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Exact analytical treatment of nuclear shape phase transitions in terms of the sextic oscillator

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I. MOTIVATION AND PRELIMINARIES

- **Dynamical symmetries** correspond to unique nuclear shape phases
  - U(5): spherical vibrator
  - O(6): γ-unstable shape
  - SU(3): axially deformed rotor
- New symmetries in nuclear collective motion
- Shape phase transitions described by the infinite square well potential
  - from spherical to deformed γ-unstable shapes: E(5) *[Iachello, 2000]*
  - from spherical to axially deformed shapes: X(5) *[Iachello, 2001]*
  - from axially deformed to triaxial shapes: Y(5) *[Iachello, 2003]*
  - from prolate to oblate shapes: Z(5) *[Bonatsos, 2004]*
- Symmetries imply predictions for the spectroscopic properties.

Use other exactly solvable potentials to study shape phase transitions

Phase transitions can then be analysed in a controllable way.

The Bohr Hamiltonian

$$H=-\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\frac{1}{4\beta^2}\sum_k\frac{Q_k^2}{\sin^2(\gamma-\frac{2}{3}\pi k)}\right)+V(\beta,\gamma)$$

Assume that $V(\beta,\gamma)=U(\beta)$ and separate the γ and angular variables:

$$\Psi(\beta,\gamma,\theta_i)=\beta^{-2}\phi(\beta)\Phi(\gamma,\theta_i)$$

Then a Schrödinger-like equation is obtained

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}\beta^2}+\left(\frac{(\tau+1)(\tau+2)}{\beta^2}+u(\beta)\right)\phi=\epsilon\phi\ ,$$

where $\epsilon=\frac{2B}{\hbar^2}E$ and $u(\beta)=\frac{2B}{\hbar^2}U(\beta)$

Note the presence of the $(\tau+1)(\tau+2)\beta^{-2}$ term

Find solvable $u(\beta)$ potentials for any τ

There is a limited variety of such potentials...

Typical γ-independent potentials proposed: *A perhaps incomplete list...*

Potential	Ref.	Minimum	Number of parameters	Solution
Square well	*Wilets, Jean 1956*	flat	0+scale	$J_\nu(z)$
Harmonic oscillator	*Bohr 1952*	$\beta = 0$	1+scale	$L_n^{(\alpha)}(z)$
H.O. $+\beta^{-2}$	*Elliott et al. 1986*	$\beta > 0$	2+scale	$L_n^{(\alpha)}(z)$
Coulomb	*Fortunato, Vitturi 2003*	$\beta = 0$ singular	1+scale	$L_n^{(\alpha)}(z)$
Kratzer	*Fortunato, Vitturi 2003*	$\beta > 0$	2+scale	$L_n^{(\alpha)}(z)$
β^4	*Arias et al. 2003*	$\beta = 0$	0+scale	numerical
Sextic oscillator	*Lévai, Arias 2004,2010*	$\beta = 0$ and/or $\beta > 0$	2+scale	QES (Heun)
Morse	*Bonatsos et al. 2008*	$\beta > 0$	1+scale	iterative

II. THE SEXTIC OSCILLATOR AS A γ-INDEPENDENT POTENTIAL

$$u(\beta) = (b^2 - 4ac^\pi)\beta^2 + 2ab\beta^4 + a^2\beta^6 + u_0^\pi \ ,$$

[G. Lévai and J. M. Arias, PRC **69**, *014304 (2004)]*

Properties:

- Two free parameters, a and b, plus a scale; c^π is fixed.
- Flexible shape: $\beta_{\rm min} = 0$ OR $\beta_{\rm min} > 0$ OR $\beta_{\rm min} > \beta_{\rm max} > 0$

  There can be two minima at $\beta = 0$ and β_{min}, but then $u(0) > u(\beta_{min})$ always holds.
- Quasi-Exactly Solvable (QES): solvable for the lowest $M + 1$ levels.

$$\phi_n(\beta) = N_n P_n(\beta^2)\beta^{\tau+2} \exp\left(-\frac{a}{4}\beta^4 - \frac{b}{2}\beta^2\right) \qquad n = 0,\ 1,\ 2,\ \ldots$$

- $a \geq 0$ is necessary for normalizability.

