Friday 15.7.: **EXCITED-STATE QUANTUM PHASE TRANSITIONS = ESQPTs**

09:00-11:00 Session

CEJNAR, Pavel (Prague) Introduction to the physics of ESQPTs [25+5]

DUKELSKY, Jorge (Madrid) Analogs of QPTs and ESQPTs in a dissipative spin model [25+5]

RELAÑO, Armando (Madrid) Constant of motion identifying excited-state quantum phases and some applications to quantum optical models [25+5]

L. CORPS, Ángel (Madrid) Theory of dynamical phase transitions driven by excited-state quantum phase transitions [25+5]

11:00-11:30 Coffee break

11:30-13:30 Session

STRÁNSKÝ, Pavel (Prague) Stabilization of quantum states at ESQPTs [25+5]

CORTINAS, Rodrigo (Yale) Pairwise kissing of excited states in a squeezed Kerr-oscillator [25+5]

SANTOS, Lea (Yeshiva Univ.), PEREZ-BERNAL, Francisco (Huelva) Detection of excited state quantum phase transition with a Kerr-nonlinear resonator [40+10]

GENERAL DISCUSSION [10]

Introduction to the physics of ESQPTs

Pavel Cejnar
Institute of Particle and Nuclear Physics
Faculty of Mathematics and Physics
Charles University, Prague, Czechia

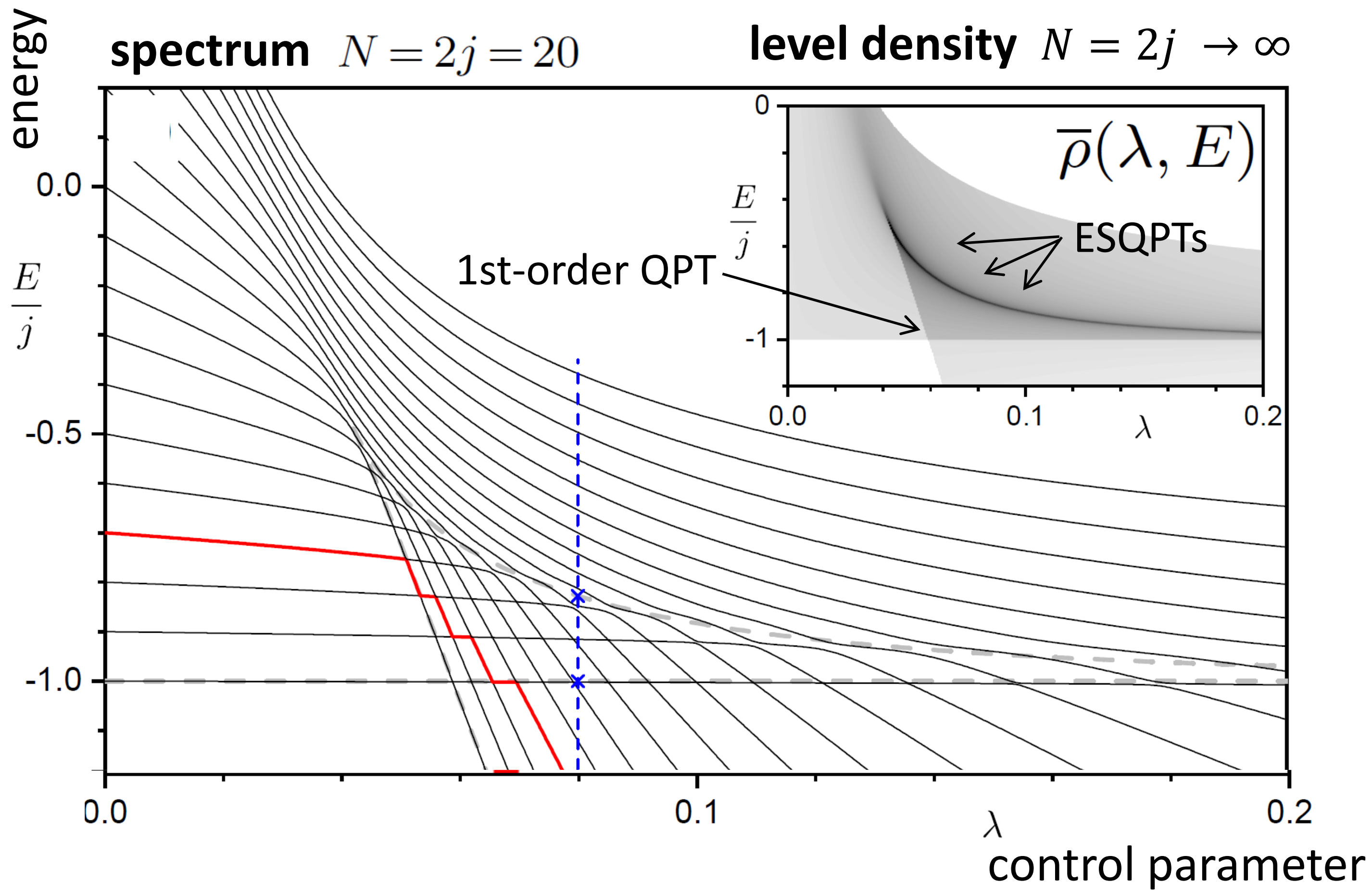
Recent review:
P. Cejnar, P. Stránský, M. Macek, M. Kloc, Journal of Physics A: Math. Theo. 54 (2021) 133001

Examples of ESQPTs

ESQPTs are singularities in quantum energy spectra (level density) and in expectation values of various observables, which become nonanalytic in the infinite-size limit of the system

Here we see an example from the **Lipkin model** with Hamiltonian $\hat{H} = \hat{J}_z - \frac{\lambda}{N}\left[\hat{J}_x + 4\left(\hat{J}_z + \frac{N}{2}\right)\right]^2$

The model describes a fully connected system of N interacting qubits using collective quasispin operators $\hat{J}_k = \frac{1}{2}\sum_{i=1}^{N}\hat{\sigma}_k^{(i)}$



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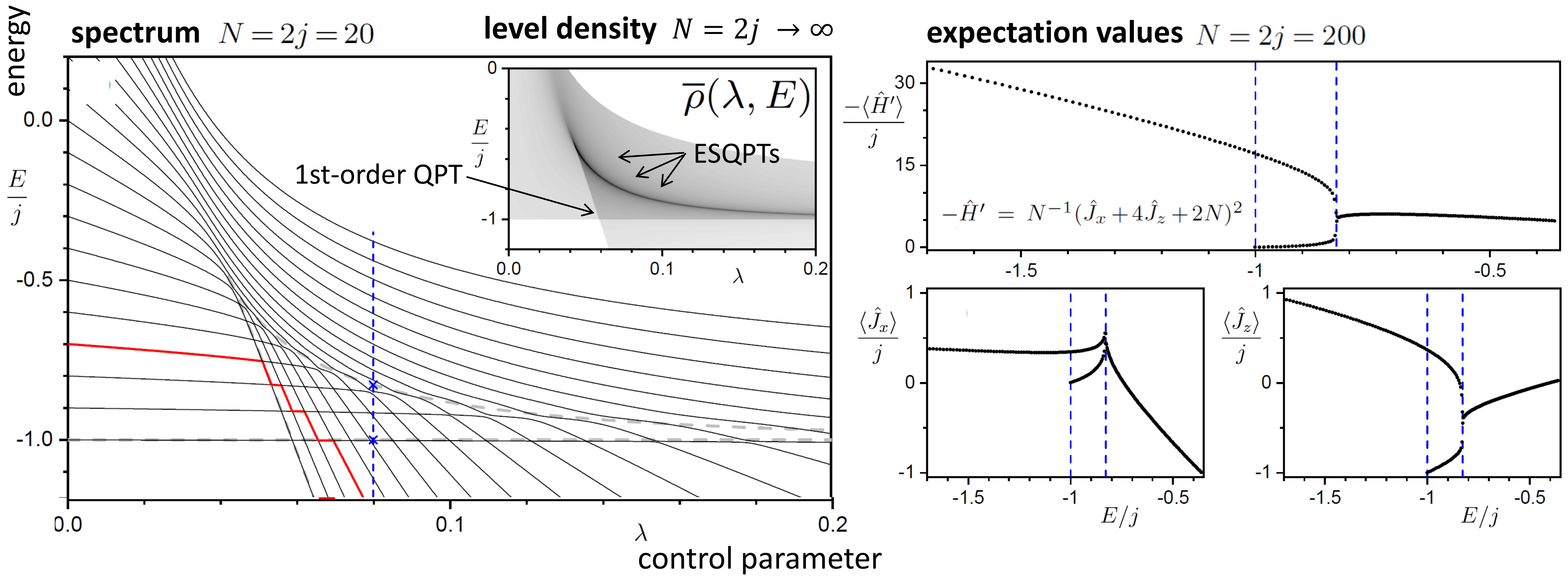

