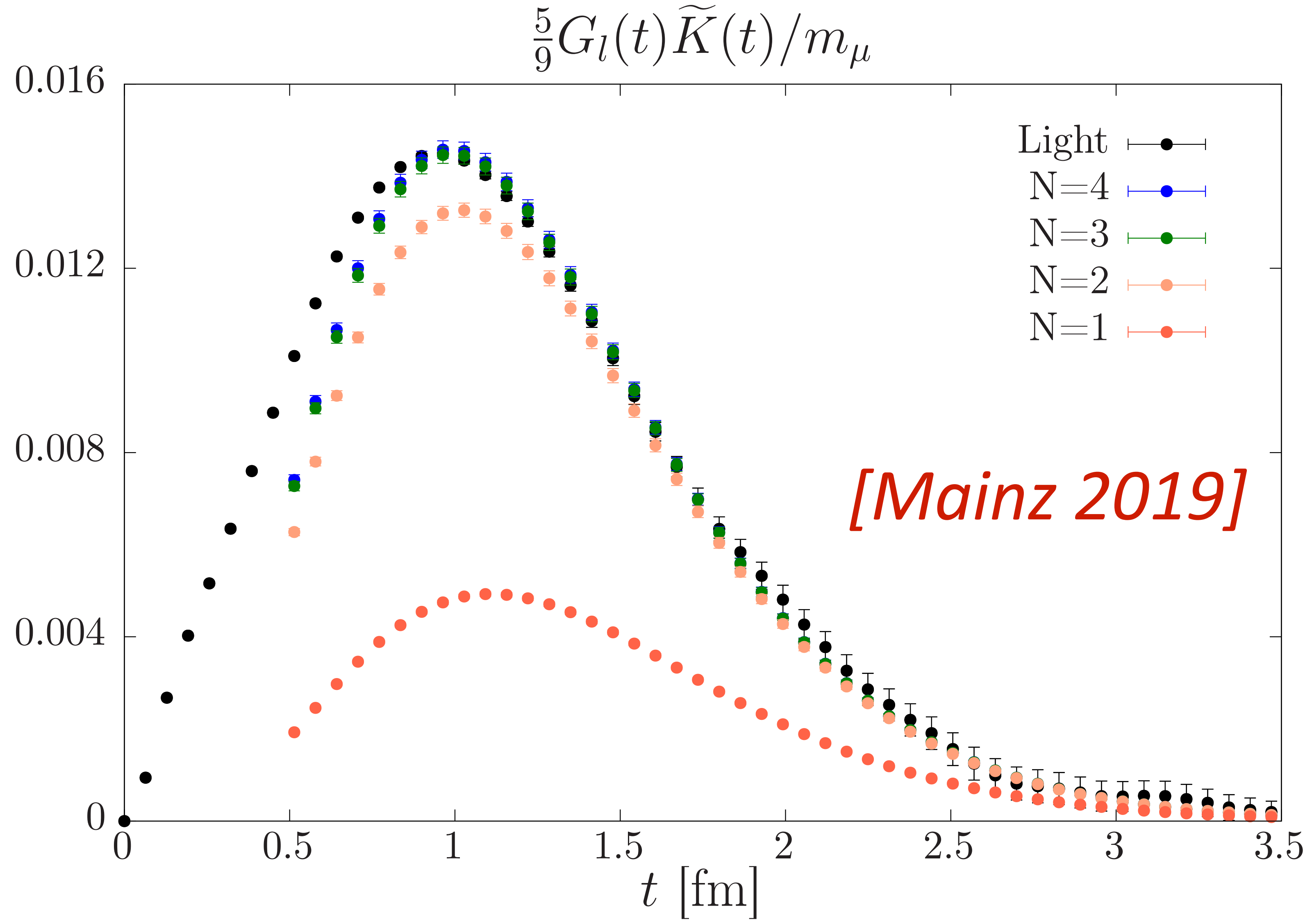


Strategies:

- Dedicated calculations of the spectrum in isovector channel and/or pion form factor $F_\pi(\omega)$
- "Bounding method":

$$0 \leq G(t) \leq G(t_c)\,\frac{G^{\pi\pi}(t)}{G^{\pi\pi}(t_c)}$$





- Noise-reduction methods: AMA, LMA, truncated solver

(can be combined)

Finite-volume effects

Mainz method (aka MLL): *[Meyer 2011, Francis et al. 2013, Della Morte et al. 2017; Lellouch & Lüscher 2001]*

$$G(t,L) \overset{t\to\infty}{=} \sum_n |A_n|^2\, \mathrm{e}^{-\omega_n t} \qquad G(t,\infty) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2)\, \mathrm{e}^{-\omega|t|}$$

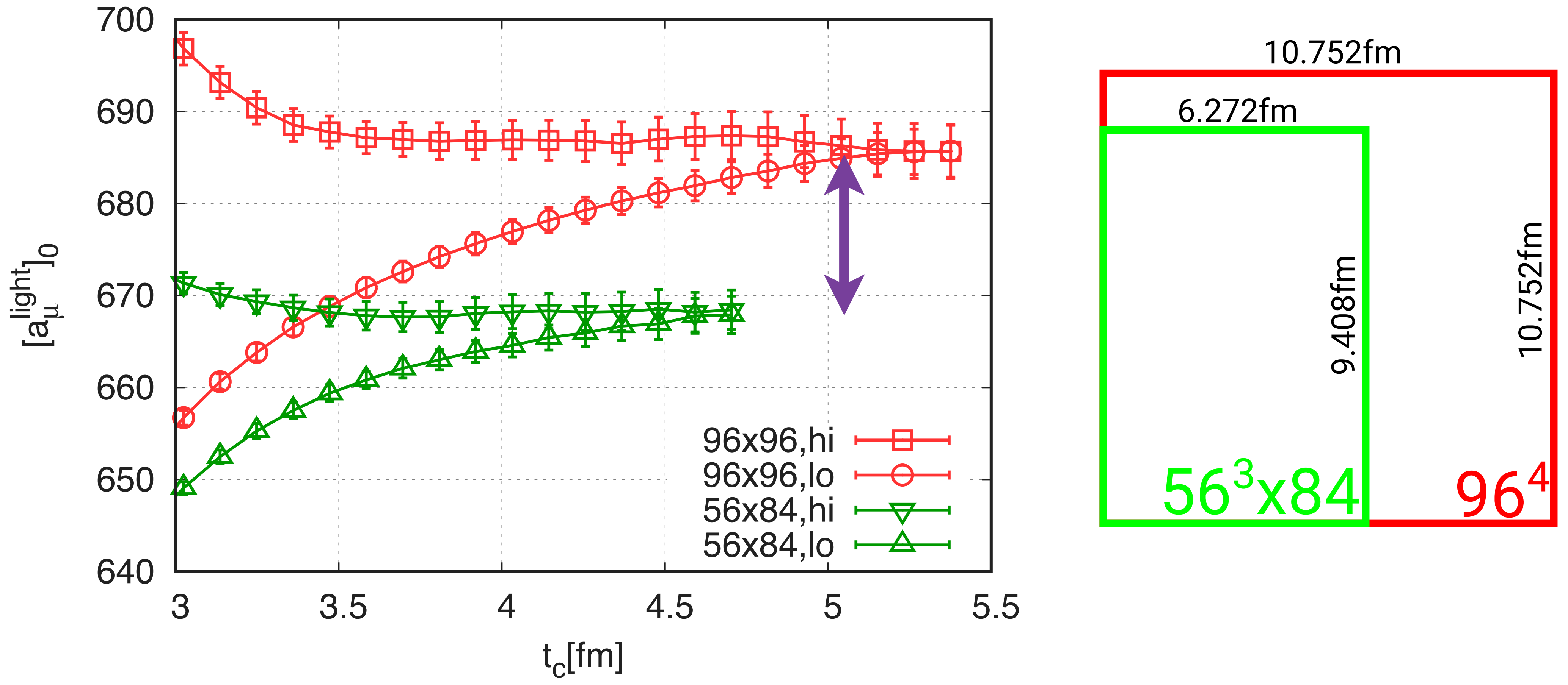
Both $|A_n|$ and $\rho(\omega^2)$ can be related to the pion form factor $F_\pi(\omega)$ $\Rightarrow$ $G(t,\infty) - G(t,L)$

Other methods:

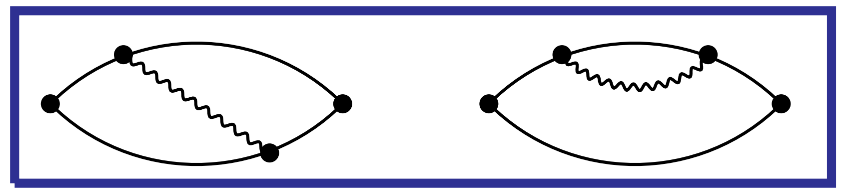
- Chiral Perturbation Theory *[Aubin et al. 2015,...]*
- Expansion in pion winding number *[Hansen & Patella]*

Correction	Comment
17.8	Gounaris-Sakurai model for $F_\pi(\omega)$
15.7	ChPT at NNLO
16.3	Expansion in pion winding number
18.1(2.4)	Direct lattice calculation

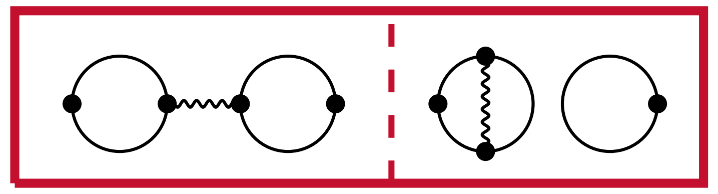
Direct lattice calculation by BMWc:



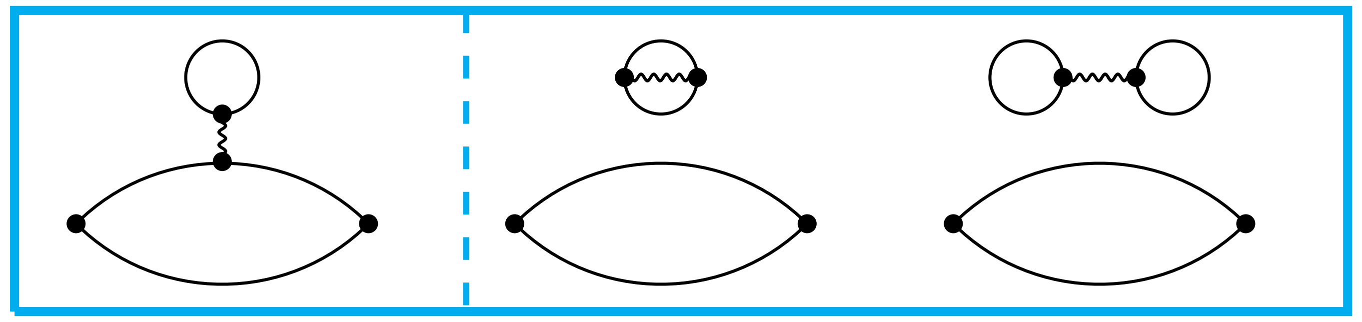
Isospin Breaking



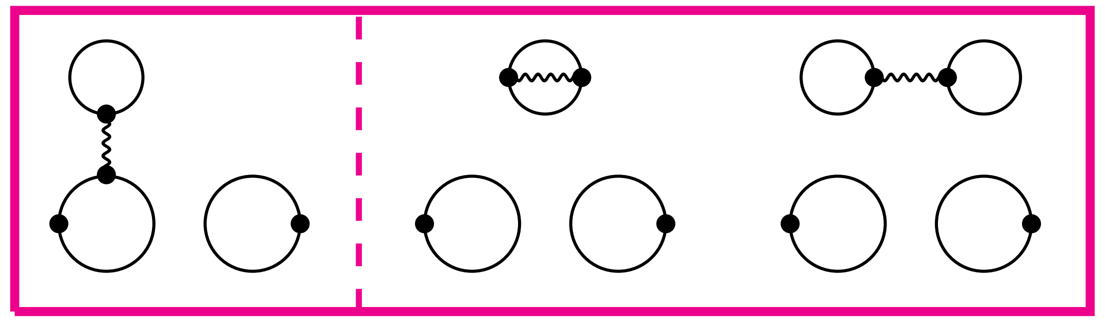
BMW $-1.27(40)(33)$
RBC/UKQCD $5.9(5.7)(1.7)$
ETM $1.1(1.0)$