  For $a = 0$ $\quad u(\beta) \Longrightarrow b^2\beta^2$ the harmonic oscillator and U(5) recovered

- N_n can be calculated exactly in terms of Kummer's functions (or ${}_1F_1(p,q;z)$)
- c^π and thus $u(\beta)$ is slightly different for even and odd values of τ:

$$\frac{1}{2}\left(\tau + 2M + \frac{7}{2}\right) \equiv c^\pi = const.$$

Set $u_0^+ = 0$ *and* u_0^- *such that the two potentials have the same minimum*

- $B(E2)$ rates can be calculated exactly from

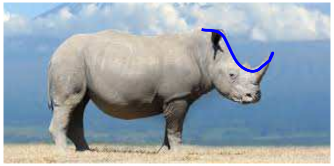
$$T^{(\mathrm{E2})} = t\alpha_{2\mu} = t\beta\left[D^{(2)}_{\mu,0}\cos\gamma + 2^{-1/2}(D^{(2)}_{\mu,2} + D^{(2)}_{\mu,-2})\sin\gamma\right] \ .$$

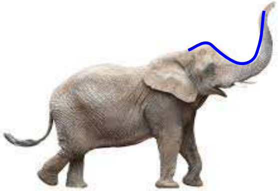
- For those interested in the special functions of mathematical physics:

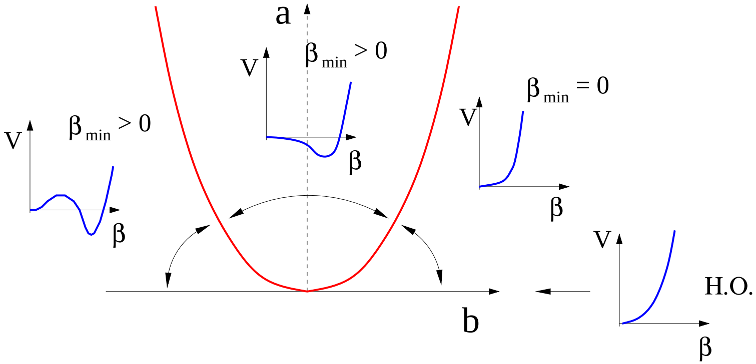
  the sextic oscillator can also be obtained from the bi-confluent Heun equation

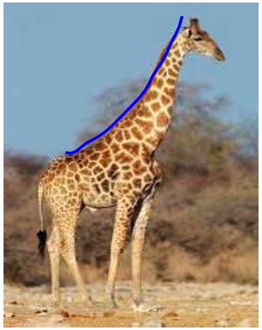
  [A. M. Ishkhanyan, G. Lévai, Physica Scripta **95**, *085202 (2020)]*

Potential shapes in regions of the (a, b) parameter space









Regions are separated by the critical parabola $a = b^2/4c^\pi$

A brief history of the model

2003 Erice conference, Franco'60: first contact of the authors at the lunch table

2004 Establishing the formalism for $M = 0$ and 1 *[PRC* **69***, 014304 (2004)]*

Model space: 6 (ξ, τ) multiplets, 9 states up to $L^\pi = 6^+$

Calculation of benchmark values of the phase transition (on the parabola)

Test calculation for ^{134}Ba

2010 Analytical calculation of matrix elements for $B(E2)$ and $B(E0)$

[PRC **81***, 044304 (2010)]*

Systematic application to 6 Ru, 5 Pd and 6 Cd nuclei

2021 Analytic extension of the model to $M = 2$ *[JPG* **48***, 085102 (2021)]*

Eigenvalues of a 3×3 matrix needed: solve a cubic algebraic equation

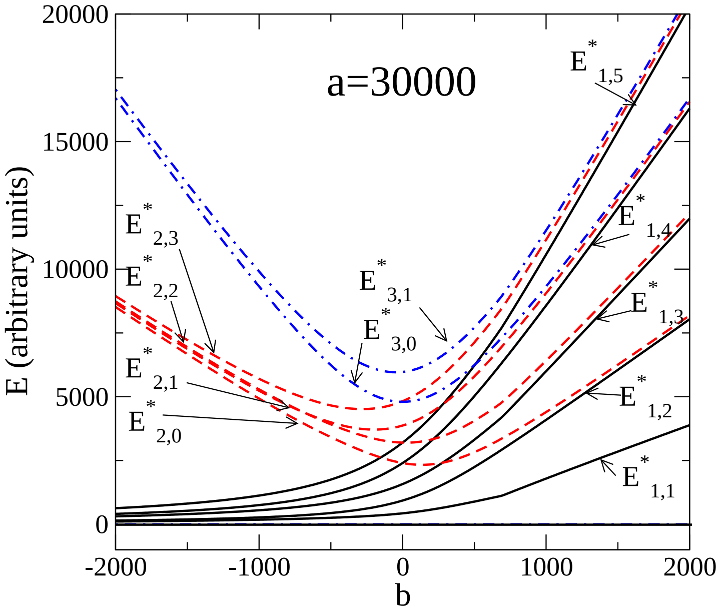
Extended model space: 12 (ξ, τ) multiplets, 29 states up to $L^\pi = 10^+$

