

Constant of motion identifying excited-state quantum phases

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Excited-state quantum phase transitions

Given a Hamiltonian with a control parameter, $\mathcal{H}(\lambda)$



An **ESQPT** is a non-analyticity
in the level density and/or the level flow in the plane $E \times \lambda$

- Very complex landscape (as Pavel just said)
- Not easy to define different **phases** at both sides of E_c

Excited-state quantum phase transitions

Typical of **collective models** with a classical analogue *corresponding to the thermodynamic limit, $N \rightarrow \infty$.*

- Quantum optics: Rabi and Dicke models
- Condensed matter: fully connected Heisenberg chains
- Nuclear physics: LMG model
- Molecular physics: vibron model

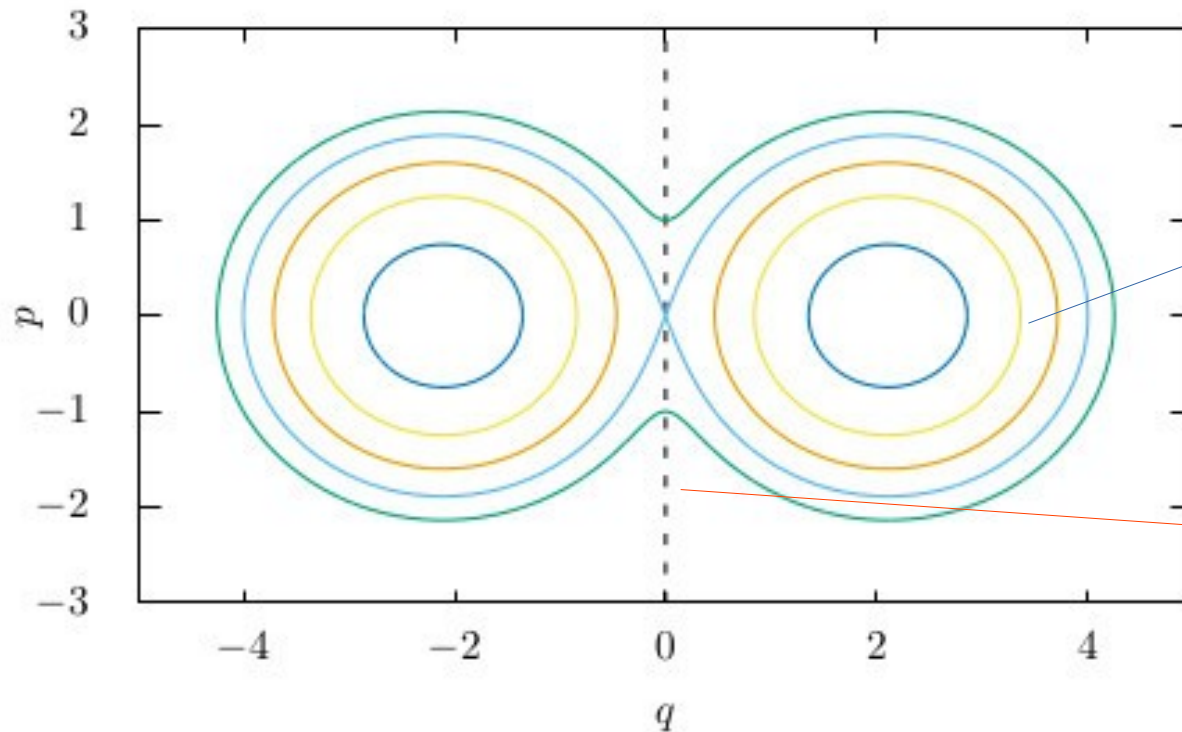
Classical definition of an ESQPT: fixed points of the Hamiltonian flow

$$\begin{aligned}\nabla \mathcal{H}|_{(\mathbf{p}_c, \mathbf{q}_c)} &= 0 \\ \mathcal{H}(\mathbf{p}_c, \mathbf{q}_c) &= E_c\end{aligned}$$

