Pseudoscalar transition form factors the muon $(g-2)$ and $P\to\bar{\ell}\ell$ decays

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Hadronic Contributions to New Physics Searches
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JOHANNES GUTENBERG UNIVERSITÄT MAINZ

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

PRISMA

Outline

1. Hadronic light-by-light: pseudoscalar pole contribution
2. A transition form factor for the HLbL
3. Updated pseudoscalar pole contribution
4. $P \to \bar{\ell}\ell$ decays: further information and new physics
5. Summary & Outlook

Section 1

Hadronic light-by-light: pseudoscalar pole contribution

The current $(g-2)_\mu$ status

$$a_\mu^{\mathrm{SM}} = (116\ 591\ 826(57)) \times 10^{-11}$$
$$a_\mu^{\mathrm{exp}} = (116\ 592\ 091(63)) \times 10^{-11}$$
$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = (265(85)) \times 10^{-11}$$

New Physics?
only 3σ

The current $(g-2)_\mu$ status

$$\begin{aligned} a_\mu^{\rm SM} &= (116\ 591\ 826(57)) \times 10^{-11} \\ a_\mu^{\rm exp} &= (116\ 592\ 091(63)) \times 10^{-11} \\ \hline a_\mu^{\rm exp} - a_\mu^{\rm SM} &= (265(85)) \times 10^{-11} \end{aligned}$$

Future $(g_\mu - 2)$ experiments

Fermilab & J-PARC: precision

$$\delta a_\mu = 16 \times 10^{-11}$$

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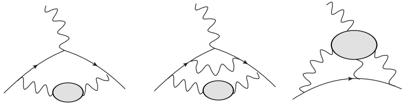
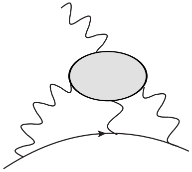
Order	Result$\times 10^{11}$
$a_\mu^{\text{HVP}\cdot\text{LO}}$	6 923(42)
$a_\mu^{\text{HVP}\cdot\text{NLO}}$	$-98.4(7)$
$a_\mu^{\text{HVP}\cdot\text{N}^2\text{LO}}$	12.4(1)
$a_\mu^{\text{HLbL}\cdot\text{LO}}$	116(39)
$a_\mu^{\text{HLbL}\cdot\text{NLO}}$	3(2)
a_μ^{QCD}	6 956(57)

Davier et al ('12), Hagiwara et al ('11), Kurz et al ('14) Jegerlehner Nyffeler ('09, '14)

Improve on the QCD side



Hadronic contributions: Hadronic Light-by-Light



- Not direct connection to data
- Dispersive proposals recently (much involved)
- Multi-scale problem $\to$ more difficulties
- Devise non-perturbative approach to QCD!

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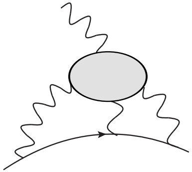

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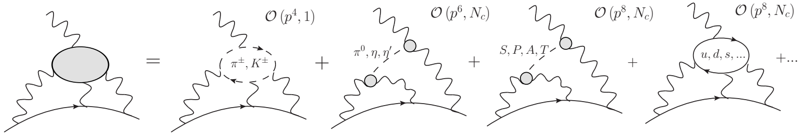

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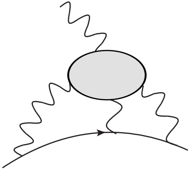

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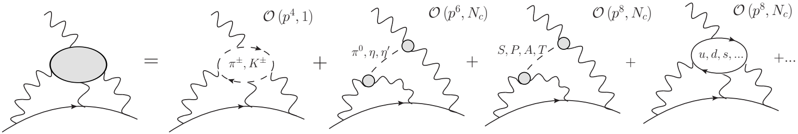