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Interacting boson systems

- **Lipkin model** (qubit systems) $s+t$
- **Vibron models** (molecules) $s+\{\tau_-,\tau_+\}$, $s+\{p_{-1},p_0,p_{+1}\}$
- **Two-site Bose-Hubbard model** (Josephson junctions) $\{b_-,b_+\}$
- **Three-mode boson model** (spinor condensates) $\{b_{-1},b_0,b_{+1}\}$
- **Interacting boson model** (nuclei) $s+\{d_{-2},d_{-1},d_0,d_{+1},d_{+2}\}$

Coupled atom-field (or other) systems

- **Dicke model** and its extensions (toy for cavity QED)
- **Tavis-Cummings model** (integrable version)
- **Rabi model** (qubit coupled to field)
- **Bosonic atom-molecule systems**

Fermi and Bose-Fermi systems

- **Fermion Lipkin model**
- **Fermion pairing models**
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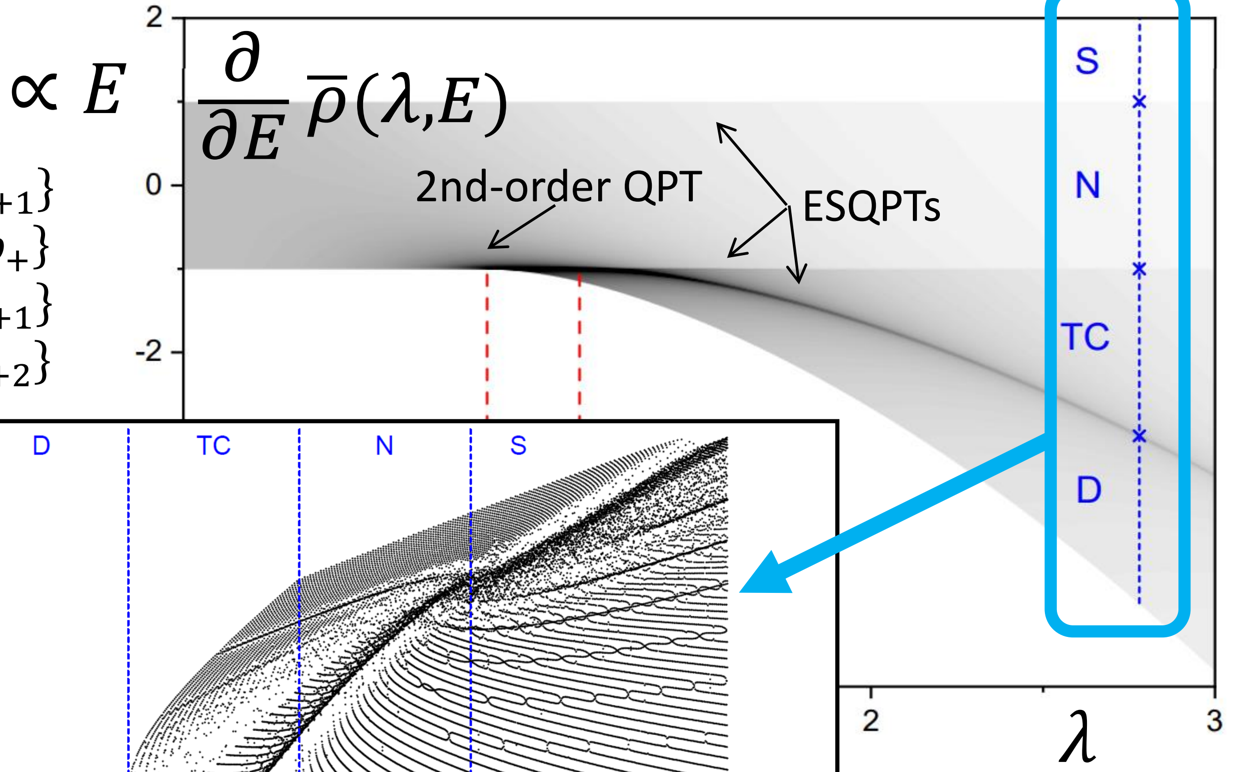