Classification in terms of their mathematical properties

Our focus

Quantum systems whose classical energy surfaces behave like this

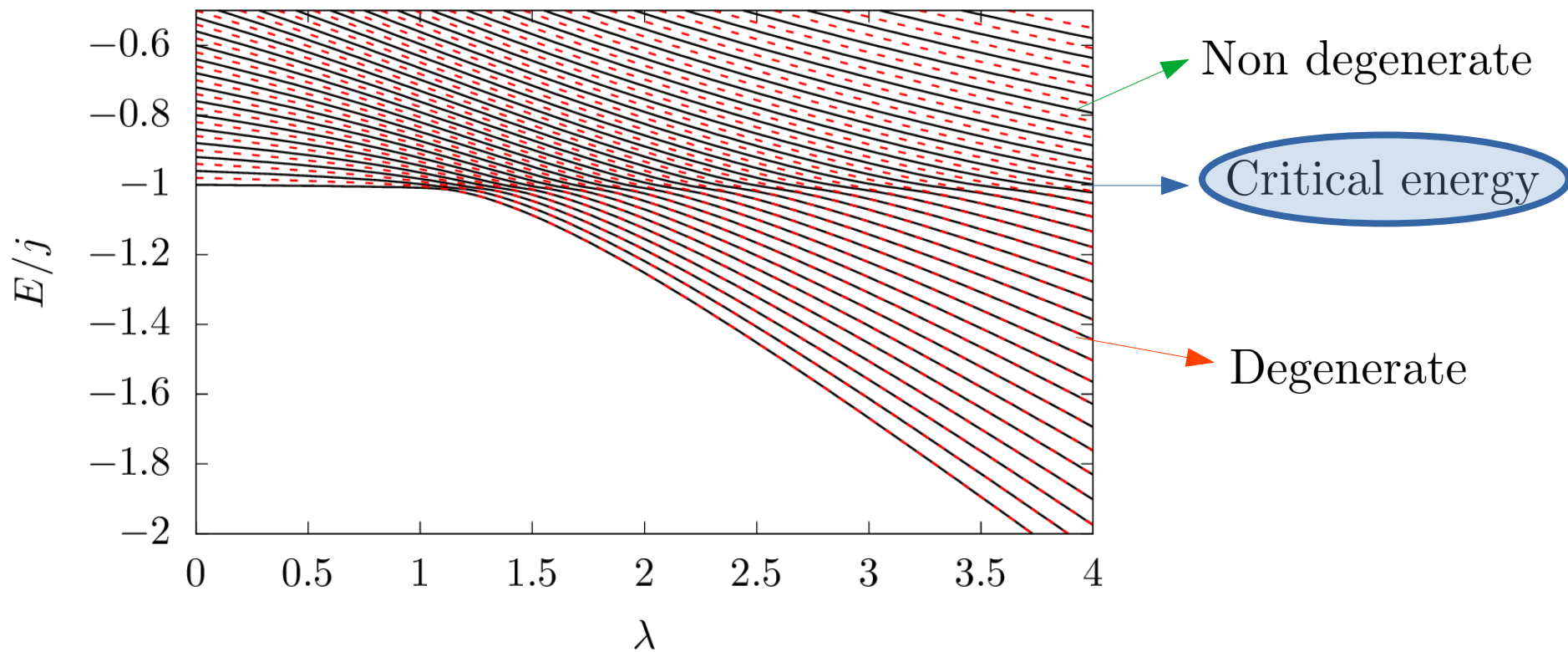


Wells maybe
not symmetric

Separatrix given by
 $f(\mathbf{q}, \mathbf{p}) = 0$

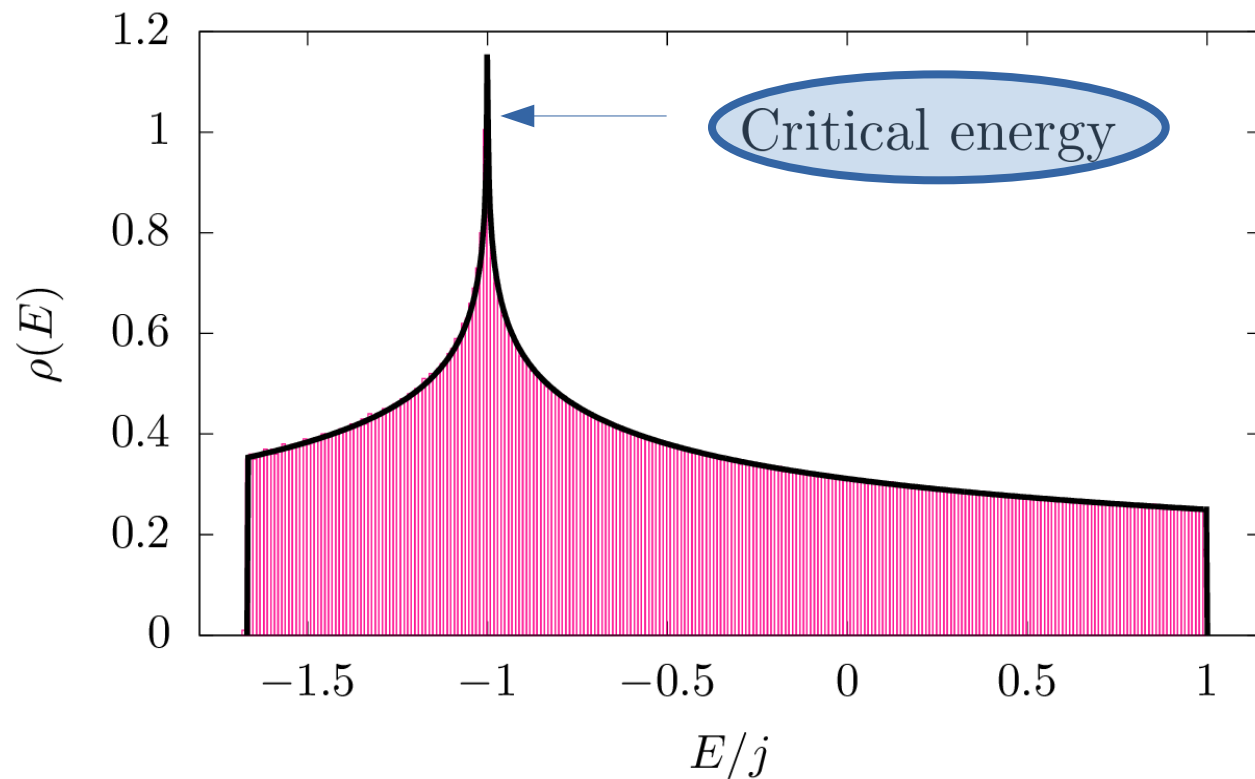