Authors	π^0, η, η'	$\pi\pi$, KK	Resonances	Quark Loop	Total
BPP	85(13)	−19(13)	−4(3)	21(3)	83(32)
HKS	83(6)	−5(8)	2(2)	10(11)	90(15)
KN	83(12)	–	–	–	80(40)
MV	114(10)	–	22(5)	–	136(25)
PdRV	114(13)	−19(19)	8(12)	2.3	105(20)
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- Most results circa 15 years old
- Some advances since then

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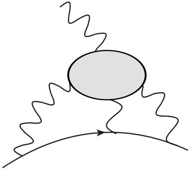

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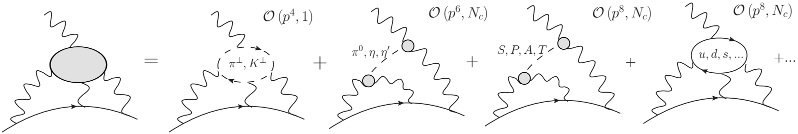

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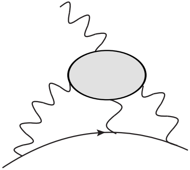

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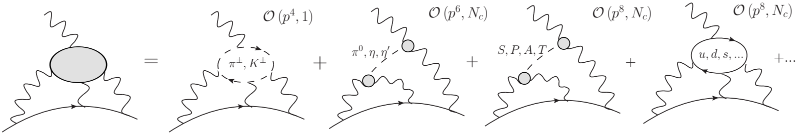

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Update
π^0, η, η'
Contributions

HLbL: the pseudoscalar-pole contribution

For the most general HLbL integral the Green's function

$$\Pi^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \int d^4x_i e^{i\mathbf{p_i}\cdot\mathbf{x_i}} \langle\Omega| T\{j^\mu(x_1)j^\nu(x_2)j^\rho(x_3)j^\sigma(x_4)\} |\Omega\rangle$$

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At low energies insert lowest-lying intermediate states (close to pole):

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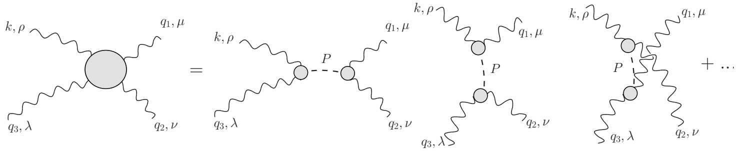
Related to physical process! Graphically, it looks like



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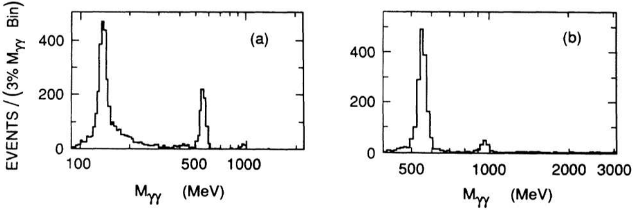
Related to physical process! Experimentally, it looks like



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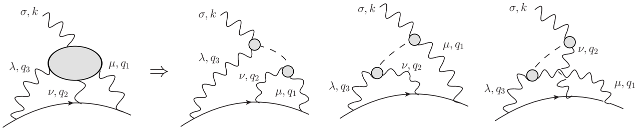
- Plug previous result into HLbL $(g-2)_\mu$ contribution



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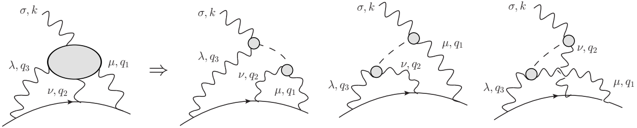

- After some fun with loops and algebra [JN Phys.Rept., 477 (2009)]

$$a_\ell^{\mathrm{HLbL};P} = \frac{-2\pi}{3}\left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt \sqrt{1-t^2}\, Q_1^3 Q_2^3$$
$$\times \left[\frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_3^2) F_{P\gamma^*\gamma}(Q_2^2, 0) I_1(Q_1, Q_2, t)}{Q_2^2 + m_P^2} + \frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) F_{P\gamma^*\gamma}(Q_3^2, 0) I_2(Q_1, Q_2, t)}{Q_3^2 + m_P^2} \right]$$