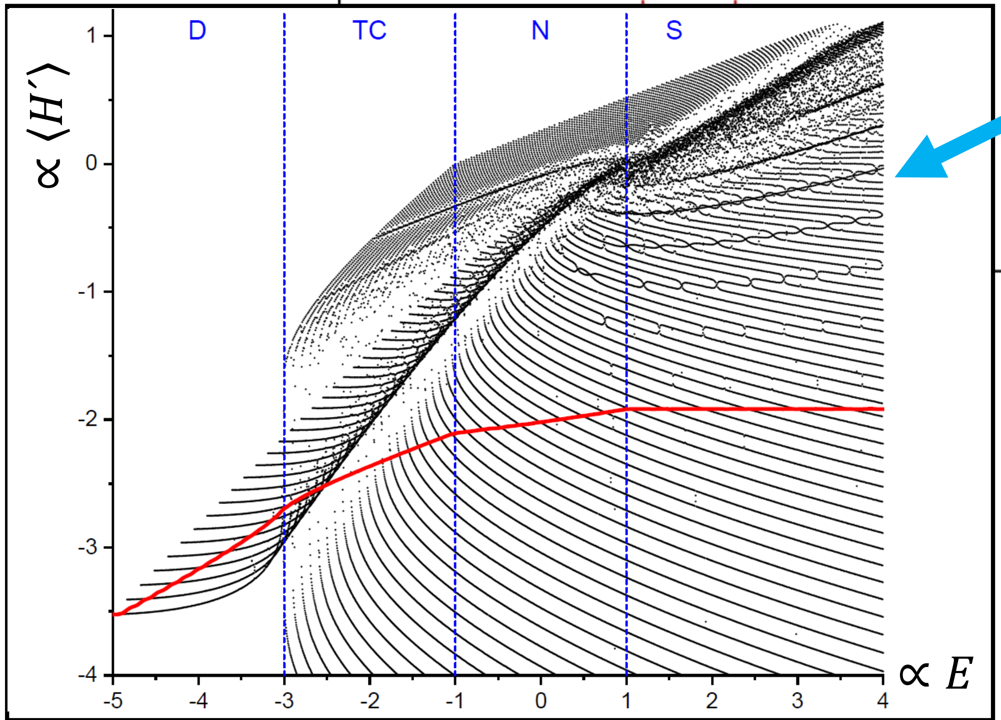

An extended Dicke model

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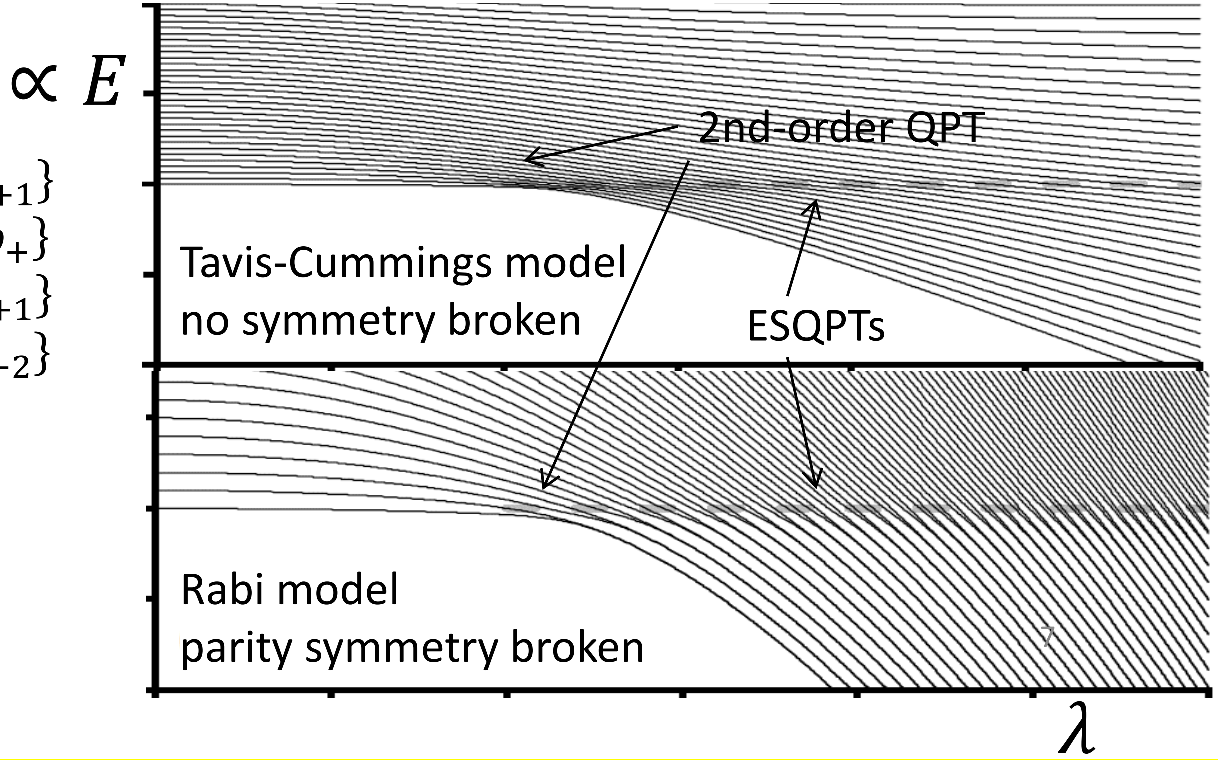


All these models have finite (moderate) numbers of degrees of freedom f (they represent fully connected many-body systems with collective dynamics). However, similar effects have been observed also in many-body systems with local interactions (hence with $f \propto N \to \infty$), e.g., in Heisenberg model in a random field... **?**

Semiclassical origin of ESQPTs

ESQPTs in the commonly observed form are rooted in the classical limit of the system because for finite-f systems the scaled infinite-size limit $N \to \infty$ is equivalent to the classical limit $\hbar \to 0$

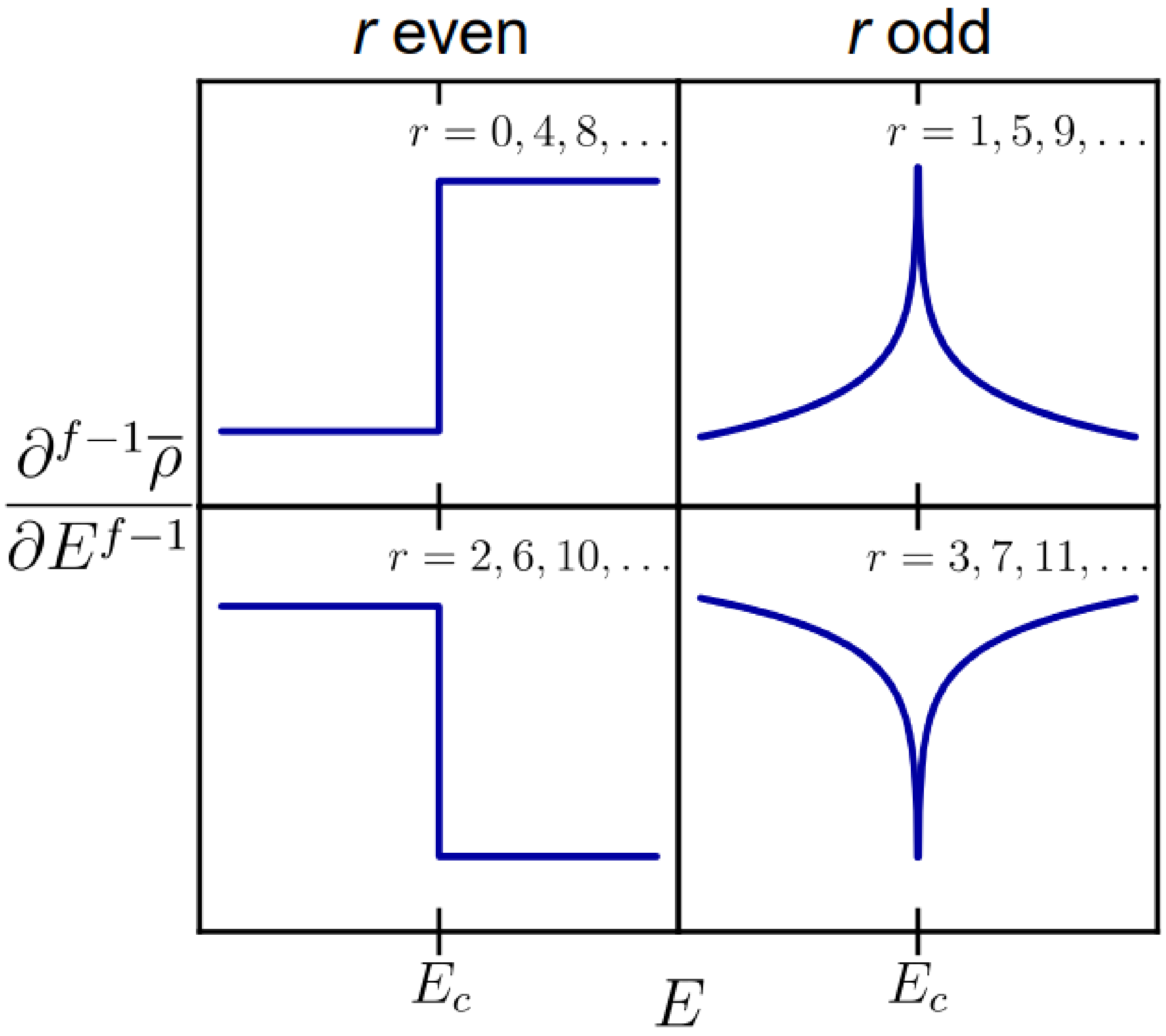
Phase space $(q_1, \dots, q_f, p_1, \dots, p_f) \equiv (\boldsymbol{q}, \boldsymbol{p})$

Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p})$

1) Generic ESQPTs are caused by classical stationary points

a) The most common stationary point is **non-degenerate** (locally quadratic), classified by **index** $r = 0,1,2, \dots, f$ (number of Hessian eigenvalues < 0)

b) The **degenerate** (locally flat) stationary points lead to stronger singularities of various types (no classification)