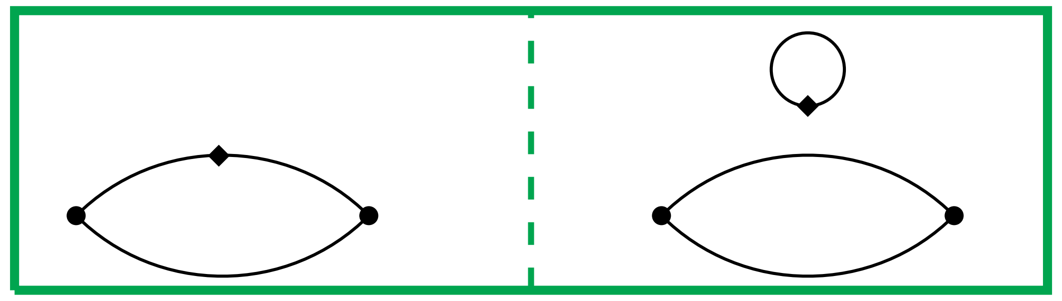
$-0.55(15)(11)$ BMW
$-6.9(2.1)(2.0)$ RBC/UKQCD



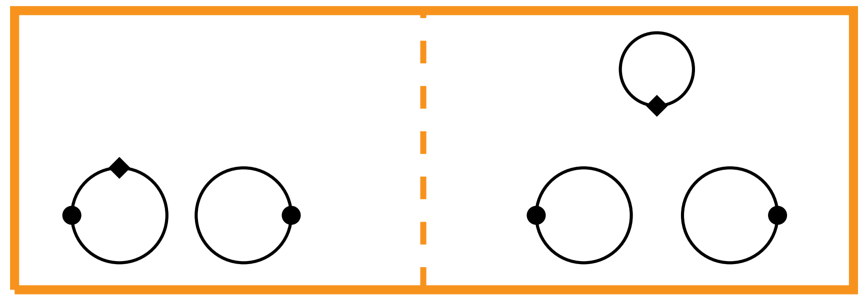
$-0.0095(86)(99)$ $0.42(20)(19)$ BMW



$0.011(24)(14)$ $-0.047(33)(23)$ BMW



$6.59(63)(53)$ BMW
$10.6(4.3)(6.8)$ RBC/UKQCD
$6.0(2.3)$ ETM
$7.7(3.7)$ $9.0(2.3)$ FHM
$9.0(0.8)(1.2)$ LM

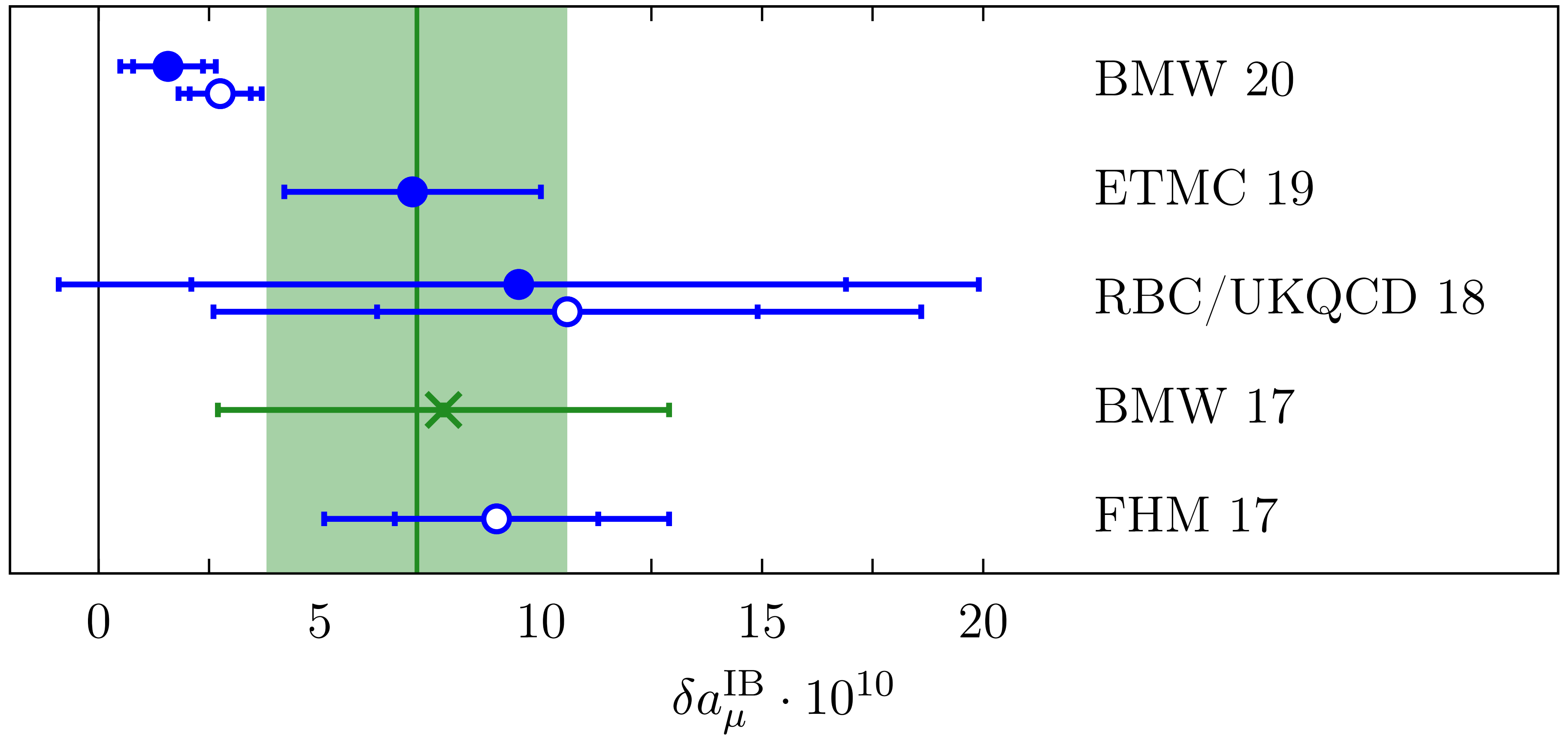


$-4.63(54)(69)$ BMW

BMW [arXiv:2002.12347]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

(Compilation by Vera Gülpers, Lattice-HVP Workshop Nov 2020)

Collection of published results:



- Small overall value result of cancellations
- Large statistical uncertainties:

$$a_\mu^{\rm IB} \lesssim 1\,\%\,, \quad \delta a_\mu^{\rm IB} \lesssim 100\,\%$$

- More precise calculations required

Discretisations of the quark action

Rooted staggered quarks:

- remnant fermion doublers — “tastes”
- correct analytically for taste-induced lattice artefacts
- low computational cost
- used by BMWc, FHM, Lehner & Meyer, Aubin et al., χQCD

Wilson quarks:

- no doublers; chiral symmetry broken explicitly
- leading lattice artefacts of $\mathrm{O}(a^2)$ after Symanzik improvement, twisted-mass formalism
- moderate computational cost
- used by ETMC, Mainz/CLS, PACS

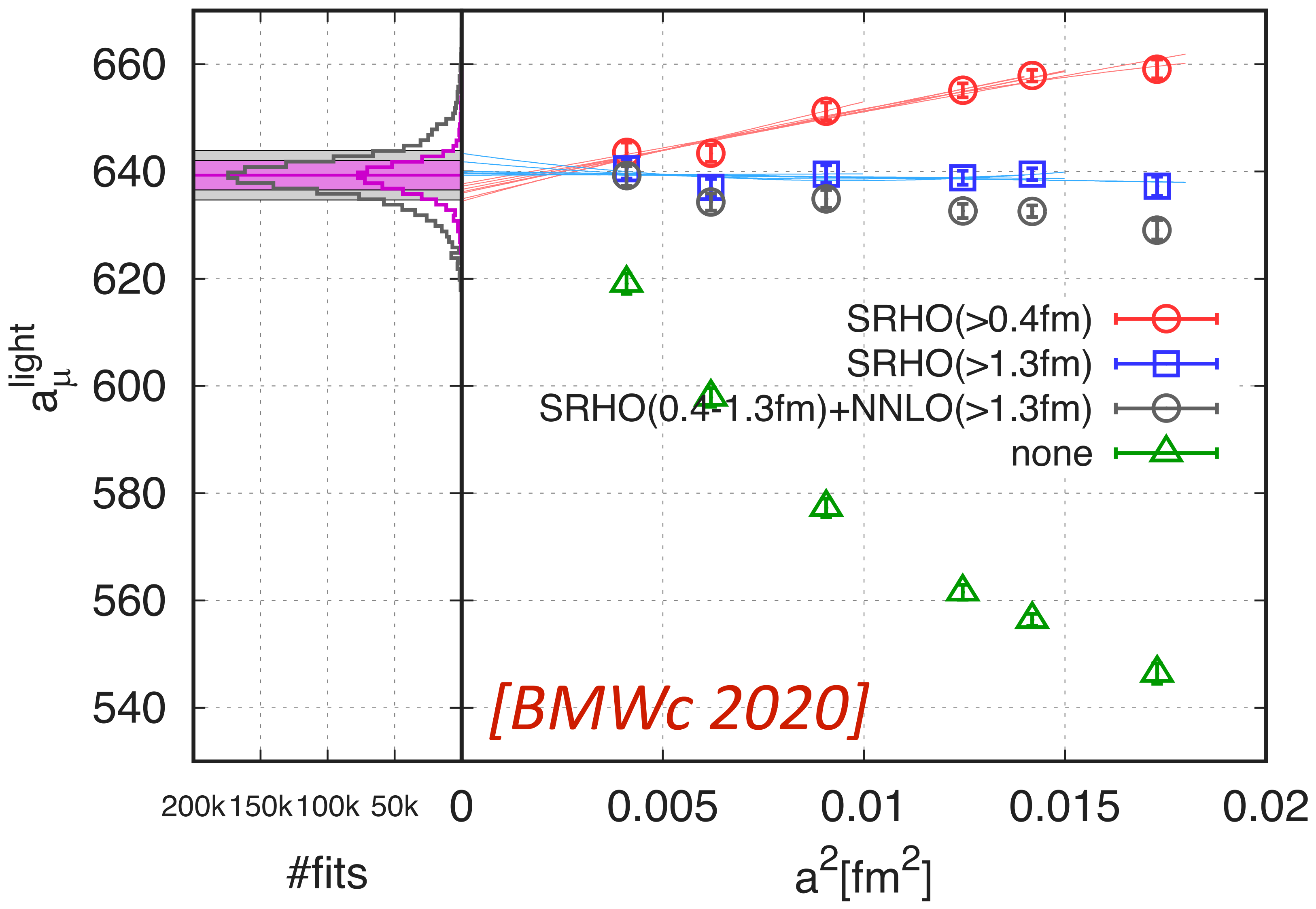
Domain wall /overlap quarks (Ginsparg-Wilson quarks):

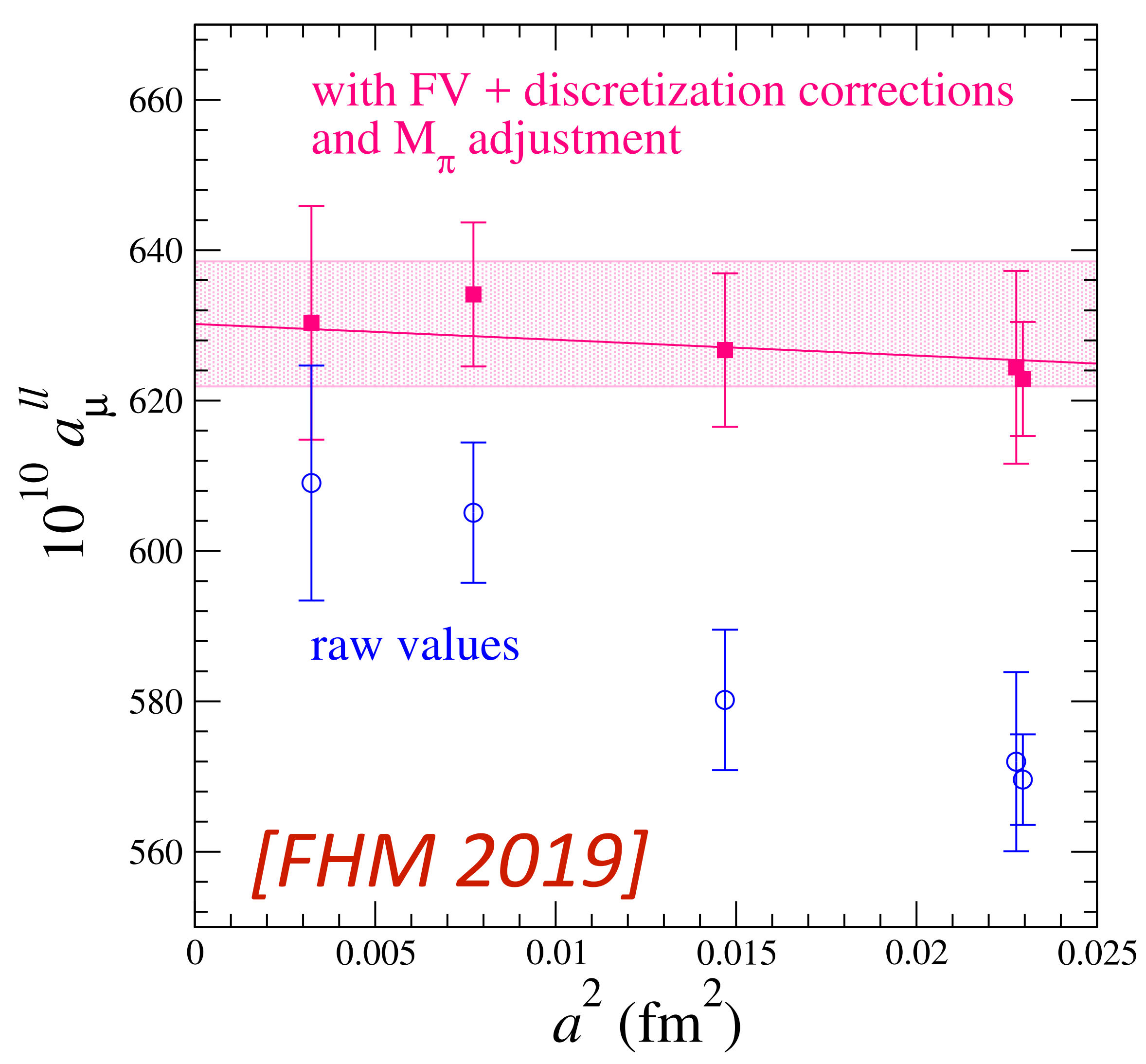
- no doublers; chiral symmetry breaking exponentially small
- live in five dimensions (dwf); evaluate sign function of “conventional” discretisation (ovlp)
- leading lattice artefacts of $\mathrm{O}(a^2)$
- high computational cost
- used by RBC/UKQCD, BMWc (Ovlp valence), χQCD (DWF sea, Ovlp valence)