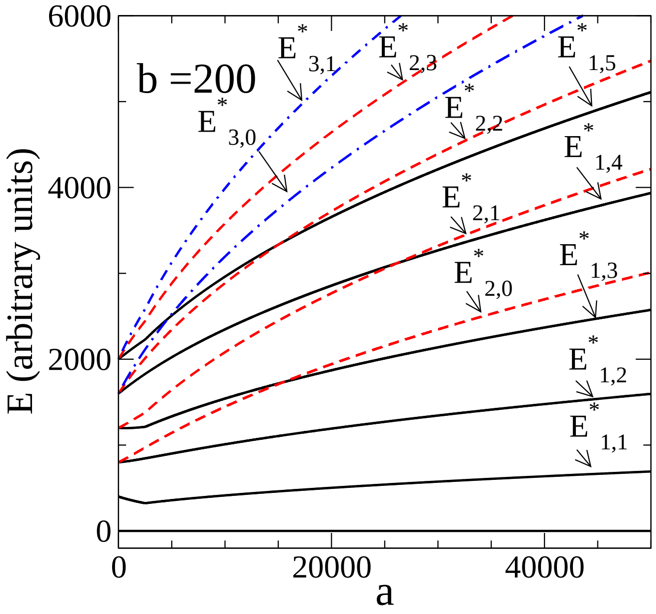
Phenomenologic $L \cdot L$ interaction added to lift the degeneracy within (ξ, τ)

Revisiting the Ru and Pd nuclei

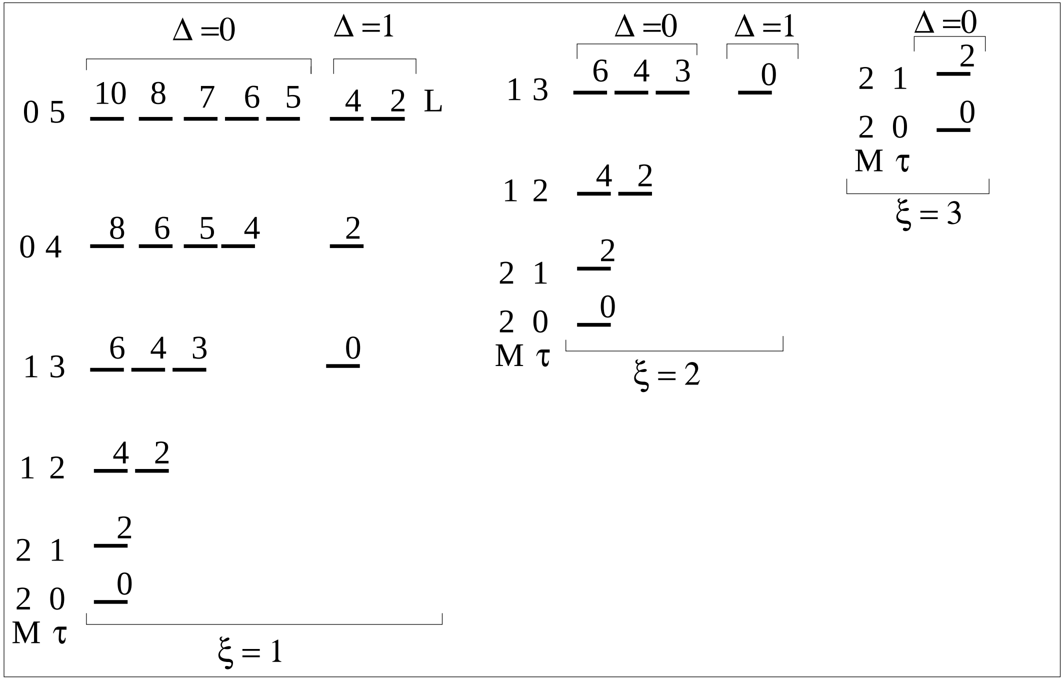
2022 Work in progress on 6 Pt and 4 Os nuclei