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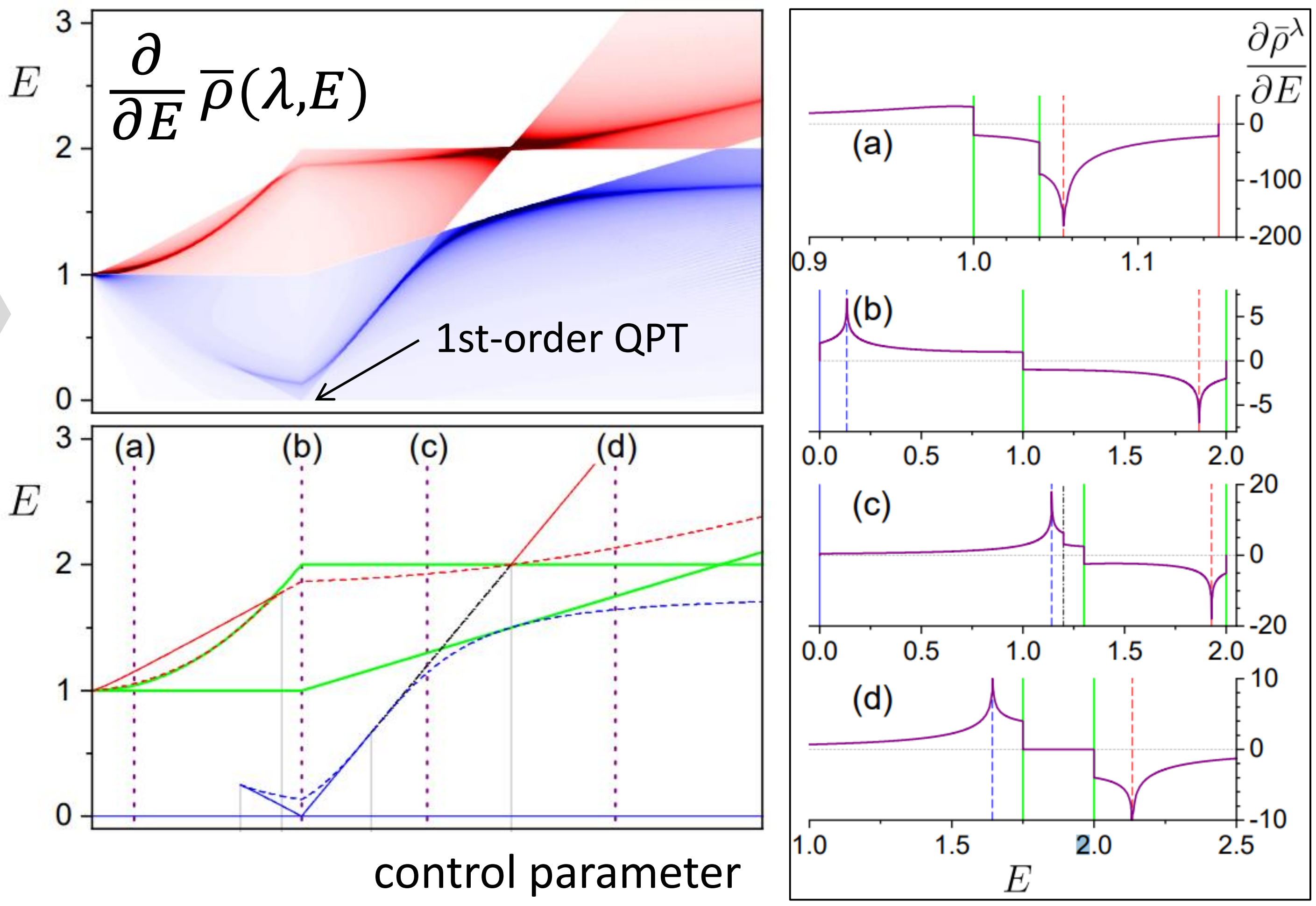
$s + d$ interacting boson model, $J = 0$ states, $f = 2$



M. Macek, P. Stránský, A. Leviatan, P. Cejnar, PRC 99 (2019) 064323

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2) Some ESQPTs are caused by saddles/extremes of $H(\boldsymbol{q}, \boldsymbol{p})$ on the phase-space boundary

For systems with bounded phase space $\Leftrightarrow$ finite Hilbert space

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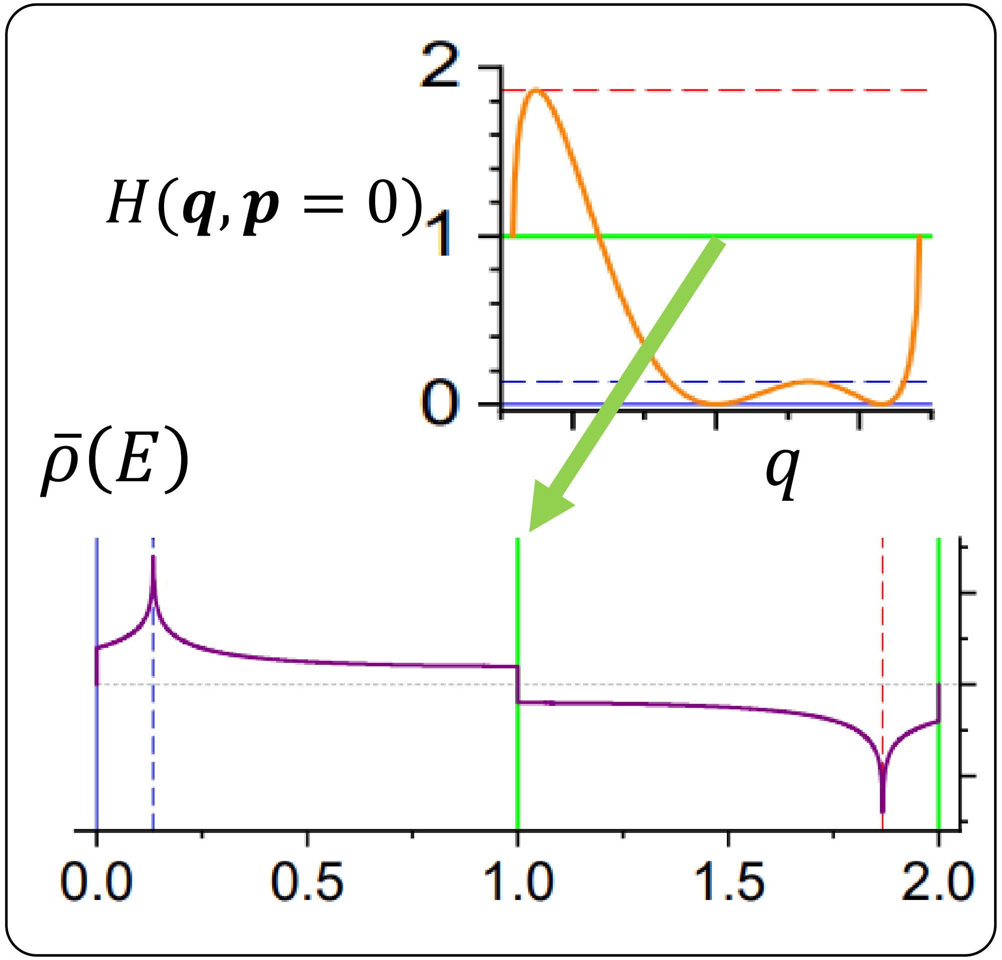