Extrapolation to the physical point

Staggered quarks:

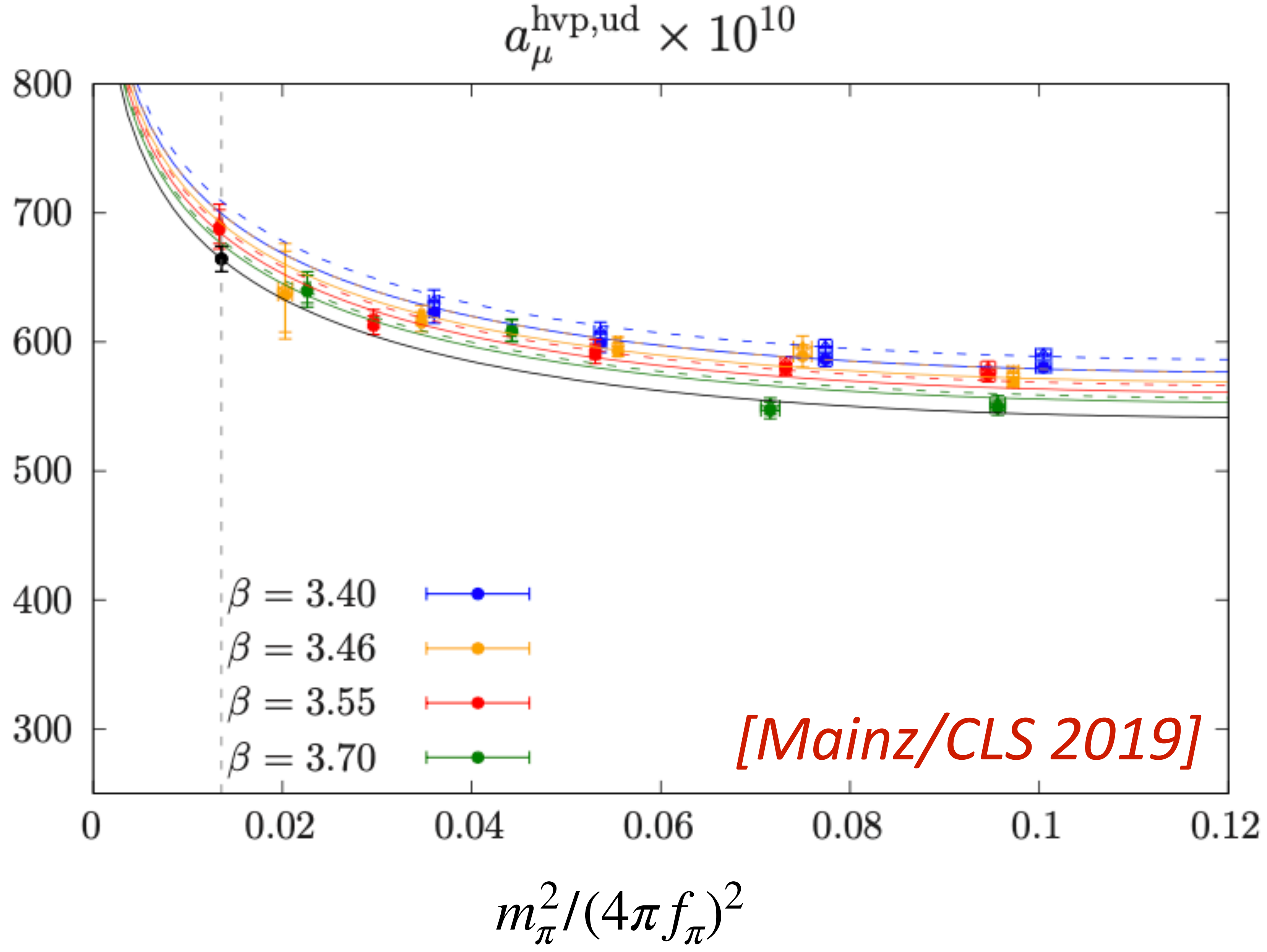
correct for taste-breaking effects





Wilson quarks:

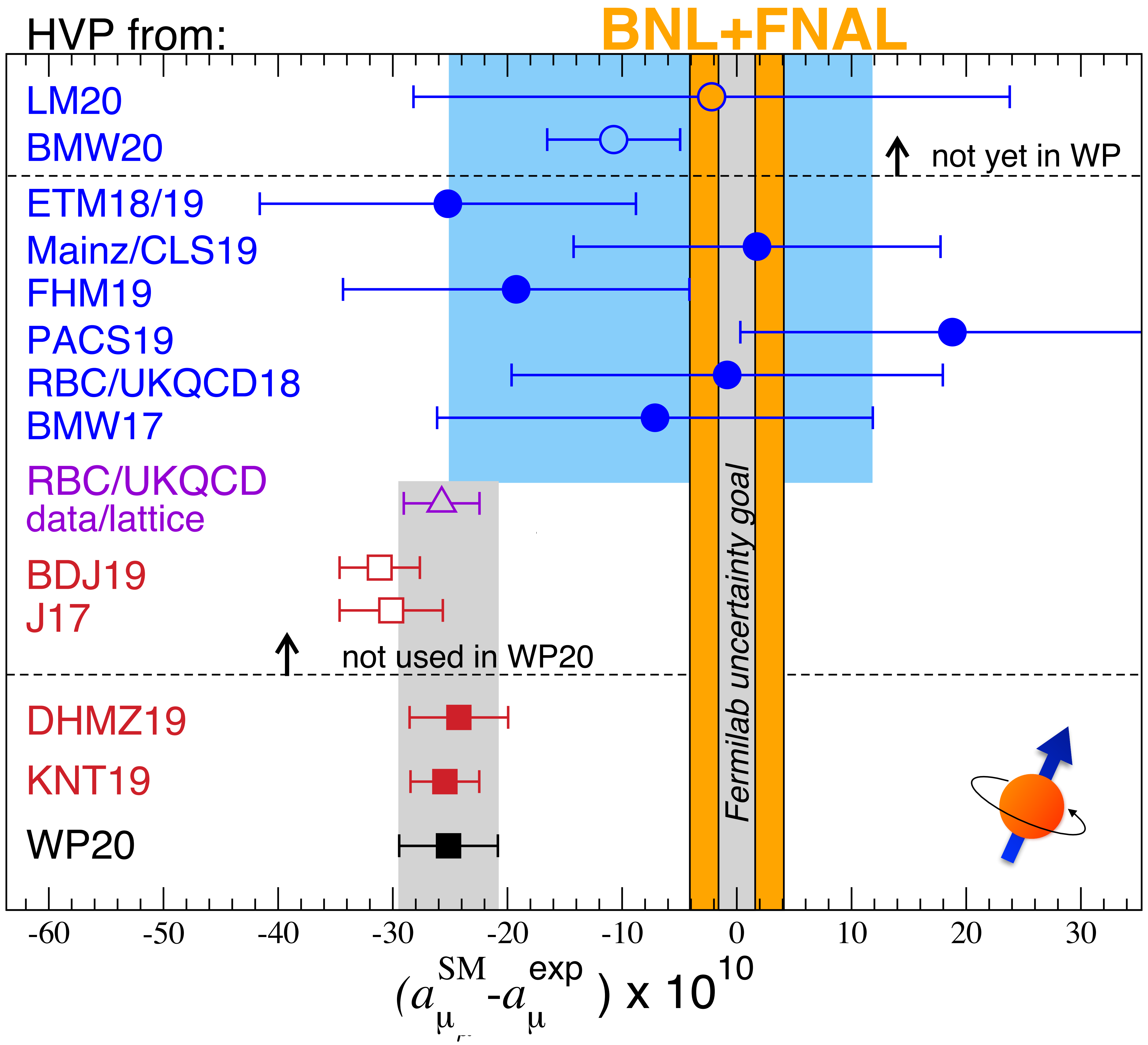
chiral & continuum extrapolation



Domain wall quarks:

two lattice spacings at $m_\pi^{\rm phys}$ and estimate of residual discretisation error *[RBC/UKQCD 2018]*

Hadronic vacuum polarisation: Data-driven approach versus lattice QCD



White Paper:

R-ratio: $a_\mu^{\rm hvp,\,LO} = (693.1 \pm 4.0) \cdot 10^{-10}$ [0.6%]

LQCD: $a_\mu^{\rm hvp,\,LO} = (711.6 \pm 18.4) \cdot 10^{-10}$ [2.6%]

Lattice QCD result by BMW Collab.:

$$a_\mu^{\rm hvp,\,LO} = (707.5 \pm 2.3 \pm 5.0) \cdot 10^{-10} \quad [0.8\%]$$

[Borsányi et al., Nature 593 (2021) 7857, arXiv:2002.12347v3]

$$\Rightarrow \quad a_\mu^{\rm exp} - a_\mu^{\rm SM}\Big|^{\rm hvp,\,LO}_{\rm BMWc} = 107(70) \times 10^{-11} \quad (1.5\,\sigma)$$

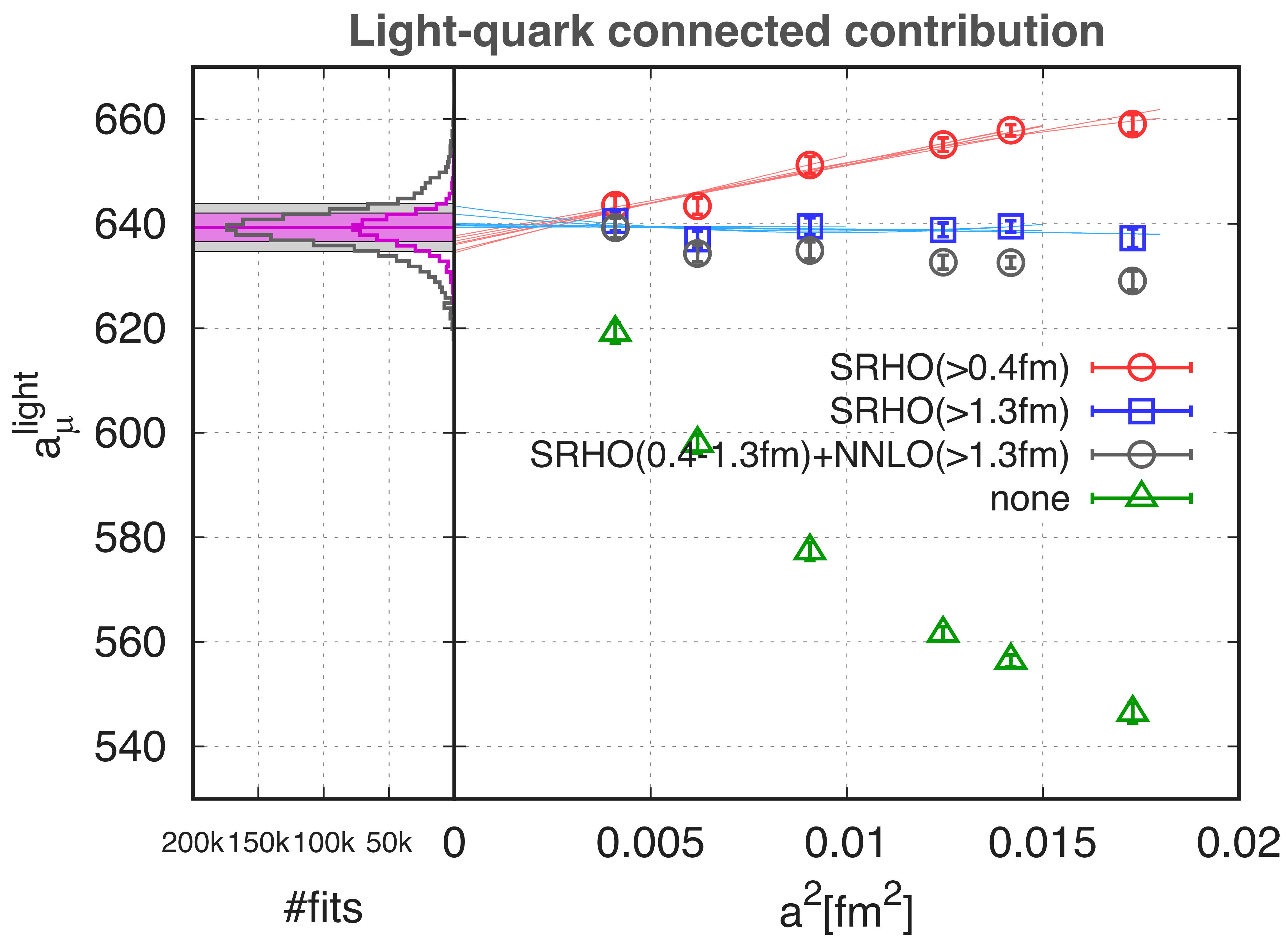
(2.1σ tension with R-ratio)