Excitation energies $E^*_{\xi,\tau} = E_{\xi,\tau} - E_{1,0}$ for fixed a and b





Schematic spectrum for $M = 0$, 1 and 2

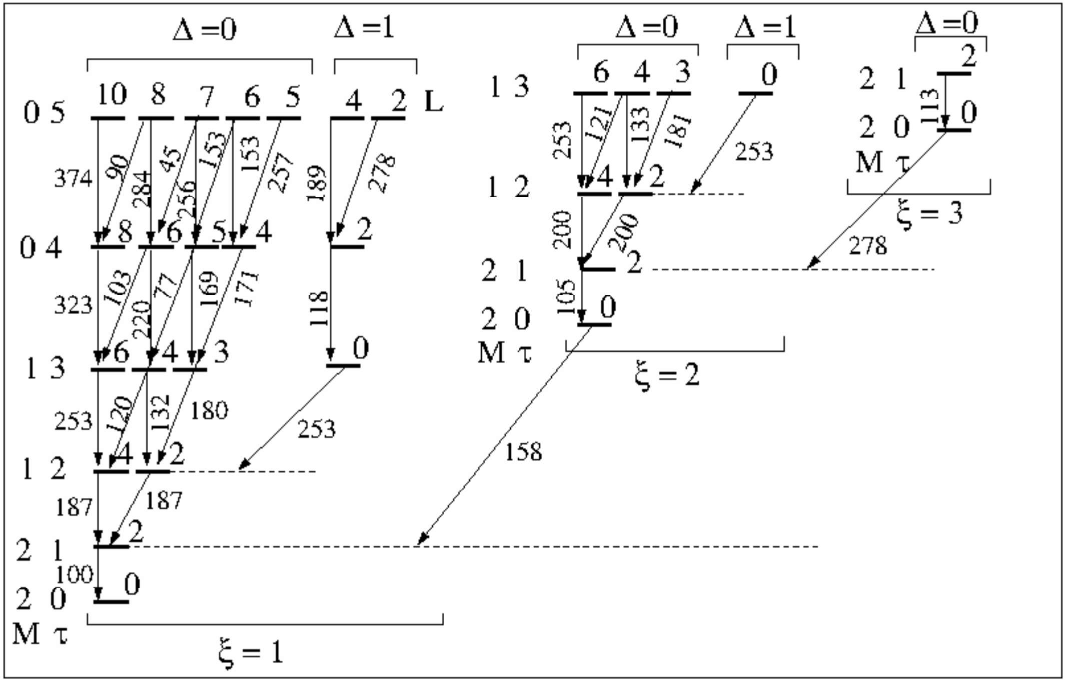


Benchmark values and their comparison with those of other models

Many key values are constant along the critical parabola

	$\frac{E(4^+_{1,2})}{E(2^+_{1,1})}$	$\frac{E(0^+_{2,0})}{E(2^+_{1,1})}$	$\frac{E(6^+_{1,3})}{E(2^+_{1,1})}$	$\frac{B(\mathrm{E2};4^+_{1,2}\to 2^+_{1,1})}{B(\mathrm{E2};2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(\mathrm{E2};0^+_{2,0}\to 2^+_{1,1})}{B(\mathrm{E2};2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(\mathrm{E2};0^+_{1,3}\to 2^+_{1,2})}{B(\mathrm{E2};2^+_{1,1}\to 0^+_{1,0})}$
U(5)	2.00	2.00	3.00	2.00	2.00	3.00
β^4	2.09	2.39	3.27	1.83	1.42	2.56
β^6	2.14	2.62	3.39	1.77	1.19	2.40
$E(5)$	2.20	3.03	3.59	1.68	0.87	2.17
sextic osc.	2.66	3.08	3.80	1.87	1.59	2.52

Benchmark $B(E2)$ values along the critical parabola



III. APPLICATION TO Ru, Pd, Pt AND Os NUCLEI

Ru ($A = 98 \ldots 108$), Pd ($A = 102 \ldots 110$): even–even nuclei below the $Z = 50$ shell closure with mid-shell N values

Pt ($A = 188 \ldots 198$), Os ($A = 186 \ldots 192$): even–even nuclei below the $Z = 82$ shell closure, close to $N = 126$

Identify candidates for the model states

Use known $B(E2)$'s and expected selection rules to identify the band structure

Two-step fitting procedure

> *First step: take the lowest two natural spin-parity states and the lowest single unnatural-parity states*
>
> *Second step: include further states to complete the assignment*