Our focus

Typical quantum consequences in the plane $E \times \lambda$

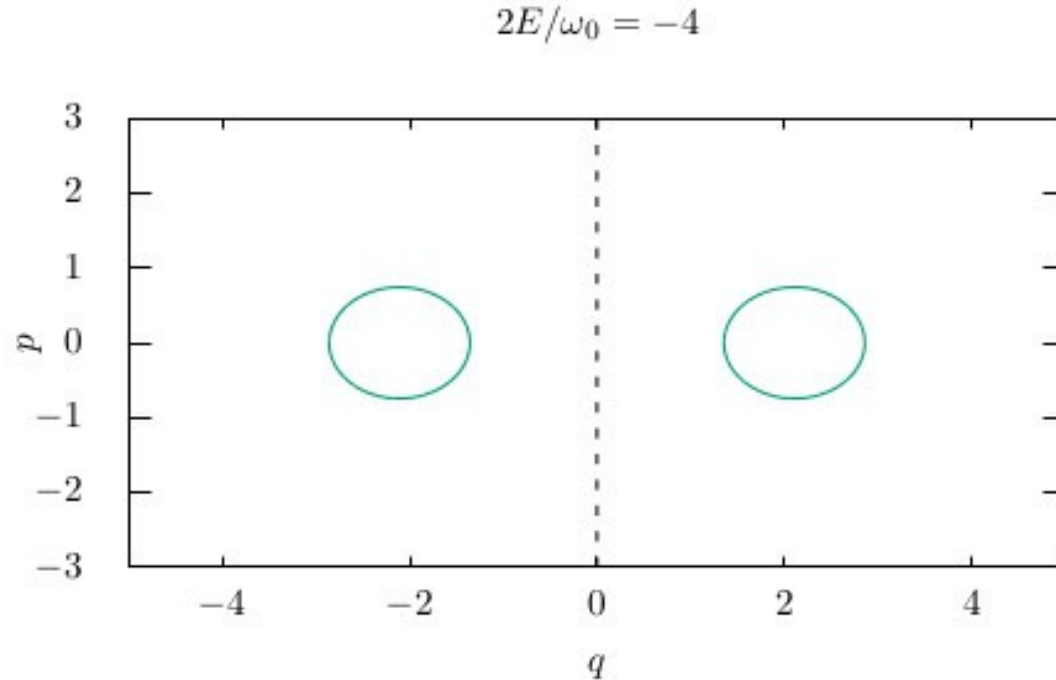


Our focus

Typical quantum consequences in the density of states ($\lambda = 3$)

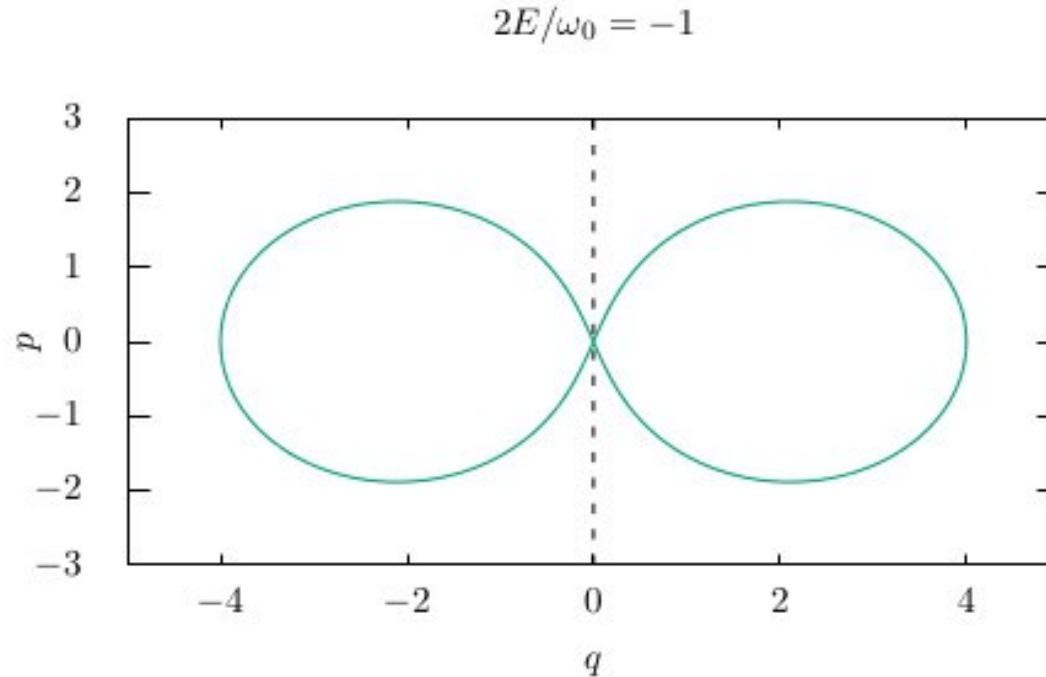


What do the classical energy surfaces tell us?