Requires independent confirmation

Staggered fermions: The BMW result

- $N_f = 2+1+1$ of rooted staggered quarks [Borsányi et al., Nature 593 (2021) 7857, arXiv:2002.12347v3]
- Six lattice spacings: $a = 0.132 - 0.064\,\text{fm}$, physical pion mass
- Correct for taste-breaking effects using EFTs: SRHO model, combined with SChPT
- Comprehensive study of finite-volume and isospin-breaking corrections
- Final result selected from distribution of different fits
- Results dominated by systematic error associated with continuum extrapolation

Can parts of the result be checked with sub-percent precision?



Window observables

Restract integration over Euclidean time to sub-intervals

$\rightarrow$ reduce/enhance sensitivity to systematic effects

$$a_\mu^{\mathrm{hvp,\,win}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{K}(t)\, G(t)\, W(t;\, t_0,\, t_1)$$

Short distance: $W^{\mathrm{SD}}(t;\, t_0) = 1 - \Theta(t,\, t_0, \Delta)$

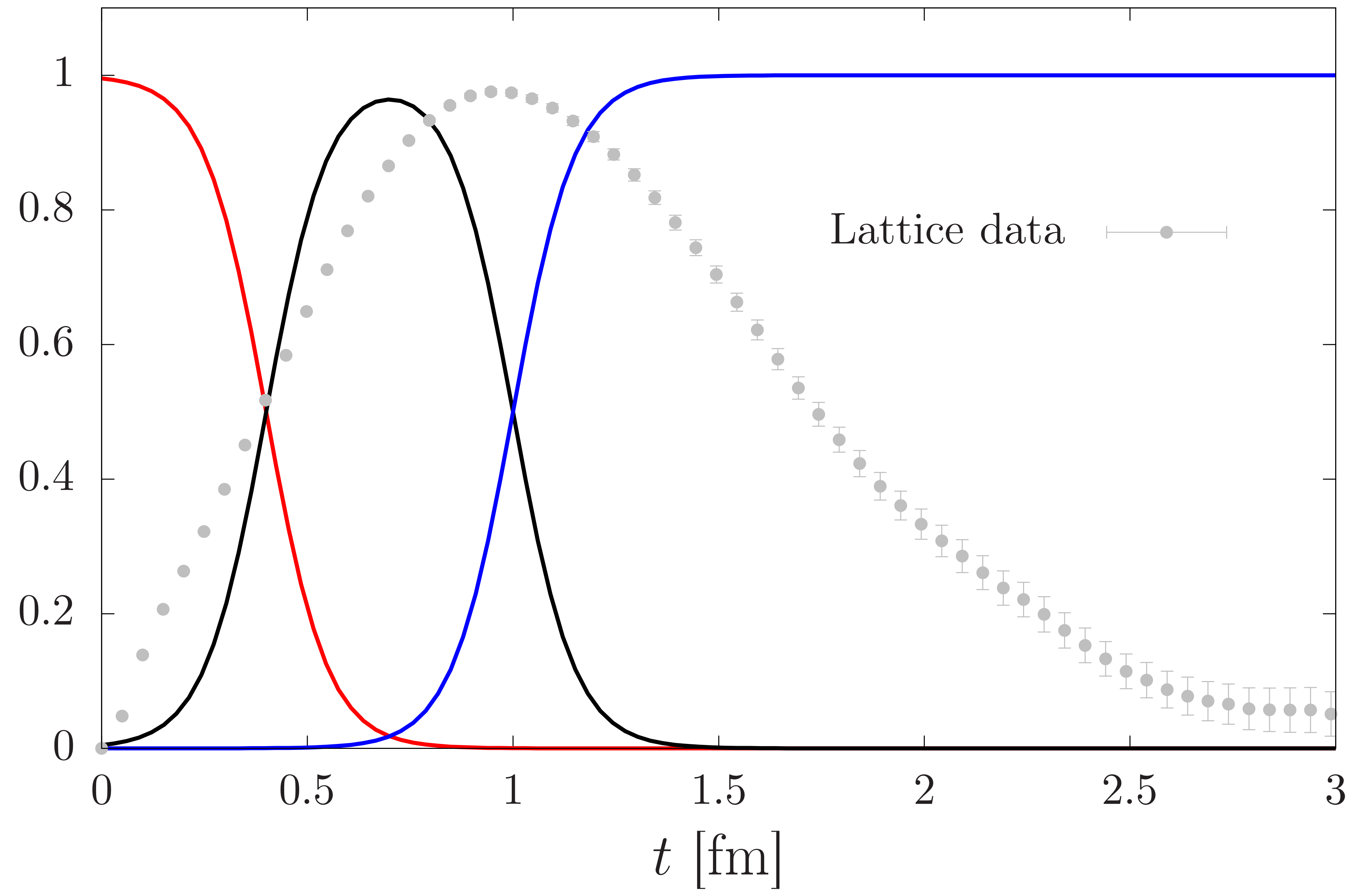
$$\Theta(t,\, t', \Delta) = \tfrac{1}{2}\left[1 + \tanh(t - t')/\Delta\right]$$

Intermediate distance: $W^{\mathrm{ID}}(t;\, t_0,\, t_1) = \Theta(t,\, t_0, \Delta) - \Theta(t,\, t_1, \Delta)$

Long distance: $W^{\mathrm{LD}}(t;\, t_1) = \Theta(t,\, t_1, \Delta)$

"Standard" choice: *[RBC/UKQCD 2018]*

$$t_0 = 0.4\,\mathrm{fm},\ t_1 = 1.0\,\mathrm{fm},\ \Delta = 0.15\,\mathrm{fm}$$



Intermediate window:

- Finite-volume correction reduced from 3% to 0.25%
- Uncertainty dominated by statistics

$\Rightarrow$ Precision test of different lattice calculations

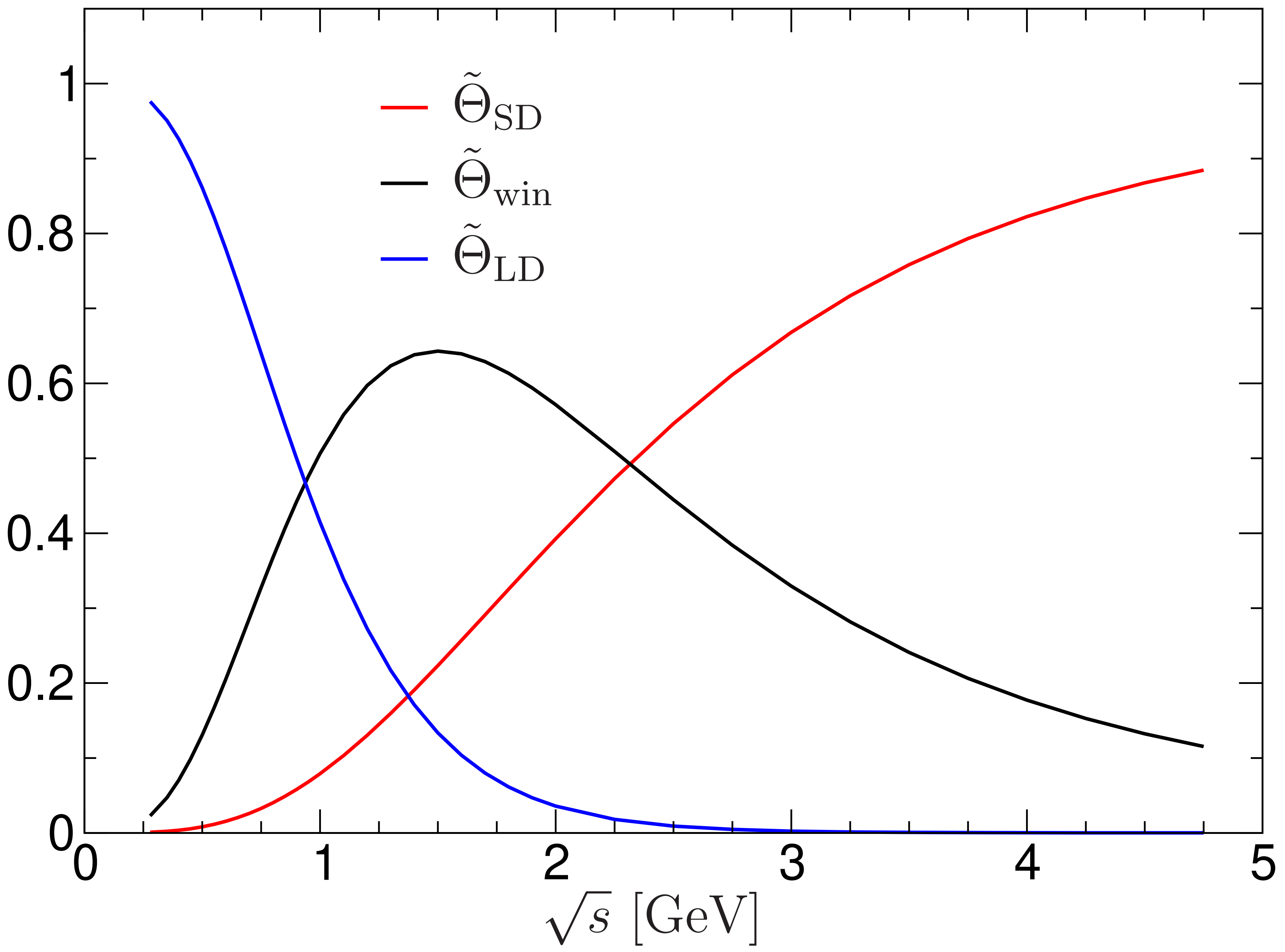
$\Rightarrow$ Comparison with corresponding R-ratio estimate

Window observables: Comparison with R-ratio

Starting point: $$G(t) = \frac{1}{12\pi^2}\int_{m_{\pi^0}^2}^{\infty} d(\sqrt{s})\,R(s)\,s\,\mathrm{e}^{-\sqrt{s}t}$$ [RBC/UKQCD 2018]

Insert $G(t)$ into expression for time-momentum representation:

$$a_\mu^{\mathrm{hvp,\,ID}} = \left(\frac{\alpha}{\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} d(\sqrt{s})\,R(s)\,\frac{1}{12\pi^2}\,s\int_0^{\infty} dt\,\tilde{K}(t)\,W^{\mathrm{ID}}(t;\,t_0,\,t_1)\,\mathrm{e}^{\sqrt{s}t}$$



Intermediate window from R-ratio following procedure for WP estimate:

$$a_\mu^{\mathrm{hvp,\,ID}} \equiv a_\mu^{\mathrm{win}} = (229.4 \pm 1.4)\cdot 10^{-10}$$

Finer decomposition allows for more detailed studies of energy dependence

[Colangelo et al., arXiv:2205.12963]

Wilson fermions: The Mainz/CLS results for $a_\mu^{\rm hvp}$ and $a_\mu^{\rm win}$