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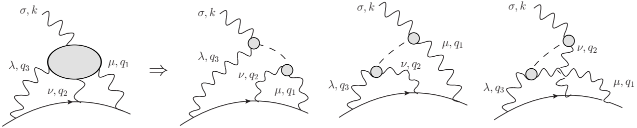

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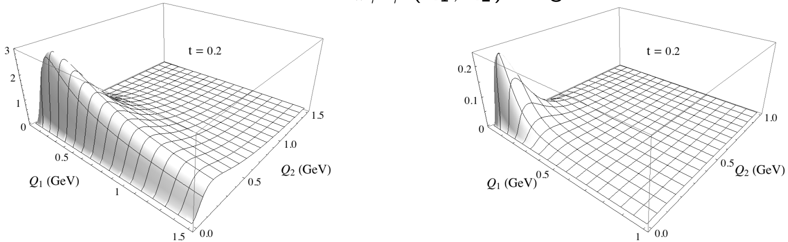
- Without the transition form factors $F_{\pi\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ integrands look like



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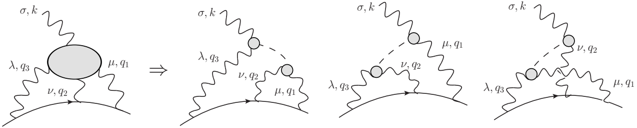

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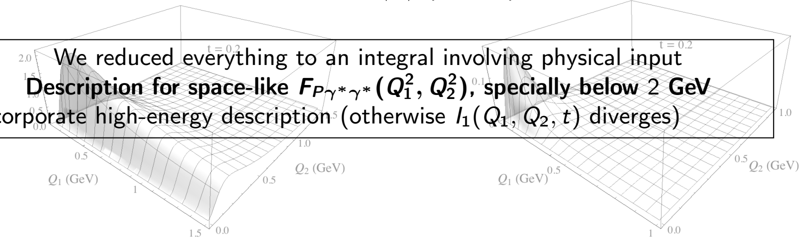


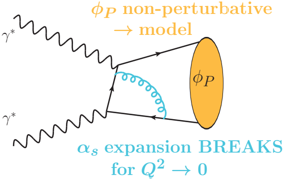
We reduced everything to an integral involving physical input
Description for space-like $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$, specially below 2 GeV
Incorporate high-energy description (otherwise $I_1(Q_1, Q_2, t)$ diverges)

Section 2

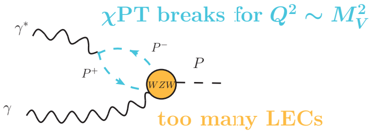
A transition form factor for the HLbL

Describing the TFF I: First principles

—High Energies: pQCD



—Low Energies: χPT



$Q^2 \to \infty$

$$\lim_{Q^2\to\infty} F_{\pi\gamma\gamma^*}(0,Q^2) = \frac{2F_\pi}{Q^2}$$

$$\lim_{Q^2\to\infty} F_{\pi\gamma^*\gamma^*}(Q^2;Q^2) = \frac{2F_\pi}{3Q^2}$$

Guarantee convergence!

?

$Q^2 \to 0$

$$F_{\pi\gamma\gamma}(0,0) = (4\pi^2 F_\pi)^{-1}$$

Describing the TFF II: Model approaches

—Lagrangian-based

Nambu Jona Lasinio • Hidden Local Symmetry • Resonance chiral th. • ...

- Nice overall picture, but not precision
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—Phenomenological Data-based

Large-N_c-based + Resonance saturation + Data-fitting

- Experiment is full QCD $\longrightarrow$ Fit it *with a model*
- Data not always available were required $\rightarrow$ extrapolation reliability?
- How to systematically improve to arbitrary *known* precision?