Consider the phenomenologic term $cL \cdot L$

Use a least-square fit of a, b and c to the energy spectrum

> *For unambigious* J^π *use* W=*0.5*

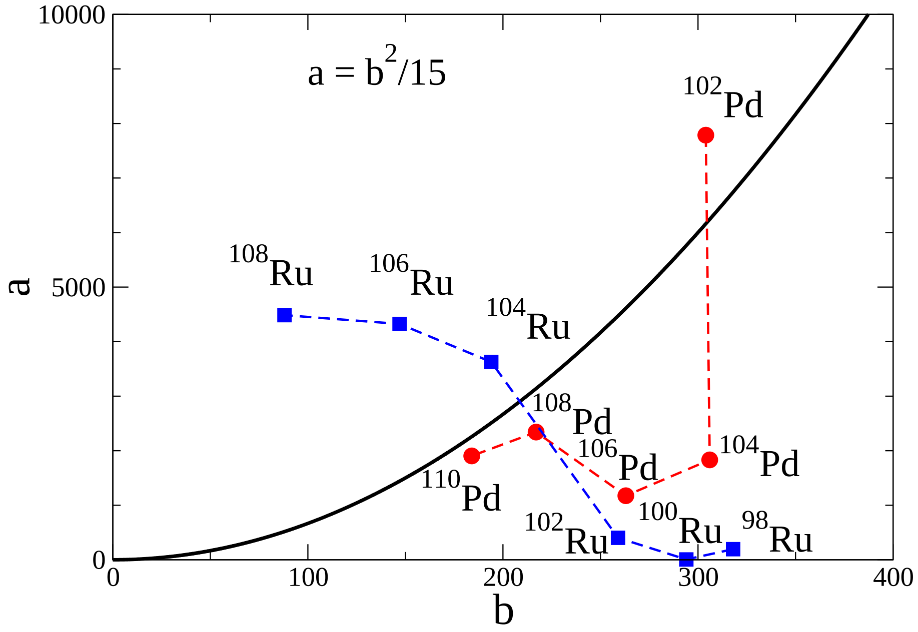
[G. Lévai and J. M. Arias, JPG **48**, *085102 (2021)]*

The Ru ($A = 98 \ldots 108$) and Pd ($A = 102 \ldots 110$) chain

The root mean square deviation: 50 to 130 keV

The c parameter varies from 3 to 12 keV

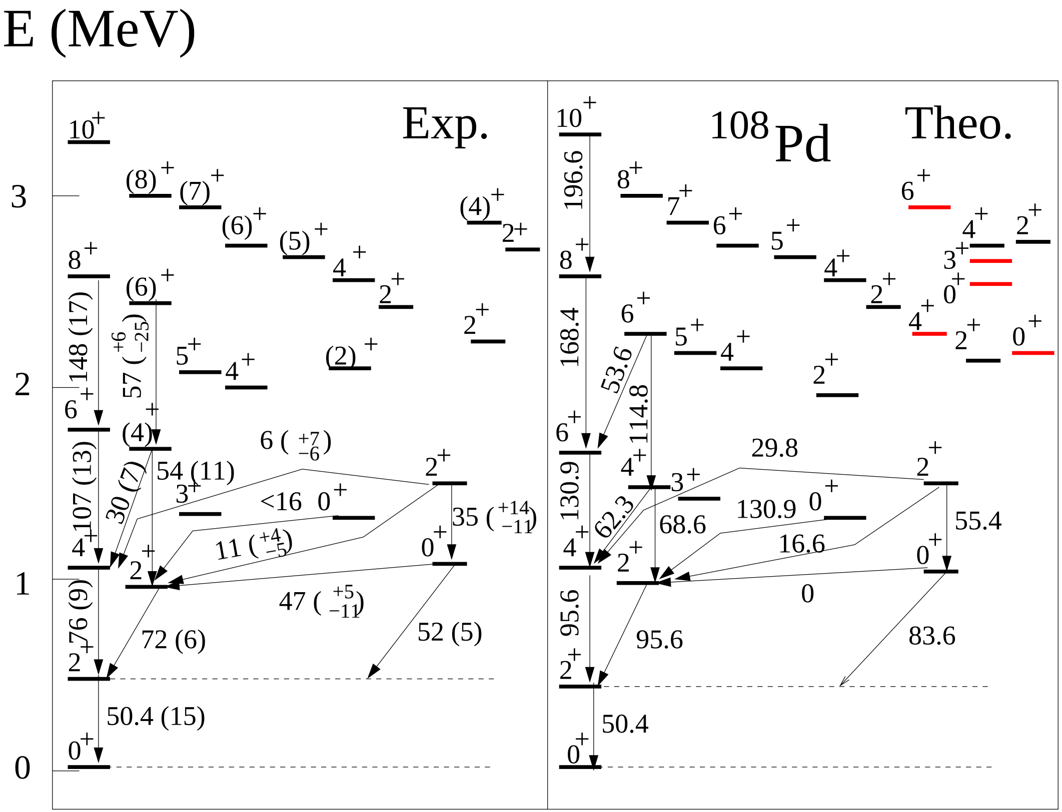
Location of the nuclei in the (a, b) parameter space