- $N_f = 2+1$ flavours of $\mathrm{O}(a)$ improved Wilson fermions
- Four lattice spacings: $a = 0.085 - 0.050\,\mathrm{fm}$; pion masses $m_\pi = 130 - 420\,\mathrm{MeV}$
- Two discretisations of the vector current: local and conserved
- Simultaneous chiral and continuum extrapolation

$$a_\mu^{\rm hvp,\,LO} = (720.0 \pm 12.6 \pm 9.9)\cdot 10^{-10}$$

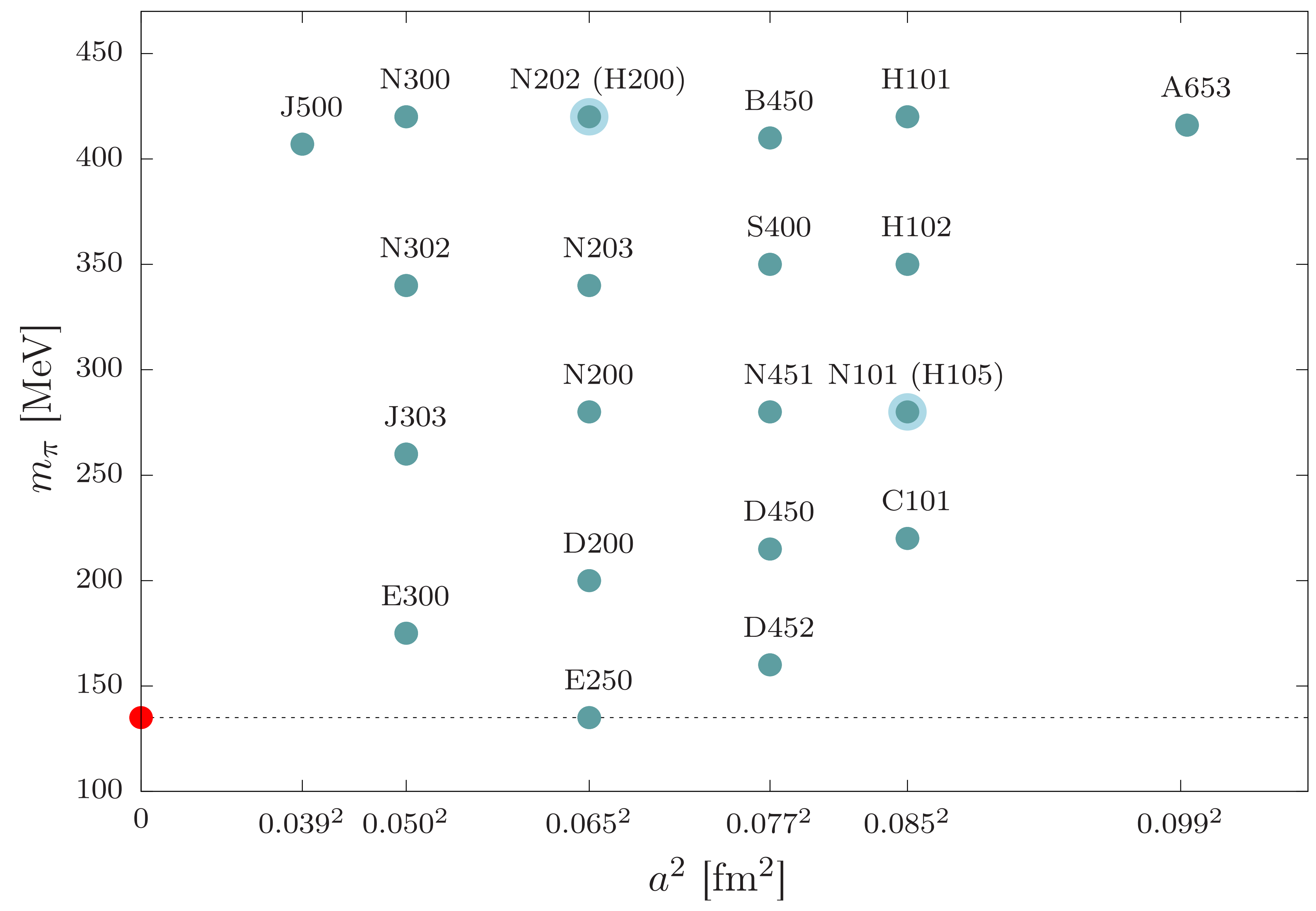
[Gérardin et al., Phys. Rev. D 100 (2019) 014510]

- Extension to six lattice spacings: $a = 0.099 - 0.039\,\mathrm{fm}$; additional ensembles with $m_\pi \gtrsim m_\pi^{\rm phys}$
- Intermediate window observable:

$$a_\mu^{\rm win,\,ud} = (207.00 \pm 0.83 \pm 1.20)\cdot 10^{-10}$$

$$a_\mu^{\rm win} = (237.30 \pm 0.79 \pm 1.22)\cdot 10^{-10}$$

[Cè et al., arXiv:2206.06582]



Mainz/CLS: Scaling test

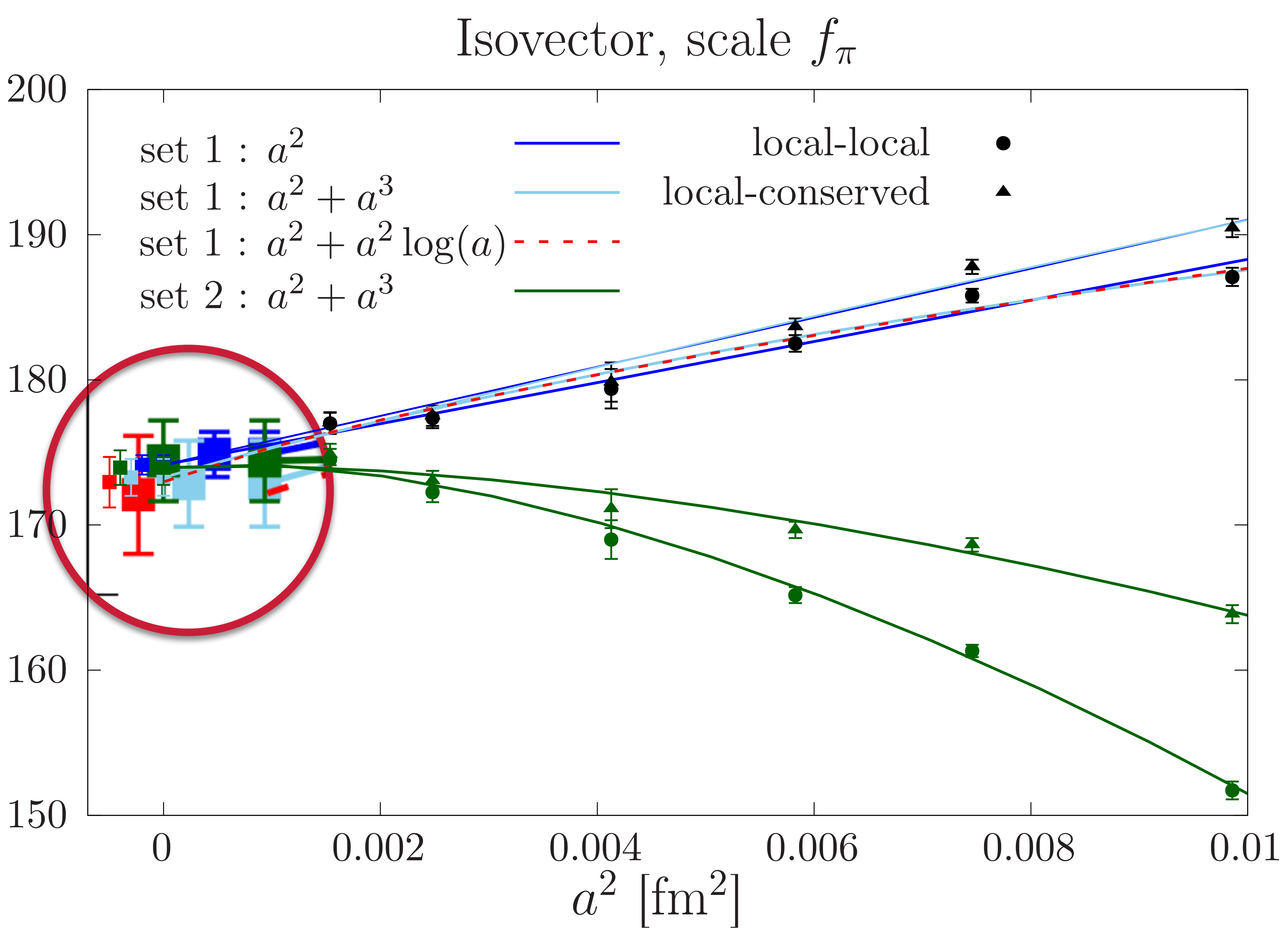
[Cè et al., arXiv:2206.06582]

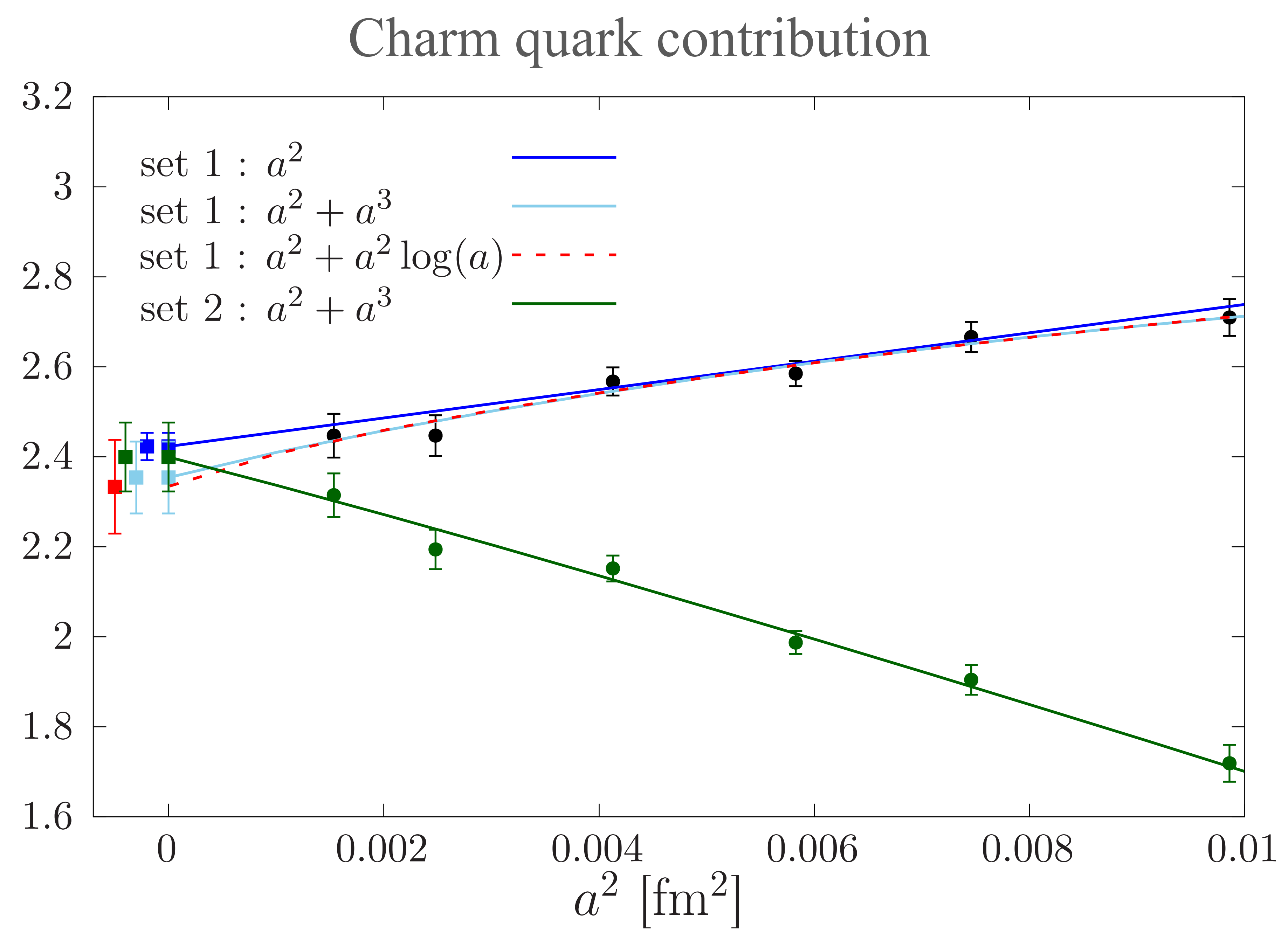
- Two independent sets of improvement coefficients for local and conserved currents

$$J_\mu^{(\alpha)} = j_\mu^{(\alpha)} + a\, c_{\rm V}(g_0)\, \tilde{\partial}_\nu \Sigma_{\mu\nu}, \qquad \alpha = L,\, C$$