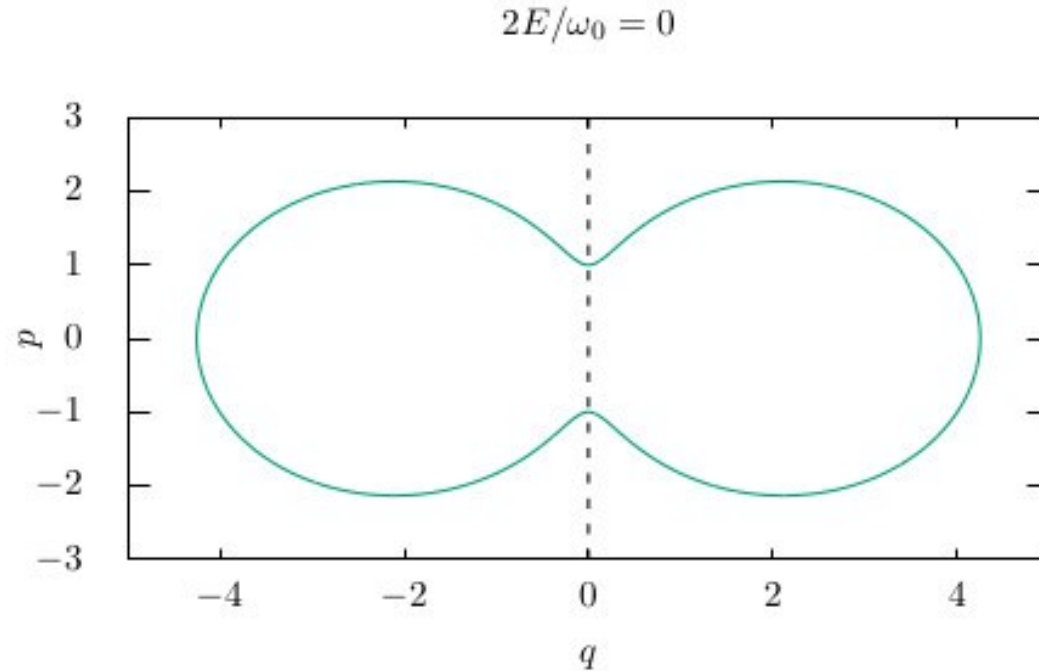
$E < E_c \longrightarrow$ Classical trajectories trapped either $q > 0$ or $q < 0$

What do the classical energy surfaces tell us?



$E = E_c \longrightarrow$ Classical trajectories *critically* trapped either $q > 0$ or $q < 0$

What do the classical energy surfaces tell us?



$E > E_c \longrightarrow$ Classical trajectories pass through both $q > 0$ and $q < 0$

And the quantum consequences of these facts?

- If the classical model shows $q(t) > 0, \forall t$, for the quantum model $\langle \hat{q}(t) \rangle > 0, \forall t$, at least in the thermodynamic limit, $N \rightarrow \infty$
- If the sign of $q(t)$ changes with time in the classical model we expect a similar behavior for $\langle \hat{q}(t) \rangle$ for the quantum model

Proposal: when $N \rightarrow \infty$, the quantum operator $\hat{C} = \text{sign}(\hat{q})$ is a constant of motion if and only if $E < E_c$

Note that $\langle \hat{C} \rangle = \langle \text{sign}(\hat{q}) \rangle \neq \text{sign} \langle \hat{q} \rangle!$

(The separatrix can be given by a more complex function $f(\mathbf{q}, \mathbf{p})$)

A first result

Dicke and Rabi models: $\mathcal{H} = \omega_0 J_z + \omega a^\dagger a + \frac{2\lambda}{\sqrt{N}} J_x (a + a^\dagger)$

- $[\mathcal{H}, J^2] = 0 \longrightarrow$ we restrict ourselves to $j = N/2$
- $[\mathcal{H}, \hat{\Pi}] = 0$, with $\hat{\Pi} = (-1)^{J_z + j + a^\dagger a} \longrightarrow \text{Spec } \hat{\Pi} = \{\pm 1\}$
- Rabi model: $N = 1$, $\frac{\omega_0}{\omega} \rightarrow \infty$ equivalent to $N \rightarrow \infty$
- Dicke model, $N > 1$, standard thermodynamic limit
- In both cases, $\hat{\mathcal{C}} = \text{sign}(a + a^\dagger)$

QPT at $\lambda_c = \frac{\sqrt{\omega\omega_0}}{2}$, ESQPT at $E_c = -j\omega_0$, if $\lambda > \lambda_c$

A first result

Numerical experiment to study if \hat{C} is a constant of motion below E_c

1.- Initial condition: $|\psi(0)\rangle = \frac{1}{\sqrt{10}} \sum_{i=M}^{M+9} |E_i\rangle$

2.- Time evolution $\langle \hat{C}(t) \rangle = \langle \psi(t) | \hat{C} | \psi(t) \rangle$