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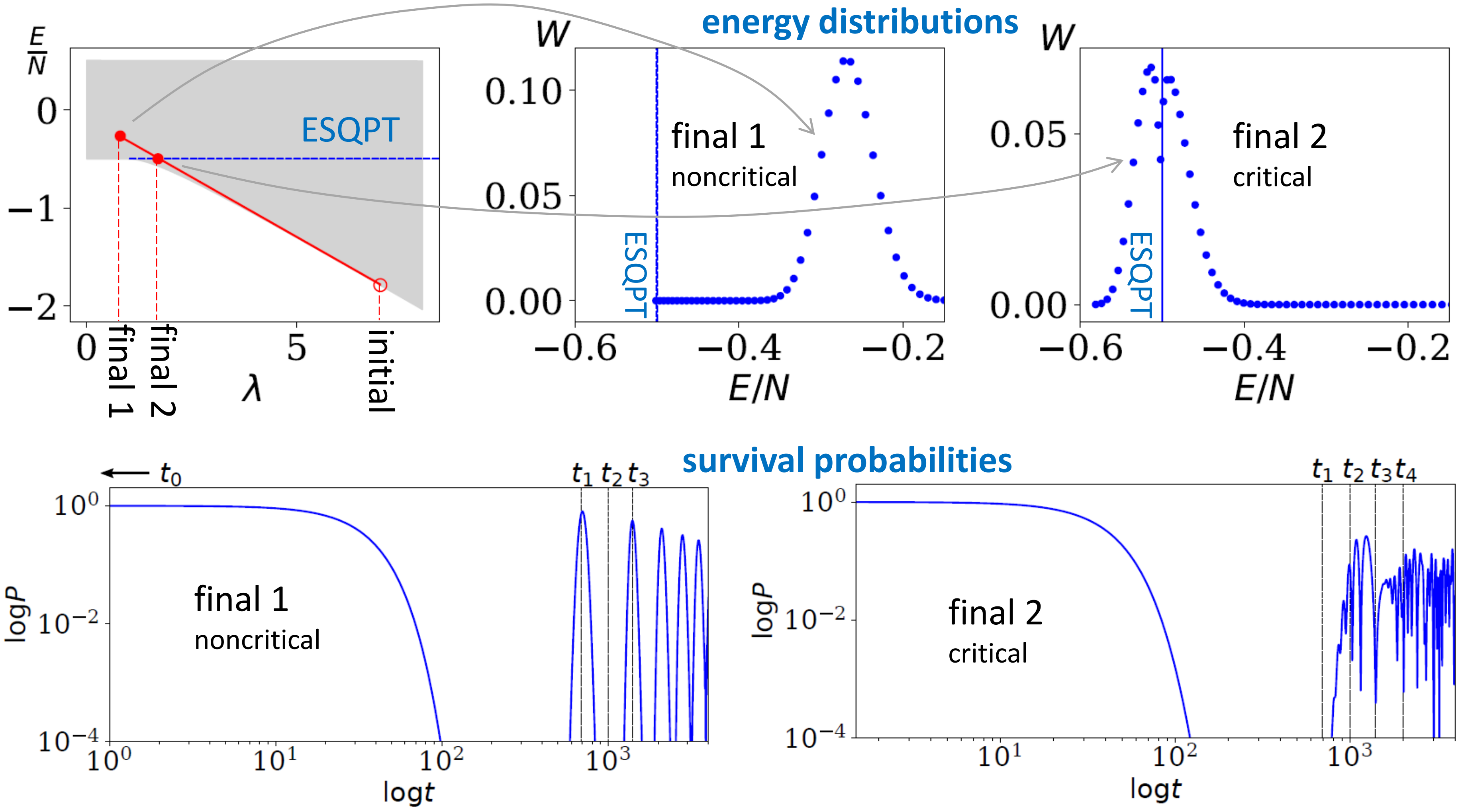
Dynamical signatures of ESQPTs

ESQPTs have various dynamical consequences, e.g., in the quantum quench dynamics …

For quenches leading to the ESQPT spectral regions the survival probability of the initial state can be both enhanced (at small times) or suppressed (at medium times)

Example: Lipkin model with 2nd-order QPT and connected ESQPT



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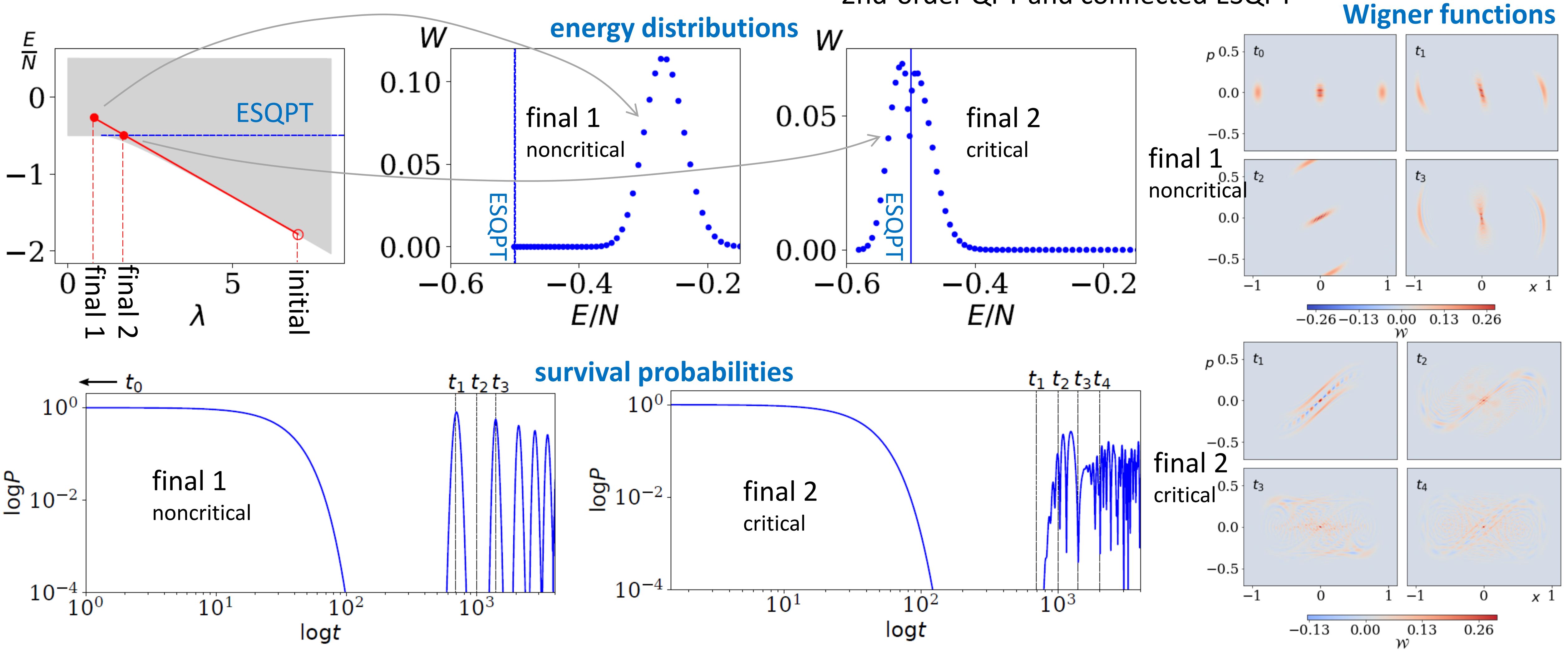