Ru chain: from spherical (and harmonic) to deformed minimum

Pd chain: Parallel with the critical parabola, on the spherical minimum side

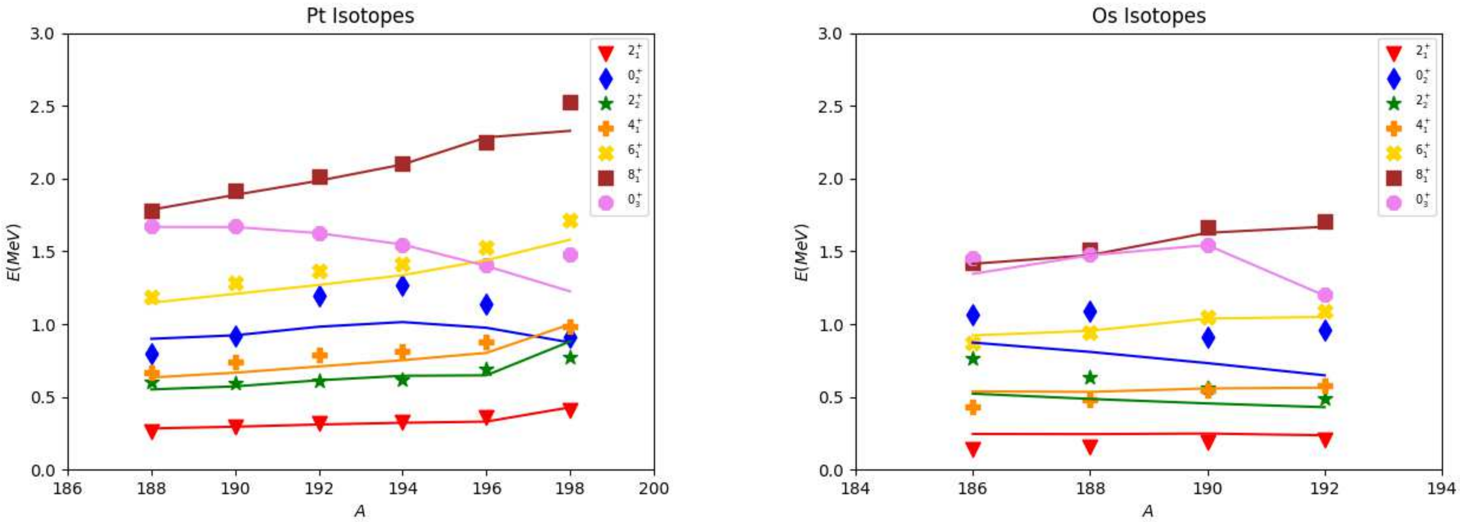
An example: the spectrum of the ^{108}Pd nucleus



Red: model states without experimental correspondent

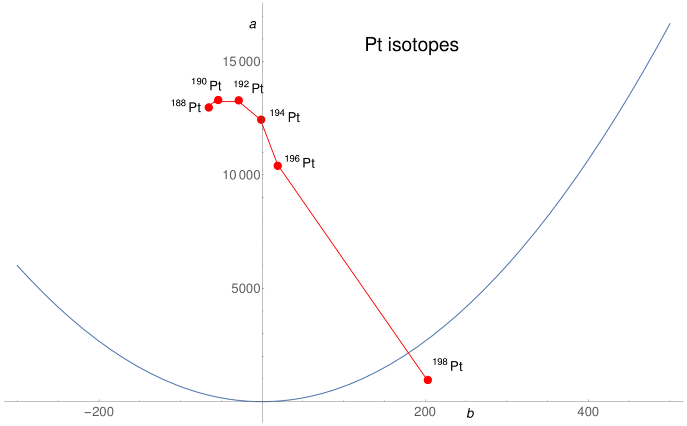
The Pt ($A = 188 \dots 198$) and Os ($A = 186 \dots 192$) chain

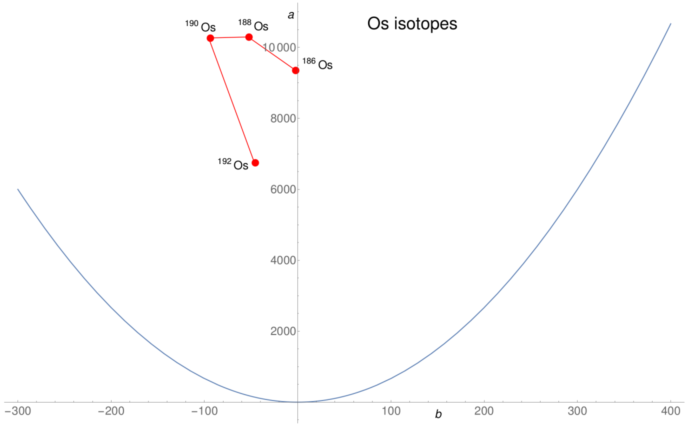
The root mean square deviation: 60 to 130 keV
The c parameter varies from 5 to 10 keV
Trends of some selected energy levels with A