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—Data-based

Dispersive reconstruction

- Data based, in principle full QCD
- In practice most of QCD contributions $\Rightarrow$ Not full Q^2 reconstruction

Objectives and strategies

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A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

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Generalize our approach to bivariate functions: Canterbury Approximants

Padé Approximants: Introduction to the method

Given a function with known series expansion

$$F_{P\gamma\gamma^*}(Q^2) = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + \ldots) \quad \text{i.e. } \chi\text{PT}$$

Its Padé approximant is defined as

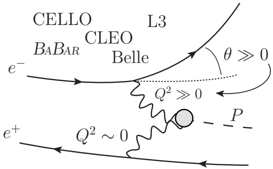
$$P^N_M(Q^2) = \frac{T_N(Q^2)}{R_M(Q^2)} = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + \ldots + \mathcal{O}(Q^2)^{N+M+1})$$

Convergence th. ⇒ Model-independency
Increase $\{N, M\}$ ⇒ Systematic error estimation

$$P^0_1 = \frac{F_{P\gamma\gamma^*}(0)}{1 - b_P Q^2} = F_{P\gamma\gamma^*}(0)(1+b_P Q^2+\mathcal{O}(Q^4)) \ \text{~~VMD~~} \to \chi\text{PT/DR} + \text{pQCD}$$

Correct low (& high) energy implementation!

Padé Approximants: Results



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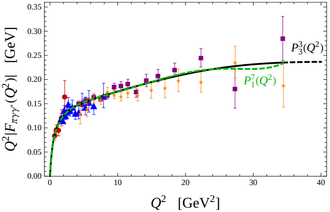

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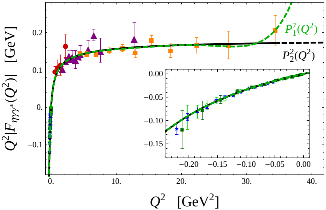
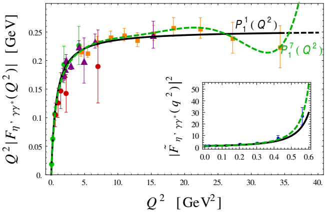
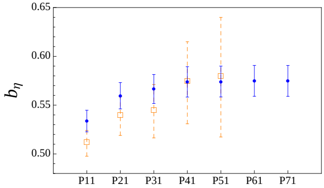


η'

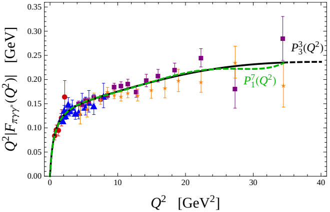


P. Masjuan '12; R. Escribano, P. Masjuan, P. S (& S. Gonzalez) '14 '15 (&16)

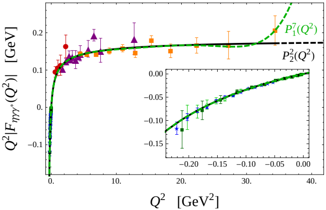
Padé Approximants: Results



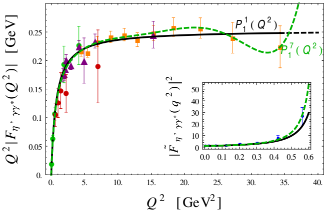
π



η



η'



P. Masjuan '12; R. Escribano, P. Masjuan, P. S (& S. Gonzalez) '14 '15 (&16)

Objectives and strategies

—What do we need?
A model-independent approach for pseudoscalar transition form factors (at least in the euclidean space-like region)

—What is the philosophy?
Toolkit allowing full use of data & QCD constraints on form factors

—How to implement for single-virtual case?
We proppose to use Padé Approximants

—How to implement the double-virtual Form Factor?
Generalize our approach to bivariate functions: Canterbury Approximants

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

- Generalization of Padé apps. $\to$ Canterbury apps. (Chisholm 1973)