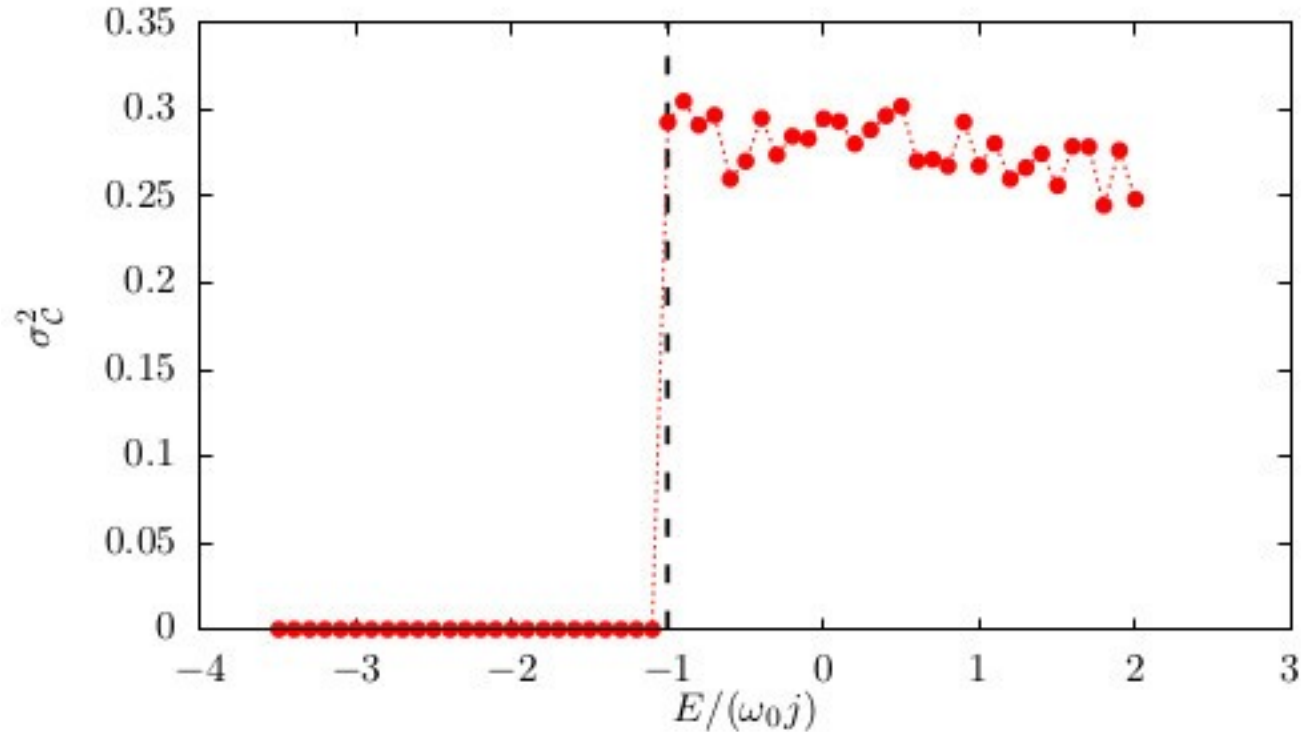
3.- Dispersion in time $\sigma_C^2 = \frac{1}{N} \sum_{i=1}^N \left(\langle \hat{C}(t_i) \rangle - \overline{\langle \hat{C} \rangle} \right)^2$

with $\overline{\langle \hat{C} \rangle} = \frac{1}{N} \sum_{i=1}^N \langle \hat{C}(t_i) \rangle$

A first result

Classical model with 1 d. o. f.

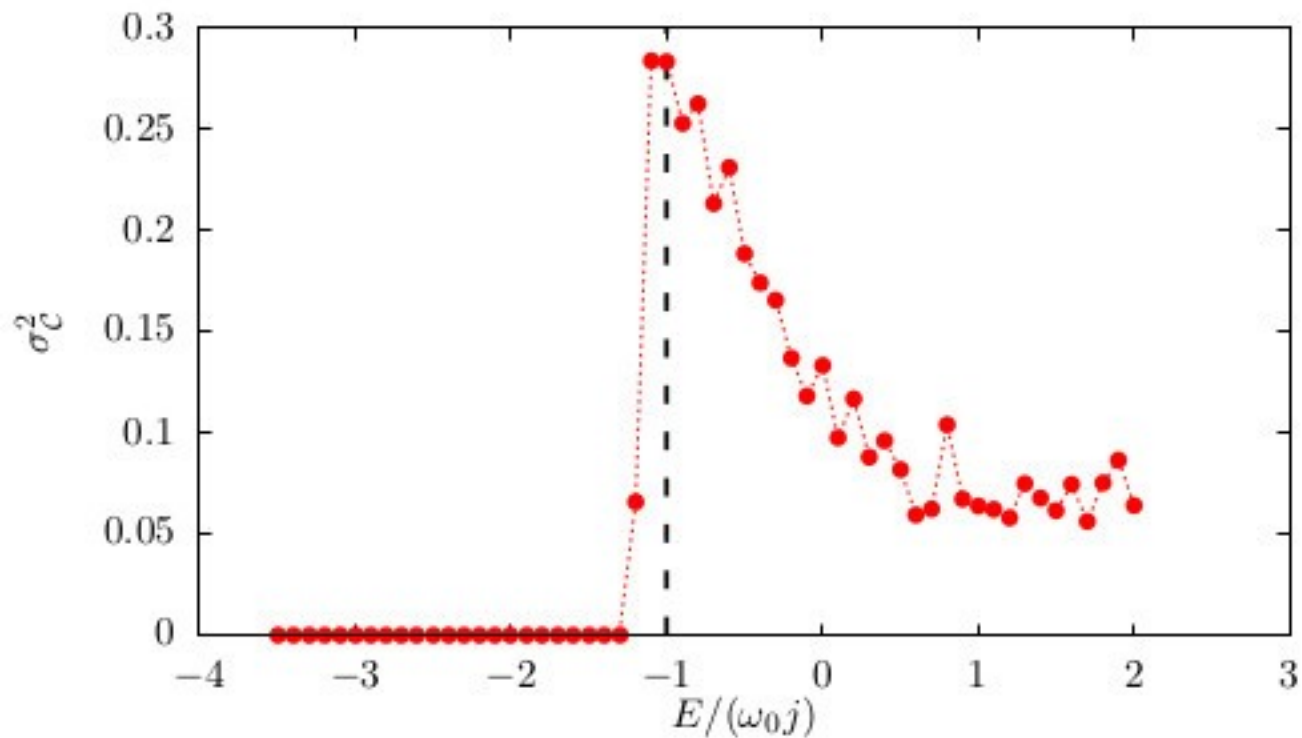
Rabi model, $\lambda = 3\lambda_c$, $\omega_o/\omega = 300$