→ four different discretisations of the current-current correlator

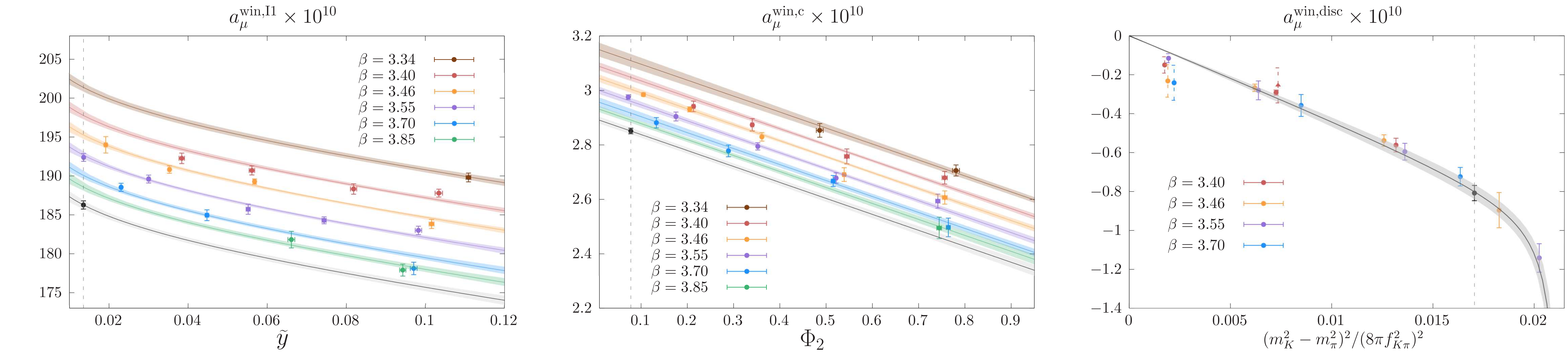
- Scaling test at $m_\pi = 420\,\mathrm{MeV}$

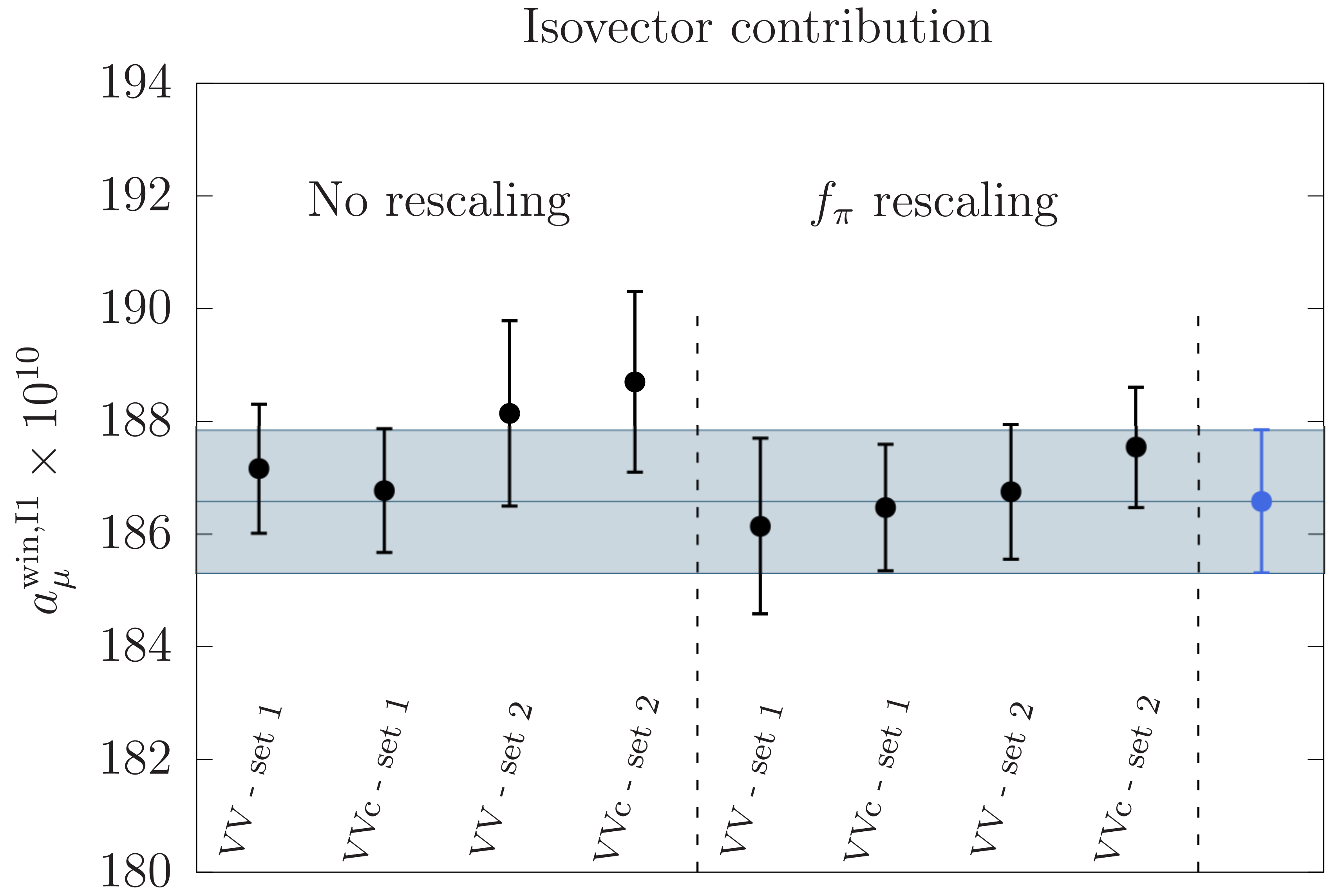




Mainz/CLS: Results at the physical point

[Cè et al., arXiv:2206.06582]





$$a_\mu^{\rm win,I1} = (186.30 \pm 0.75_{\rm stat} \pm 1.08_{\rm syst}) \times 10^{-10}\,,$$

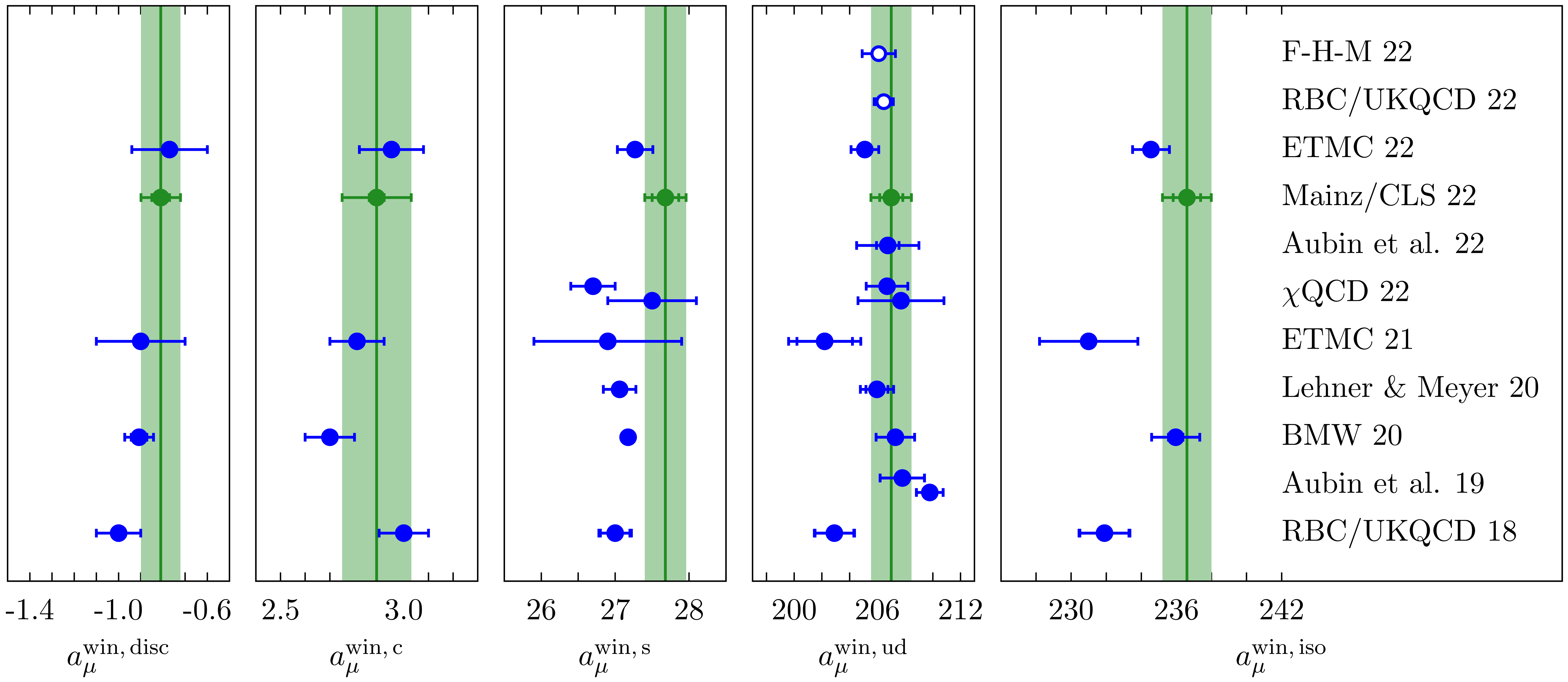
$$a_\mu^{\rm win,I0} = a_\mu^{\rm win,I0,\not{c}} + a_\mu^{\rm win,c} = (50.30 \pm 0.23_{\rm stat} \pm 0.32_{\rm syst}) \times 10^{-10}\,,$$

$$a_\mu^{\rm win,iso} = a_\mu^{\rm win,I1} + a_\mu^{\rm win,I0} = (236.60 \pm 0.79_{\rm stat} \pm 1.13_{\rm syst} \pm 0.05_{\rm Q}) \times 10^{-10}$$

Include shift of $+(0.70 \pm 0.47) \cdot 10^{-10}$ due to isospin-breaking:

$$a_\mu^{\rm win} = (237.30 \pm 0.79_{\rm stat} \pm 1.13_{\rm syst} \pm 0.05_{\rm Q} \pm 0.47_{\rm IB}) \times 10^{-10}$$

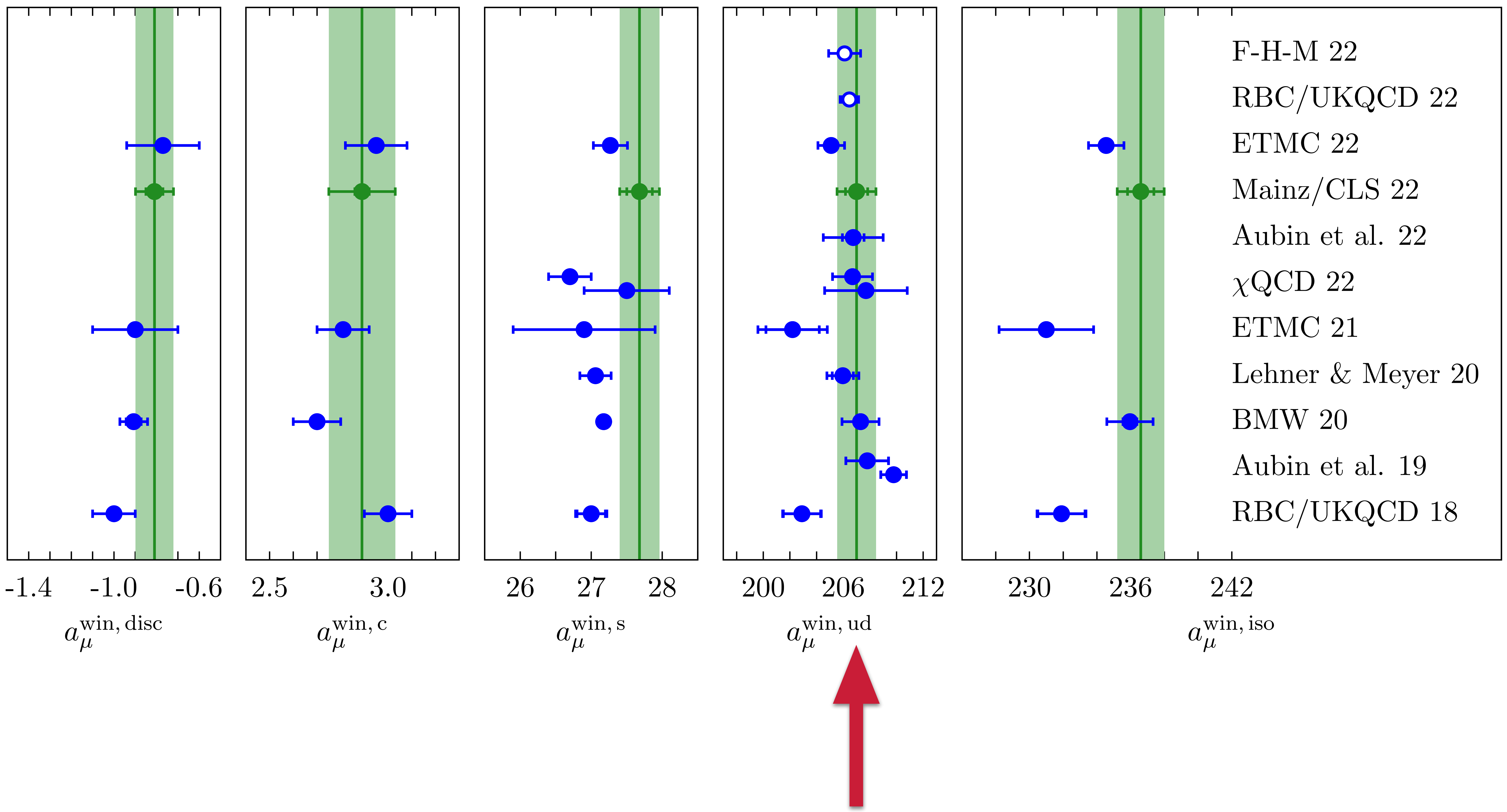
Intermediate window observable in isosymmetric QCD