For a symmetric function with Taylor expansion

$$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + c_{1,0}(Q_1^2 + Q_2^2) + c_{2,0}(Q_1^4 + Q_2^4) + c_{1,1}Q_1^2Q_2^2 + ...)$$

Its Canterbury appproximant is defined as

$$C_M^N(Q_1^2, Q_2^2) = \frac{T_N(Q_1^2, Q_2^2)}{Q_M(Q_1^2, Q_2^2)} = \frac{\sum_{i,j}^N a_{i,j} Q_1^{2i} Q_2^{2j}}{\sum_{k,l}^M b_{k,l} Q_1^{2k} Q_2^{2l}}$$

Fulfilling the conditions that

$$\sum_{i,j}^{M} b_{i,j} Q_1^{2i} Q_2^{2j} \sum_{\alpha,\beta}^{\infty} c_{\alpha,\beta} Q_1^{2\alpha} Q_2^{2\beta} - \sum_{k,l}^{N} a_{k,l} Q_1^{2k} Q_2^{2l} = \sum_{\gamma,\delta}^{\infty} d_{\gamma,\delta} Q_1^{2\gamma} Q_2^{2\delta},$$

$$\begin{aligned}
d_{\gamma,\delta} &= 0 \quad 0 \leq \gamma+\delta \leq M+N \\
d_{\gamma,\delta} &= 0 \quad 0 \leq \gamma \leq \max(M,N), \\
&\qquad\quad 0 \leq \delta \leq \max(M,N) \\
d_{\gamma,\delta} &= 0 \quad 1 \leq \gamma \leq \min(M,N), \\
&\qquad\quad \delta = M+N+1-\gamma.
\end{aligned}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

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—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

$$C_1^0(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + b_P(Q_1^2 + Q_2^2) + a_{P;1,1}Q_1^2 Q_2^2 + \ldots)$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants

$$C_1^0(Q^2, 0) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P Q^2} = P_1^0(Q^2) \Rightarrow F_{P\gamma\gamma}(0,0) \ \ \& \ \ b_P \ \ \text{determined}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants (already determined)

3.Systematically implement double virtuality: $a_{P;1,1}$ (Exp. unknown)

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

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—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants (already determined)

3.Systematically implement double virtuality: $a_{P;1,1}$ (Exp. unknown)

3a. χPT leading logs suggest factorization at low energies

$$C_1^0(Q_1^2, Q_2^2)|_{\chi PT} = \frac{F_{P\gamma\gamma}(0,0)}{(1 + b_P Q_1^2)(1 + b_P Q_2^2)}; \quad (a_{P;1,1} \equiv b_P^2) \ \textit{Factorization}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants (already determined)

3.Systematically implement double virtuality: $a_{P;1,1}$ (Exp. unknown)

3a. χPT leading logs suggest factorization at low energies

3b. Can incorporate QCD constraints from OPE

$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}; \quad (a_{P;1,1} \equiv 2b_P^2)\ OPE$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Simplest approach: $C_1^0(Q_1^2, Q_2^2)$

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}.$$

—Reconstruction

1.Reproduce original series expansion $\Rightarrow$ low energies

2.Reduce to Padé Approximants (already determined)

3.Systematically implement double virtuality: $a_{P;1,1}$ (Exp. unknown)

3a. χPT leading logs suggest factorization at low energies

3b. Can incorporate QCD constraints from OPE

Theoretically, we expect $a_{P;1,1} \in \{b_P^2 \div 2b_P^2\}$
Precise value ultimately from experiment (implements low energies)

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1+\alpha_1(Q_1^2+Q_2^2)+\alpha_{1,1}Q_1^2Q_2^2)}{1+\beta_1(Q_1^2+Q_2^2)+\beta_2(Q_1^4+Q_2^4)+\beta_{1,1}Q_1^2Q_2^2+\beta_{2,1}Q_1^2Q_2^2(Q_1^2+Q_2^2)+\beta_{2,2}Q_1^4Q_2^4}$$