Ground-state band: 2_1^+, 4_1^+, 6_1^+, 8_1^+	$(1,1)2^+$, $(1,2)4^+$, $(1,3)6^+$, $(1,3)8^+$
$K^\pi = 2^+$ bandhead: 2_2^+	$(1,2)2^+$
$K^\pi = 0^+$ bandheads: 0_2^+, 0_3^+	$(1,3)0^+$ OR $(2,0)0^+$

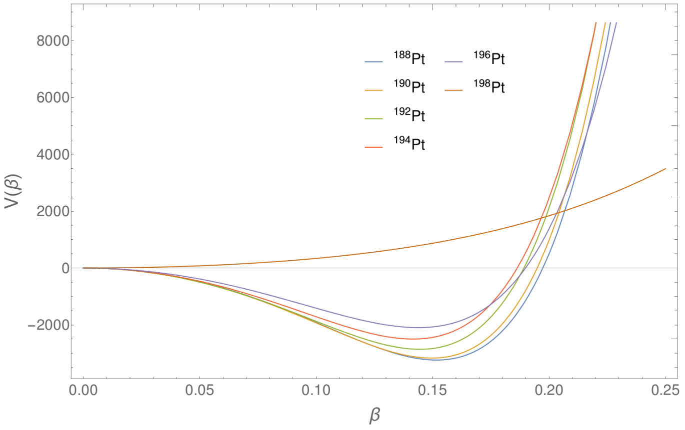
Location of the nuclei in the (a, b) parameter space

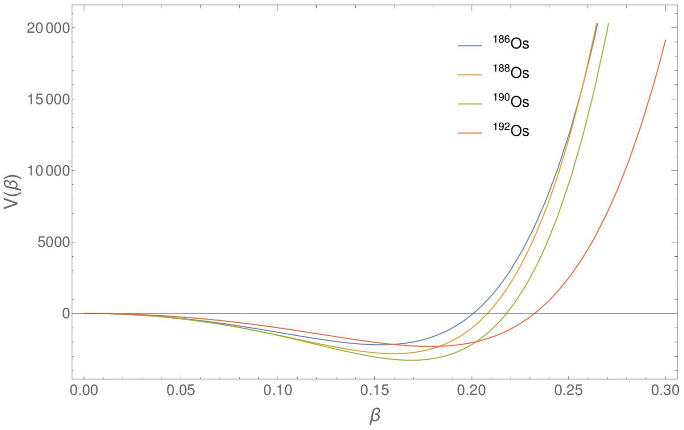




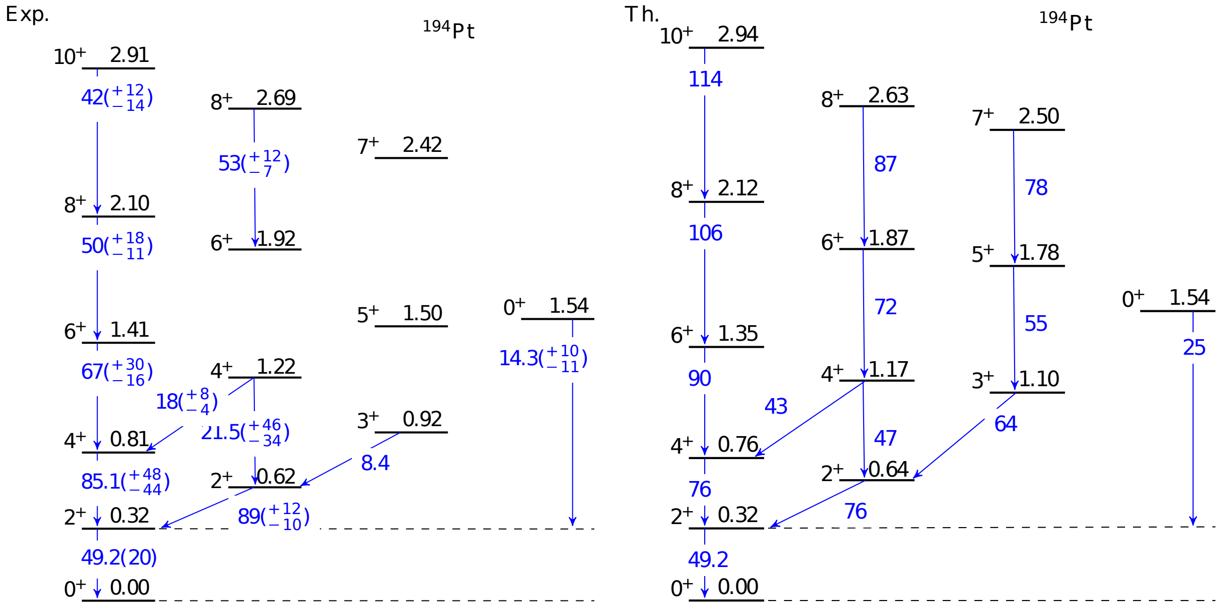
Pt chain: from deformed to spherical minimum (^{198}Pt close to phase transition)
Os chain: all isotopes remain in the domain of deformed minimum