[S. Gottlieb @ Benasque 2022]
[C. Lehner @ Edinburgh 2022]
[Alexandrou et al., arXiv:2206.15084]
[Cè et al., arXiv:2206.06582]

- Broad agreement among most lattice calculations; exceptions: RBC/UKQCD 18 and ETMC 21
- New calculation by ETMC (twisted-mass Wilson fermions) consistent with Mainz/CLS 22
- Preliminary results by RBC/UKQCD (domain wall fermions; added third lattice spacing) and Fermilab-HPQCD-MILC (staggered) confirm recent results for light-connected contribution

Intermediate window observable in isosymmetric QCD



[S. Gottlieb @ Benasque 2022]
[C. Lehner @ Edinburgh 2022]
[Alexandrou et al., arXiv:2206.15084]
[Cè et al., arXiv:2206.06582]

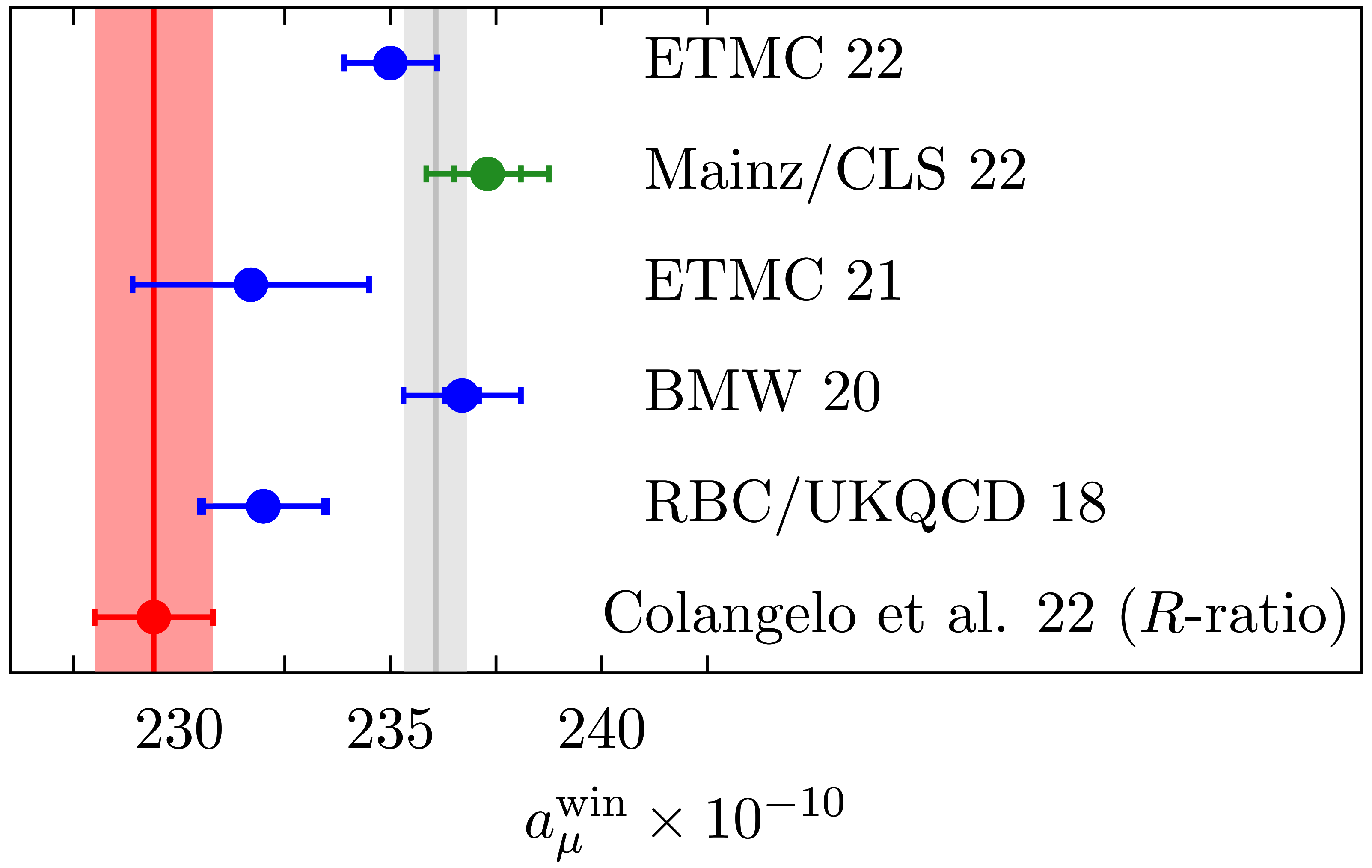
Result for the dominant, light-quark connected contribution confirmed for wide range of different discretisations with sub-percent precision

Intermediate window observable: Comparison with R-ratio

R-ratio estimate: $a_\mu^{\rm win} = (229.4 \pm 1.4) \cdot 10^{-10}$

Mainz/CLS 22: $a_\mu^{\rm win} = (237.30 \pm 1.46) \cdot 10^{-10}$

$$\Rightarrow \quad a_\mu^{\rm win}\Big|_{\rm Mainz} - a_\mu^{\rm win}\Big|_{R-\rm ratio} = (7.9 \pm 2.0) \cdot 10^{-10} \quad [3.9\,\sigma]$$



Lattice average: $a_\mu^{\rm win} = (236.08 \pm 0.74) \cdot 10^{-10}$

(ETMC 22, Mainz/CLS 22, BMW 20)

$$\Rightarrow \quad a_\mu^{\rm win}\Big|_{\rm Lat-av.} - a_\mu^{\rm win}\Big|_{R-\rm ratio} = (6.7 \pm 1.6) \cdot 10^{-10} \quad [4.2\,\sigma]$$

Subtract R-ratio prediction for $a_\mu^{\rm win}$ from White Paper estimate and replace by lattice average:

$$a_\mu^{\rm exp} - a_\mu^{\rm SM}\Big|^{\rm win}_{\rm Lat-av.} = (184 \pm 58) \cdot 10^{-11} \quad [3.2\,\sigma]$$

Hadronic running of electromagnetic coupling

Correlation between $a_\mu^{\rm hvp}$ and the hadronic running of $\Delta\alpha_{\rm had}$:

$$\Delta\alpha_{\rm had}^{(5)}(q^2) = -\frac{\alpha\, q^2}{3\pi} \fint_{m_{\pi^0}^2}^{\infty} ds\, \frac{R(s)}{s(s-q^2)}, \qquad R(s) = \frac{3s}{4\pi\,\alpha(s)}\,\sigma(e^+e^- \to \text{hadrons})$$

$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R(s)\,\hat{K}(s)}{s^2}, \qquad 0.63 \lesssim \hat{K}(s) \leq 1$$

Hadronic running at Z-pole: $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ $\rightarrow$ key quantity in global electroweak fit

Euclidean momenta

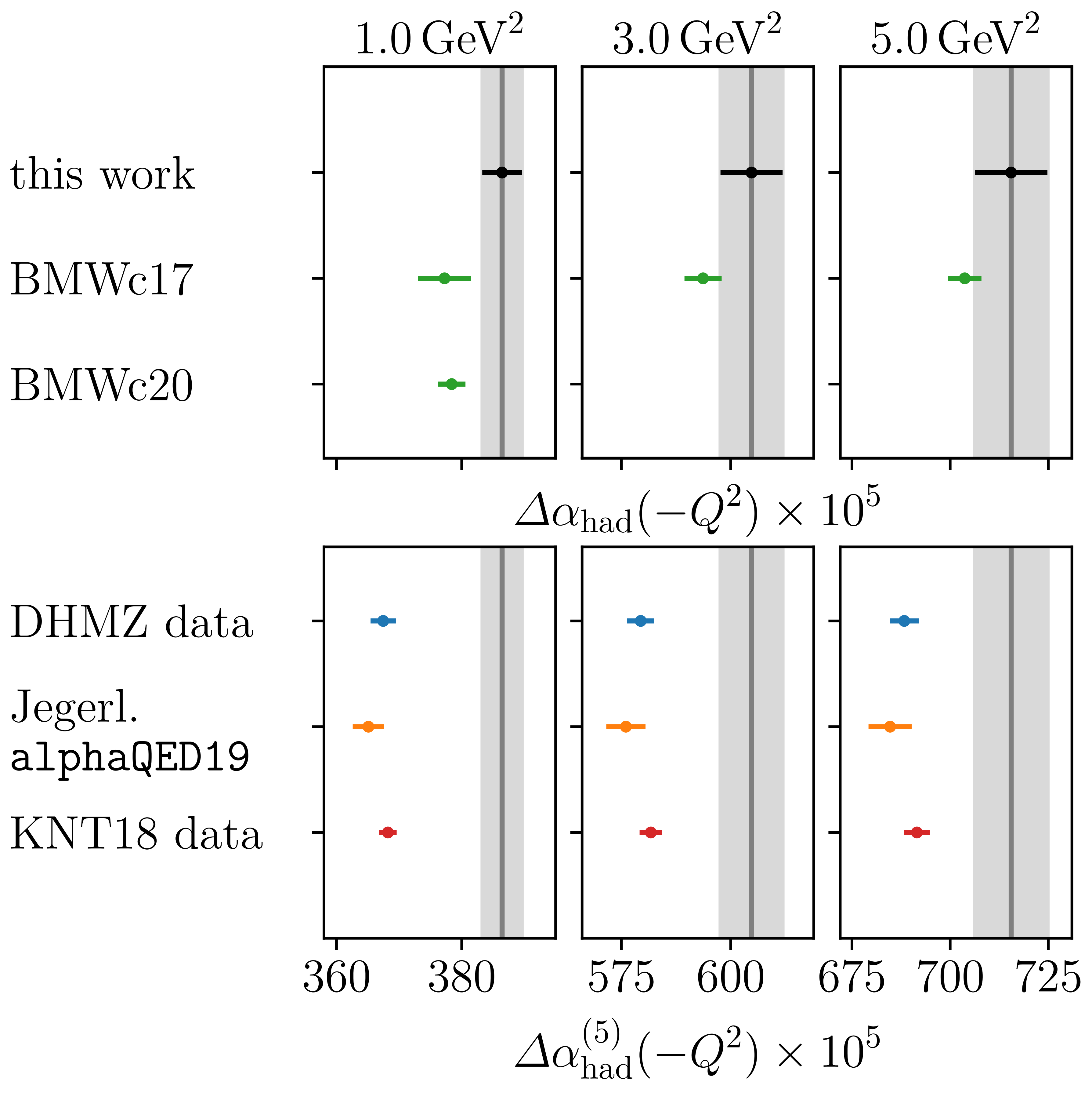
$\Delta\alpha_{\rm had}(-Q^2)$ accessible in lattice QCD via the same correlator $G(t)$ with a different kernel function:

$$\Delta\alpha_{\rm had}(-Q^2) = \frac{\alpha}{\pi}\frac{1}{Q^2}\int_0^{\infty} dt\, G(t)\left[Q^2t^2 - 4\sin^2\left(\tfrac{1}{2}Q^2t^2\right)\right]$$

Hadronic running of $\alpha_{\mathrm{e.m.}}$ in lattice QCD

Direct lattice calculation of $\Delta\alpha(-Q^2)$ on the same gauge ensembles used in Mainz/CLS 22

[Cè et al., JHEP 08 (2022) 220, arXiv:2203.08676]



- Mainz/CLS and BMWc (2017) differ by 2–3% at the level of 1–2σ
- Tension between Mainz/CLS and data-driven evaluation of $\sim 3\sigma$ for $Q^2 \gtrsim 3\,\mathrm{GeV}^2$