Analogs of ESQPTs

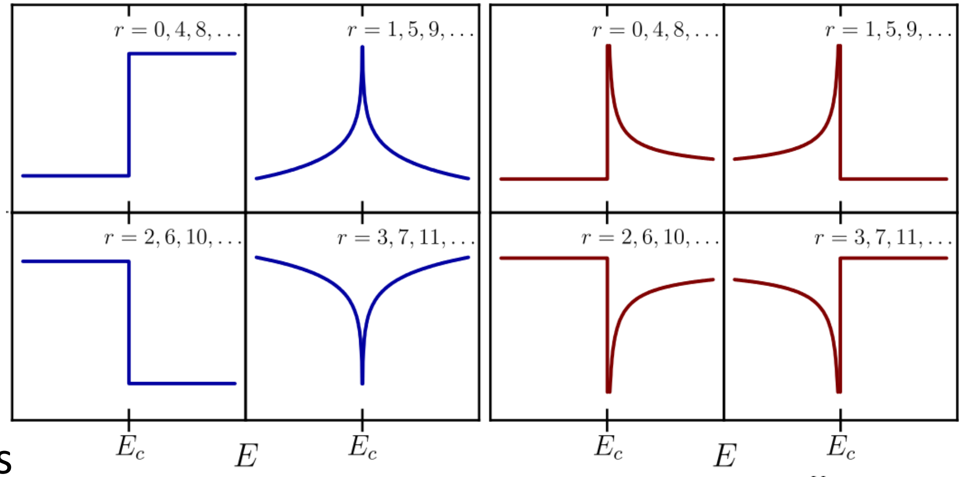
Periodic lattice systems

Vibrations of *D*-dimensional periodic lattices with *n* phonons

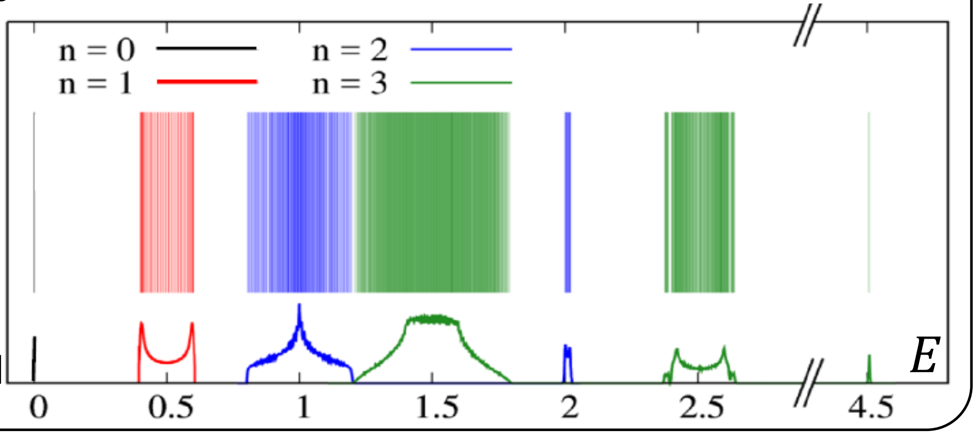
phase space *d=2f*	$\boldsymbol{q},\boldsymbol{p} \longrightarrow \boldsymbol{\wp}$	*d=nD* quasimomentum
Hamiltonian	$H(\boldsymbol{q},\boldsymbol{p}) \longrightarrow E(\boldsymbol{\wp})$	dispersion relation

$d = 2k$ $\quad$ $d = 2k-1$

$$\frac{\partial^{k-1}\bar{\rho}}{\partial E^{k-1}}$$



Extended typology of singularities in the density of states



N-site Bose-Hubbard model

Analogs of ESQPTs

V.M. Bastidas, G. Engelhardt, P. Pérez-Fernández, M. Vogl, T. Brandes, PRA 90 (2014) 063628

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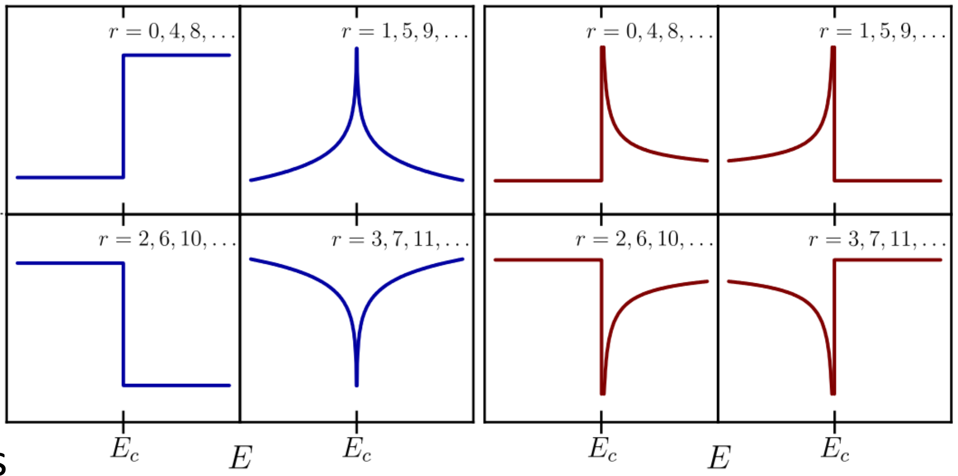

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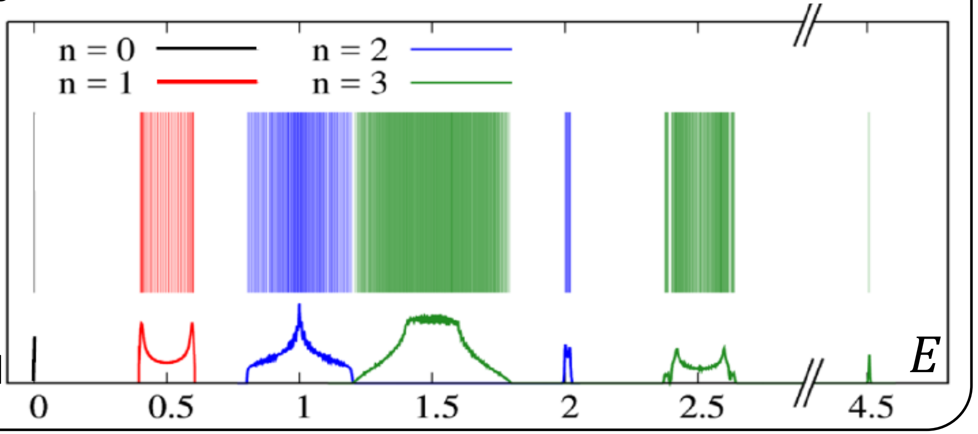
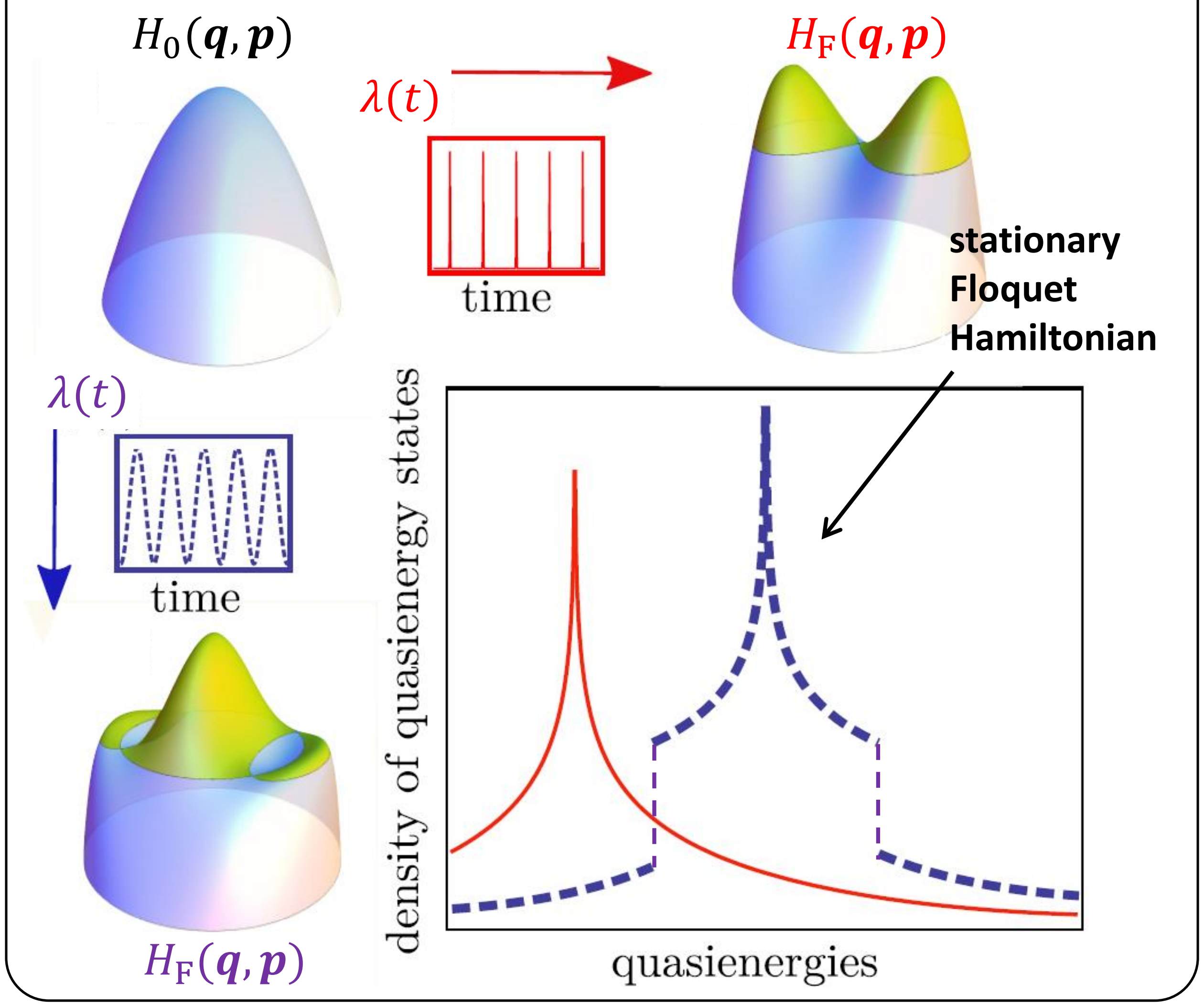