A first result

Classical model with 2 d. o. f.

Dicke model, $\lambda = 3\lambda_c$, $N = 40$



Towards a more stringent test

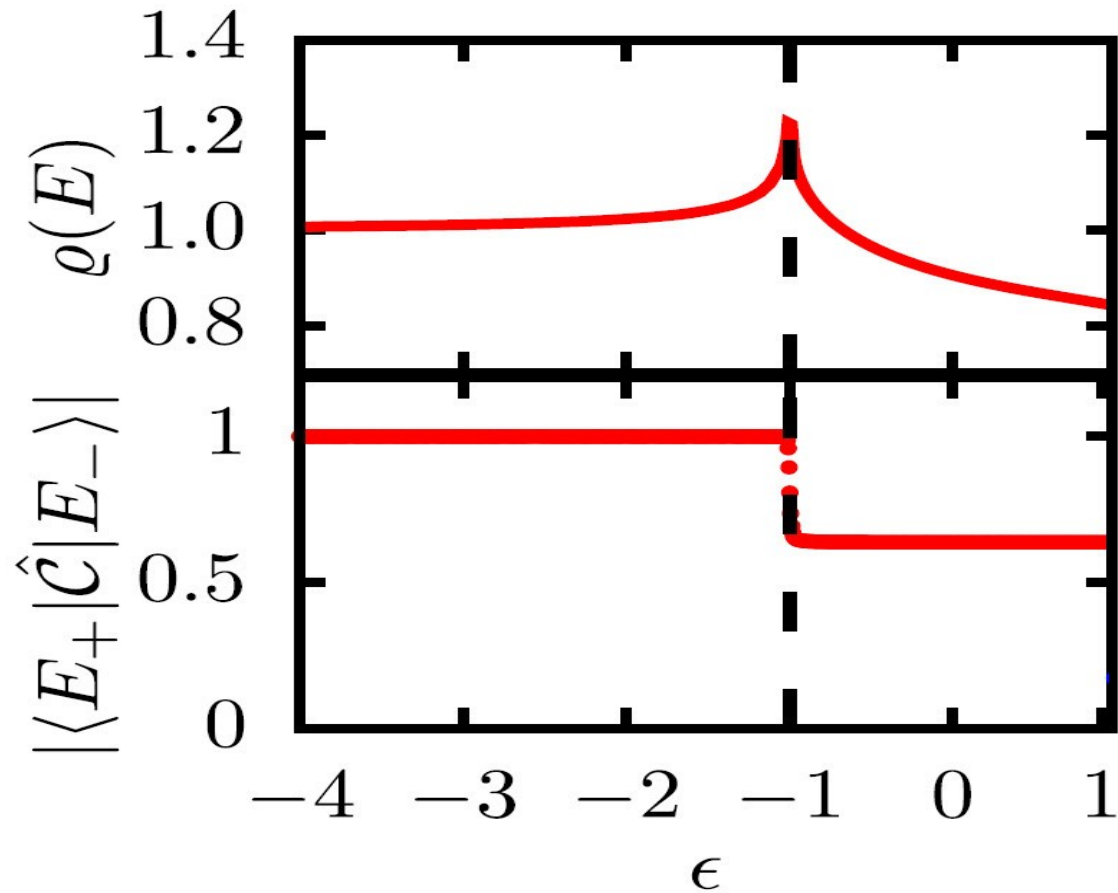
- Two important properties: $\begin{cases} \hat{C} \text{ changes the parity, so } [\hat{\Pi}, \hat{C}] \neq 0 \\ \text{Spec } \hat{C} = \{\pm 1\} \end{cases}$
- Our proposal implies $[\hat{P}_n, \hat{C}] = 0$, if $E_n < E_c$

(\hat{P}_n is the projector to the eigenspace of energy E_n)

These two facts have important consequences

- Below E_c , $E_{n,+} = E_{n,-}$, when $N \rightarrow \infty$
- Below E_c , $\hat{C} |E_{n,\pm}\rangle = |E_{n,\mp}\rangle$, when $N \rightarrow \infty$
- None of them hold above E_c