  ⇒ consistent with observed tension for window observable
- Tension increases to $\gtrsim 5\sigma$ for $Q^2 \lesssim 2\,\mathrm{GeV}^2$ (smaller statistical error due to ansatz for continuum extrapolation)

→ convert lattice result for $\Delta\alpha^{(5)}_{\mathrm{had}}(-Q^2)$ to $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ and compare to global electroweak fit

Evaluation of $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ and consistency of the Standard Model

Method 1: Direct dispersion relation (DR)

$$\Delta\alpha_{\rm had}^{(5)}(q^2) = -\frac{\alpha\, q^2}{3\pi} \fint_{m_{\pi^0}^2}^{\infty} ds\, \frac{R(s)}{s(s-q^2)} \qquad \text{for } q^2 = M_Z^2$$

→ use combination of perturbation theory and experimental data for R-ratio

Method 2: Adler function approach, aka. "Euclidean split technique"

$$\Delta\alpha_{\rm had}^{(5)}(M_Z^2) = \Delta\alpha_{\rm had}^{(5)}(-Q_0^2) \qquad \leftarrow \text{ lattice QCD or DR for } q^2 = -Q_0^2$$

$$+[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)] \qquad \leftarrow \text{ Adler function in pQCD or DR}$$

$$+[\Delta\alpha_{\rm had}^{(5)}(M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-M_Z^2)] \qquad \leftarrow \text{ pQCD}$$

[Chetyrkin et al., Nucl Phys B482 (1996) 213; Eidelman et al., Phys Lett B454 (1999) 369; Jegerlehner, hep-ph/9901386, arXiv:0807.4206]

Euclidean split technique and the Adler function

Adler function: $$D(-s) = \frac{3\pi}{\alpha}\, s\, \frac{d}{ds}\, \Delta\alpha_{\rm had}(s)$$

$D(Q^2)$ known in massive QCD perturbation theory at three loops

$$\left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right]_{\rm pQCD/Adler} = \frac{\alpha}{3\pi}\int_{Q_0^2}^{M_Z^2} \frac{dQ^2}{Q^2}\, D(Q^2)$$

Relation of $D(Q^2)$ and R-ratio: $$D(Q^2) = Q^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R(s)}{(s+Q^2)^2}$$

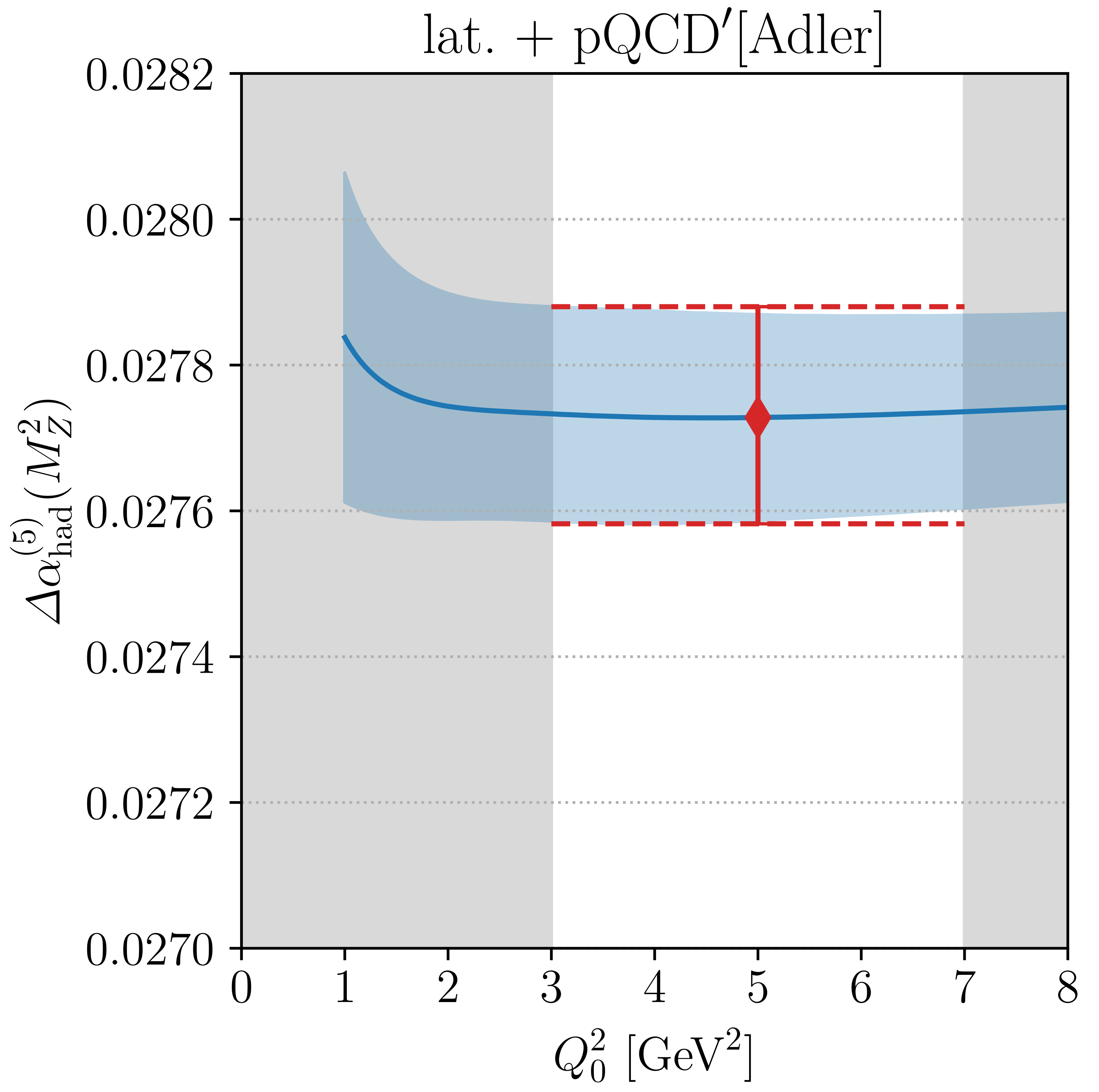
Direct DR: $$\left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right]_{\rm DR} = \frac{\alpha(M_Z^2 - Q_0^2)}{3\pi}\int_{m_{\pi^0}^2}^{\infty} ds\, \frac{R(s)}{(s+Q_0^2)(s+M_Z^2)}$$

Perturbation theory: $$\left[\Delta\alpha_{\rm had}^{(5)}(M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-M_Z^2)\right] = 0.000\,045(2)$$ *[Jegerlehner, CERN Yellow Report, 2020]*

Evaluation of $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ from lattice QCD data

Input: Lattice result for $\Delta\alpha_{\rm had}(-Q_0^2)$ for $Q_0^2 = 3-7\,\mathrm{GeV}^2$ *[Cè et al., JHEP 08 (2022) 220, arXiv:2203.08676]*

Evaluate $\left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right]_{\rm pQCD/Adler}$ using Jegerlehner's software package **`pQCDAdler`**



Final estimate: lattice + pQCD/Adler

$$\Delta\alpha_{\rm had}^{(5)}(M_Z^2) = 0.027\,73(9)_{\rm lat}(2)_{\rm btm}(12)_{\rm pQCD}$$
$$= 0.027\,73(15)$$

(error contains ambiguity in the choice of Q_0^2)

Comparison with phenomenology and electroweak fit

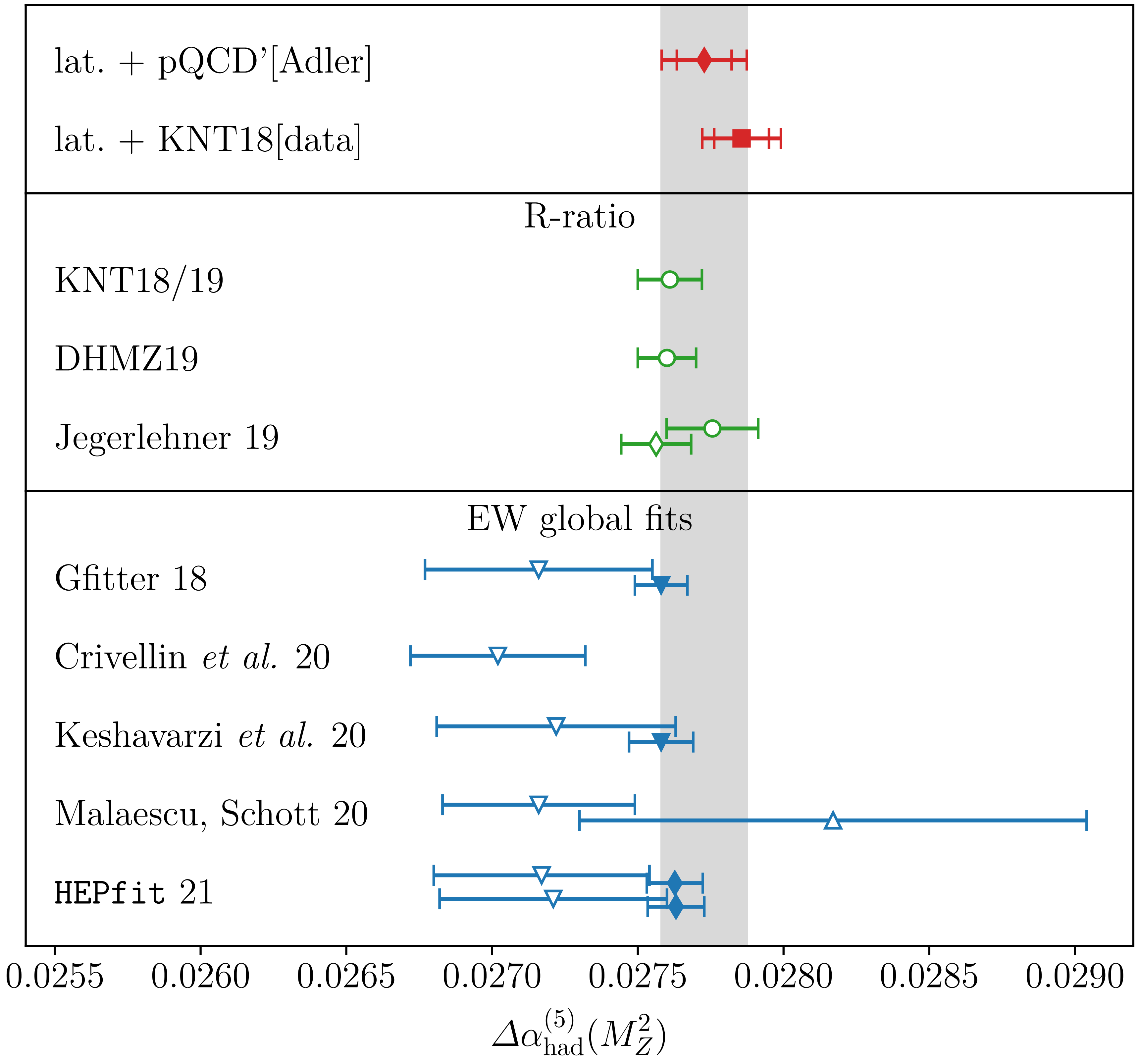
Mainz/CLS: $\Delta\alpha_{\rm had}(M_Z^2) = 0.027\,73(15)$

(pQCD/Adler + lattice input)

Jegerlehner 19: $\Delta\alpha_{\rm had}(M_Z^2) = 0.027\,53(12)$

(pQCD/Adler + R-ratio input)

- Agreement within errors at Z-pole obscures the fact that there is a tension of $\sim 3\sigma$ for $Q_0^2 \sim (3-7)\,\mathrm{GeV}^2$
- Running from $-Q_0^2$ to $-M_Z^2$ is correlated
- Global EW fits yield smaller estimates but have larger errors



[Cè et al., JHEP 08 (2022) 220, arXiv:2203.08676]

Standard Model can accommodate a larger value for a_μ without contradicting electroweak precision data

Summary — Conclusions — Outlook

* There is a 4.2σ tension between the measurement of a_μ and the SM prediction derived from the data-driven approach to the HVP contribution
* There is a tension between data-driven and lattice evaluations of a sub-contribution to $a_\mu^{\mathrm{LO,\,hvp}}$, confirmed by several lattice calculations using different discretisations
* There is no straightforward strategy to trace the tension to a specific energy range in the hadronic cross section; it is unlikely to come from the dominant two-pion channel
* There is a corresponding tension between lattice and data-driven evaluations of the hadronic running of $\alpha_{\mathrm{e.m.}}$
* The global electroweak fit is not sensitive enough to resolve the tension between lattice and data-driven evaluations of $\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2)$

What next?

* Resolve the tension between experimental data for hadronic cross sections for the two-pion channel

* Perform more and more precise lattice calculations for the complete HVP contribution

* Deviation of order $100 \cdot 10^{-11}$ between SM and experiment is a large one!

$\Rightarrow$ Precision must be increased further

Results from future experiments will be decisive!