Periodically driven systems

Time-dependent Hamiltonian $\hat{H}(\lambda(t)) = \hat{H}_0 + \lambda(t)\hat{H}'$



ESQPTs in scattering (tunneling) systems

The level density formalism can be applied also in systems with continuous energy spectra …

Basic ingredients:

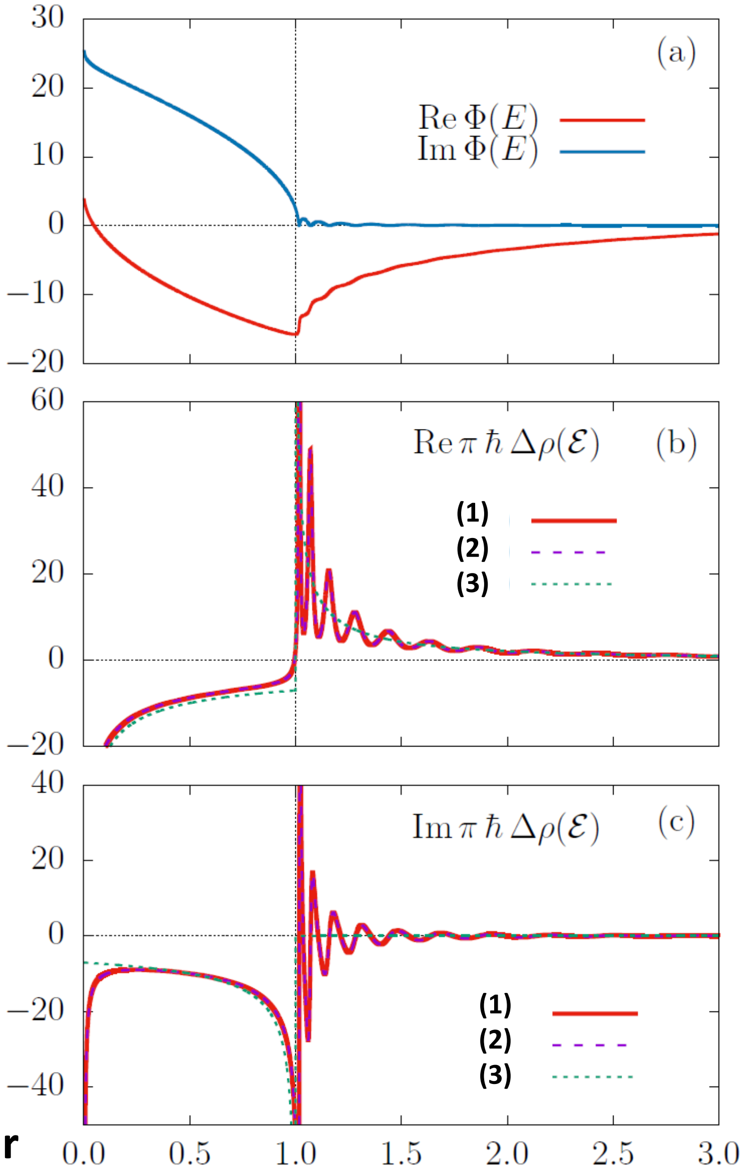
- Gamow-Siegert complex energy formalism $\mathcal{E} = E - \frac{i}{2}\Gamma$
- Green operators of the full and free Hamiltonians
- **Complex continuum level density** for $\mathcal{E} = E$

$$\mathrm{Re}\Delta\rho(E) + i\,\mathrm{Im}\Delta\rho(E) = \Delta\rho(E) = \frac{i}{\pi}\lim_{\Gamma\to 0-}\mathrm{Tr}\left[\hat{G}(\mathcal{E}) - \hat{G}_0(\mathcal{E})\right] \quad \textbf{(1)}$$

with $\hat{G}(\mathcal{E}) = \frac{1}{\mathcal{E} - \hat{H}}$, $\hat{G}_0(\mathcal{E}) = \frac{1}{\mathcal{E} - \hat{H}_0}$

… it is connected with the **complex transmission amplitude**

$$\beta(E) = e^{i\Phi(E)} = e^{-\mathrm{Im}\Phi(E)}e^{i\mathrm{Re}\Phi(E)} \qquad \Delta\rho(E) = \frac{1}{\pi}\frac{d}{dE}\Phi(E) \quad \textbf{(2)}$$



Example: Tunneling through a rectangular potential barrier

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… and with the **complex-extended Eisenbud-Wigner tunneling time (time shift)**

$$\Delta\rho(E) = \frac{\Delta t(E)}{\pi\hbar} \quad \textbf{(3)}$$

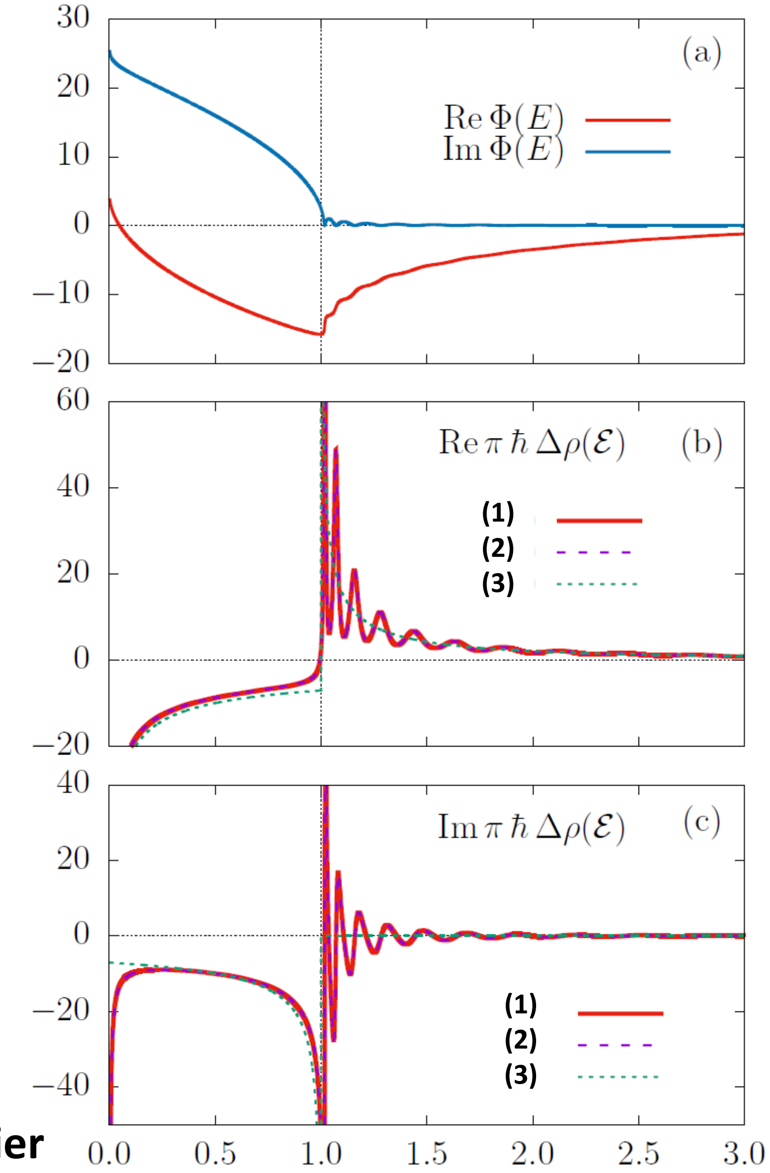
… whose **smoothed form** can be **calculated classically**

$$\mathrm{Re}\,\overline{\Delta t}(E) = t_+(E) - t_0(E) \qquad \mathrm{Im}\,\overline{\Delta t}(E) = t_-(E) - t_0(E)$$