Results for the Rabi model



A quantitative test

Previous results are not enough to conclude that \hat{C} becomes a constant of motion *exactly* at E_c

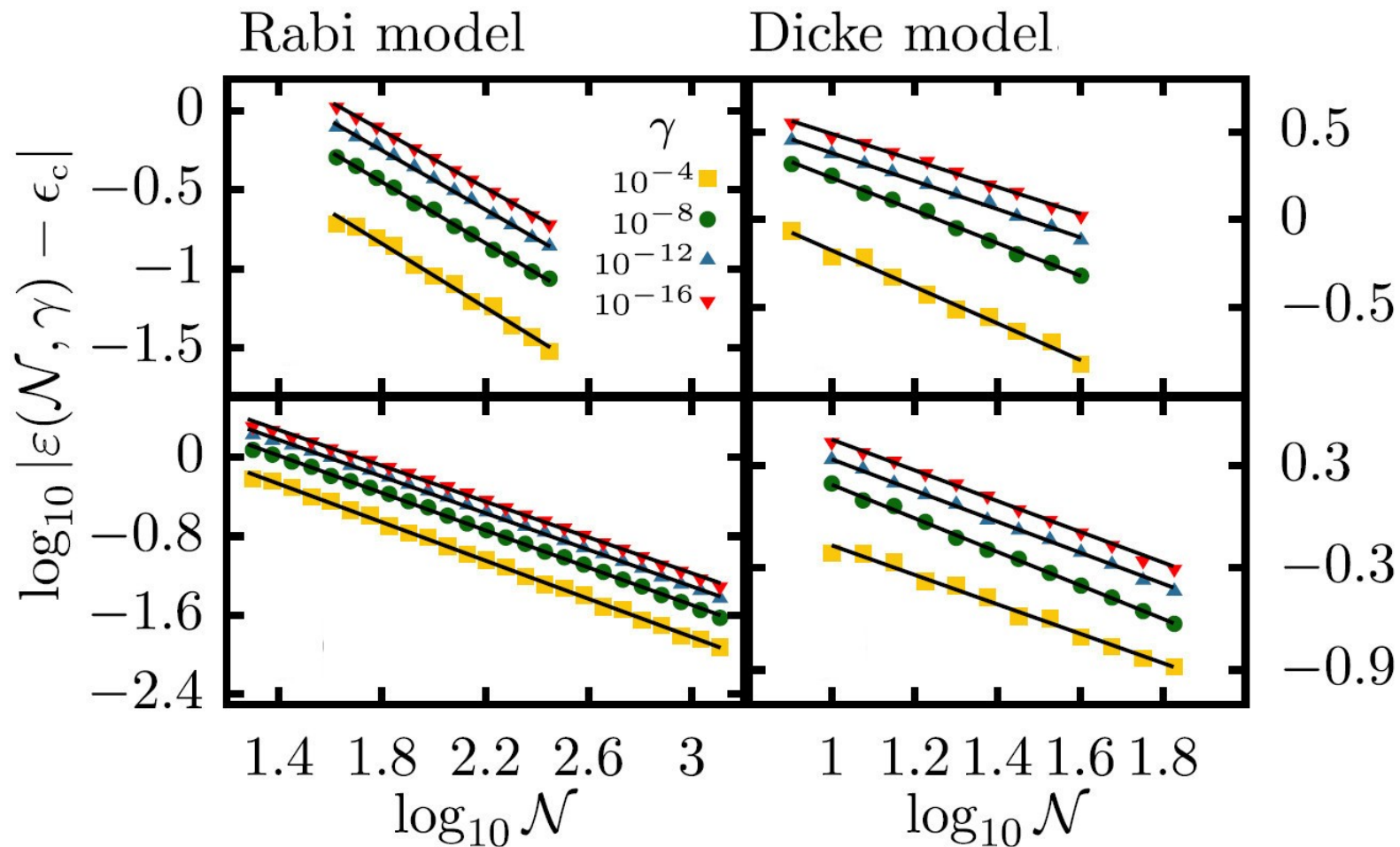
- We choose a small bound, $\gamma \sim 10^{-\alpha}$, $\alpha \in [4, 16]$
- We seek the smallest energy fulfilling

$$(i) \left| \frac{E_{n,+} - E_{n,-}}{\langle s \rangle} \right| > \gamma, \quad (ii) \quad 1 - \left| \langle E_{n,+} | \hat{C} | E_{n,-} \rangle \right| > \gamma$$

- We study how this energy, $E(\gamma, N)$, changes with system size, N

If $E(\gamma, N) \rightarrow E_c$ when $N \rightarrow \infty$, our proposal works

Results for the Rabi and Dicke models



Dynamics below the critical energy

From previous results, below E_c we have eigenspaces of $\dim \mathcal{H}_n = 2$, and spanned by $\{|E_{n,+}\rangle, |E_{n,-}\rangle\}$

In this basis:

$$\hat{\Pi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Something missing?