The corresponding potentials

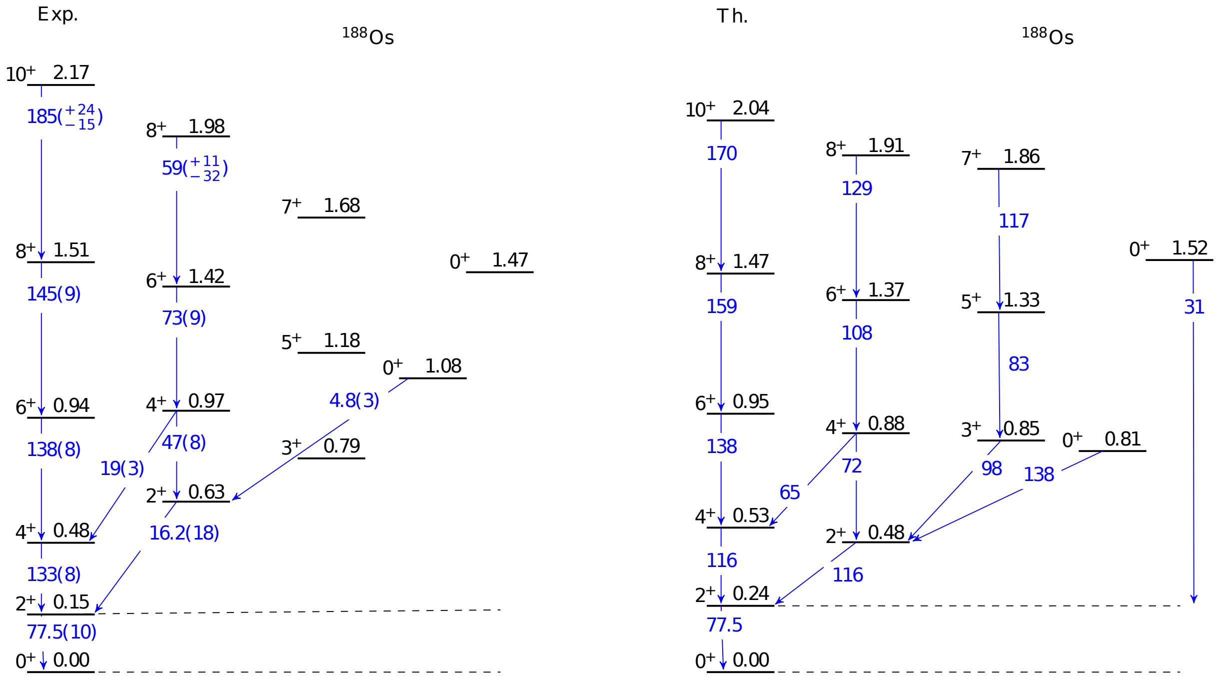




An example: the spectrum of the ^{194}Pt nucleus



Another example: the spectrum of the ^{188}Os nucleus



IV. DISCUSSION

- The sextic oscillator is able to reproduce γ-independent potentials
- The transition from spherical to deformed shape phase can be described analytically
- The extended model ($M \leq 2$) describes 29 states up to $L^{\pi} = 10^{+}$

  This covers the most important states and bands
- Energy eigenvalues and $B(\mathrm{E2})$ values can be determined **exactly**.
- Realistic spectra are reproduced for Ru, Pd, Pt and Os nuclei

  Some of them seem to be close to phase transition
- The model can be used in numerical studies too ($M > 2$)

  Such studies indicate that $M = 2$ is the most suitable: see Lahbas et al. Mod. Phys. Lett. A35 (2020) 2050085
- The sextic oscillator has been used in other models too:

  $X(5)$, $Z(5)$, $Z(4)$, $X(3)$ *see Budaca et al. Ann. Phys. 375 (2016) 65*