$t_+(E) \equiv$ passage time through **allowed regions** of the potential $V(x)$ at energy E

$t_-(E) \equiv$ passage time through **forbidden regions** of the potential, i.e., passage time of particle with energy $-E$ through allowed regions of $-V(x)$

$t_0(E) \equiv$ passage time of a free particle



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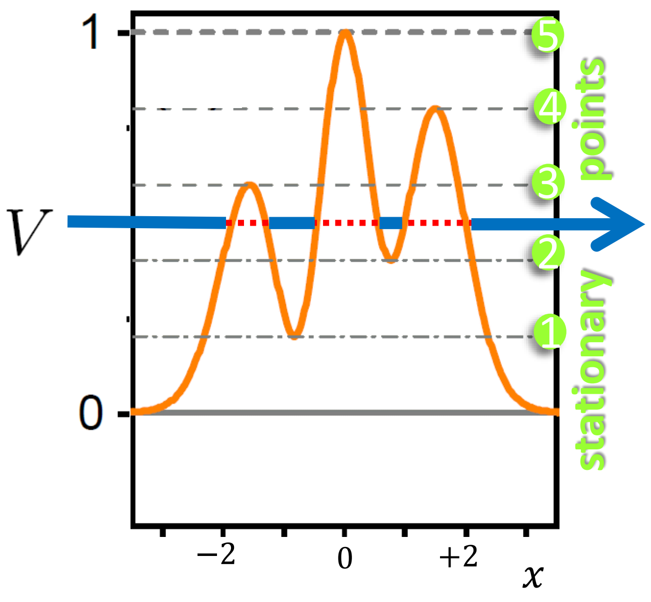
ESQPTs in scattering (tunneling) systems

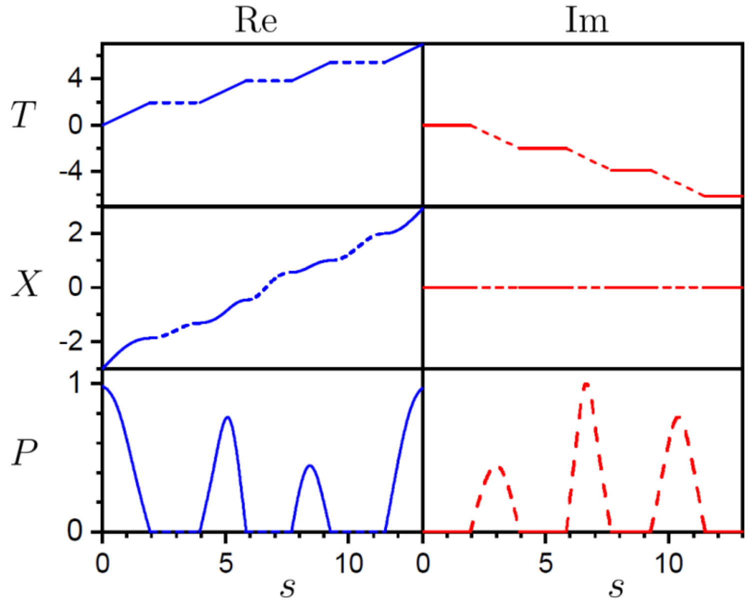
Stationary points of the tunneling potential imply singularities in the complex continuum level density.

- The singularities become nonanalytic in the semiclassical limit $\hbar/\sqrt{M} \to 0$
- The singularities in the real and imaginary parts of $\overline{\Delta\rho}(E)$ are different: e.g., a local minimum of $V(x)$ behaves as a minimum for $\mathrm{Re}\,\overline{\Delta\rho}(E)$, but acts as a local maximum for $\mathrm{Im}\,\overline{\Delta\rho}(E)$. This represents a dual extension of ESQPTs known from bound quantum systems with $f = 1$
- Generalizations to systems with $f > 1$ is possible, but not yet elaborated in general (it is obvious in special cases of quasi-1D systems, like separable or spherically symmetric) …

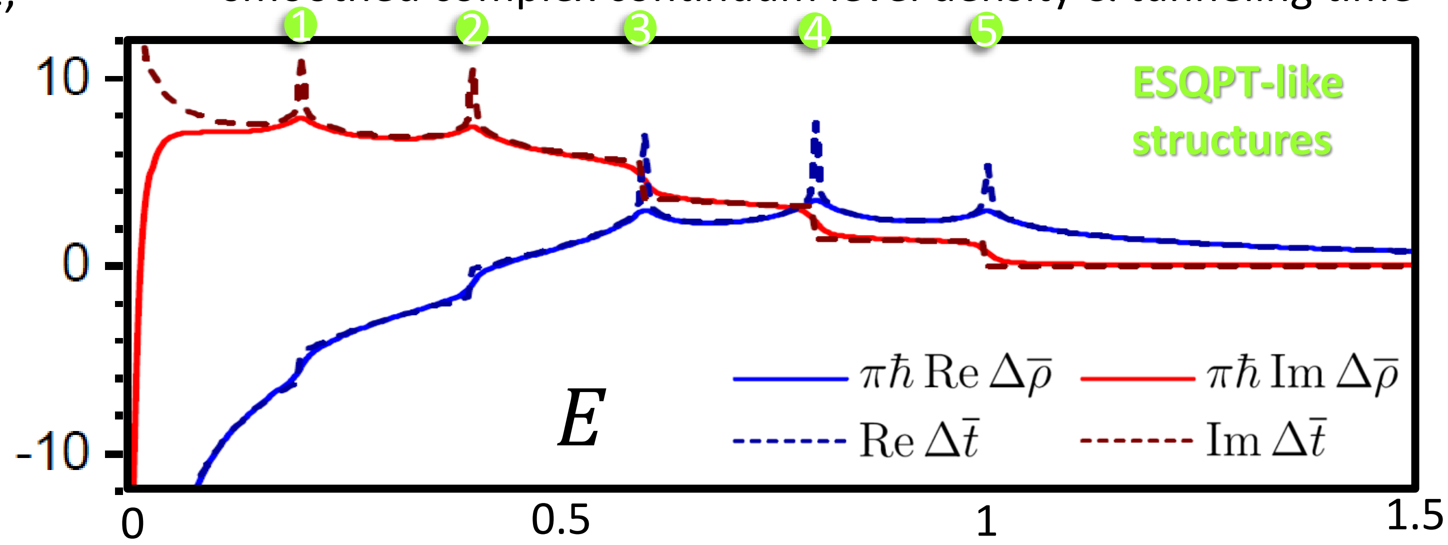
tunneling potential



complex-time tunneling trajectory



smoothed complex continuum level density & tunneling time



P. Stránský, M. Šindelka, M. Kloc, P. Cejnar,
PRL 125 (2020) 020401

P. Stránský, M. Šindelka, P. Cejnar,
PRA 103 (2021) 062207

Thank you for attention

Thanks to collaborators

Pavel Stránský (Prague)

Milan Šindelka (Prague)

Michal Kloc (Basel)

Michal Macek (Brno)

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