Dynamics below the critical energy

There exists an extra constant of motion defined by

$$\hat{\mathcal{K}} = \frac{i}{2} [\hat{\mathcal{C}}, \hat{\Pi}]$$

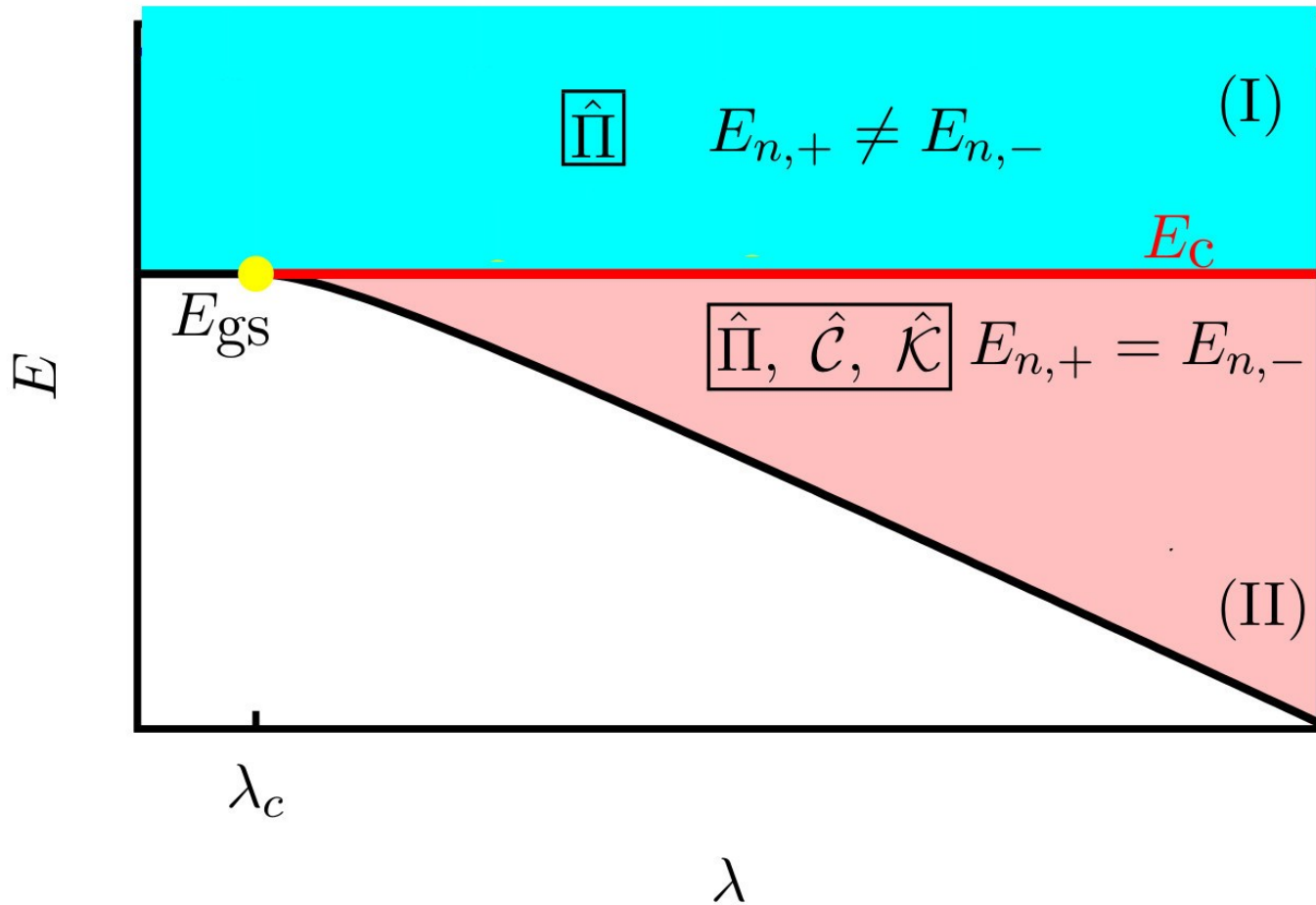
In the eigenbasis $\{|E_{n,+}\rangle, |E_{n,-}\rangle\}$:

$$\hat{\Pi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\mathcal{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\mathcal{K}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Excited-state phase diagram



Dynamical consequences of the phase diagram

- Below E_c , quantum evolution conserves E , Π , \mathcal{C} , and \mathcal{K}

A proper equilibrium ensemble, $\rho_{\text{eq}}(E, \Pi, \mathcal{C}, \mathcal{K})$,
must depend on the initial value of these four observables

- Above E_c , quantum evolution only conserves E and Π

A proper equilibrium ensemble, $\rho_{\text{eq}}(E, \Pi)$,
must depend on the initial value of just these two observables

Dynamical consequences of the phase diagram

- Below E_c , it is possible to find symmetry-breaking steady states

$$\text{Tr} [\rho_{\text{eq}}(E, \Pi, \mathcal{C}, \mathcal{K}) \cdot J_x] \neq 0$$

- Above E_c , it is *not* possible to find symmetry-breaking steady states

$$\text{Tr} [\rho_{\text{eq}}(E, \Pi) \cdot J_x] = 0$$

Summary and outlook

- A very usual kind of ESQPTs gives rise to two dynamical regimes
 - Below E_c , there appear two discrete constants of motion, acting like partial \mathbb{Z}_2 symmetries
 - Above E_c , this is no longer true

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- Both equilibrium and non-equilibrium consequences are expected

See Ángel's talk!

Thanks for your attention!!