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—Reconstruction

1.Reduce to Padé Approximants

$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \to$ from PAs

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1+\alpha_1(Q_1^2+Q_2^2)+\alpha_{1,1}Q_1^2Q_2^2)}{1+\beta_1(Q_1^2+Q_2^2)+\beta_2(Q_1^4+Q_2^4)+\beta_{1,1}Q_1^2Q_2^2+\beta_{2,1}Q_1^2Q_2^2(Q_1^2+Q_2^2)+\beta_{2,2}Q_1^4Q_2^4}$$

—Reconstruction

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$F_{P\gamma\gamma}(0,0), \alpha_1, \beta_1, \beta_2 \to$ from PAs

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—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

$$F_{\pi\gamma^*\gamma^*} = \frac{1}{3Q^2}(2F_\pi)\left(1-\frac{8}{9}\frac{\delta^2}{Q^2}+\mathcal{O}(\alpha_s(Q^2))\right) \Rightarrow \beta_{2,2}=0, \alpha_{1,1}, \beta_{2,1}$$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1+\alpha_1(Q_1^2+Q_2^2)+\alpha_{1,1}Q_1^2Q_2^2)}{1+\beta_1(Q_1^2+Q_2^2)+\beta_2(Q_1^4+Q_2^4)+\beta_{1,1}Q_1^2Q_2^2+\beta_{2,1}Q_1^2Q_2^2(Q_1^2+Q_2^2)}$$

—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}$)

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1 + \alpha_1(Q_1^2 + Q_2^2) + \alpha_{1,1} Q_1^2 Q_2^2)}{1 + \beta_1(Q_1^2 + Q_2^2) + \beta_2(Q_1^4 + Q_2^4) + \beta_{1,1} Q_1^2 Q_2^2 + \beta_{2,1} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2)}$$

—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}$)

Previous estimate $b_P^2 \leq a_{P;1,1} \leq 2b_P^2 \Rightarrow$ limited if avoiding poles

Be generous: all configurations with no poles $\Rightarrow a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$

What about the double-virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

—Next element: $C_2^1(Q_1^2, Q_2^2)$

$$C_2^1(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)(1+\alpha_1(Q_1^2+Q_2^2)+\alpha_{1,1}Q_1^2Q_2^2)}{1+\beta_1(Q_1^2+Q_2^2)+\beta_2(Q_1^4+Q_2^4)+\beta_{1,1}Q_1^2Q_2^2+\beta_{2,1}Q_1^2Q_2^2(Q_1^2+Q_2^2)}$$

—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}^{\min} < a_{P;1,1} < a_{P;1,1}^{\max}$)

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—Reconstruction

1.Reduce to Padé Approximants

2.Reproduce the OPE behavior (high energies)

3.Reproduce the low energies ($a_{P;1,1}^{\mathrm{min}} < a_{P;1,1} < a_{P;1,1}^{\mathrm{max}}$)

Low- and high energies implemented
Full use of data and theory constraints
Double-virtual data for $a_{P;1,1}$ (and δ^2) desirable
Systematization up to required precision ($C_3^2(Q_1^2, Q_2^2) \to C_{N+1}^N(Q_1^2, Q_2^2)$)

Section 3

Updated pseudoscalar pole contribution

Seeing is believing: toy models and systematics

—Regge Model—

$$F^{\rm Regge}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	55.2	59.7	60.4	60.6
OPE	65.7	60.8	60.7	60.7
Fit^{OPE}	66.3	62.7	61.1	60.8
Exact		60.7		

—Logarithmic Model—

$$F^{\rm log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	56.7	64.4	66.1	66.8
OPE	65.7	67.3	67.5	67.6
Fit^{OPE}	79.6	71.9	69.3	68.4
Exact		67.6		

Seeing is believing: toy models and systematics

—Regge Model—

$$F^{\text{Regge}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{aF_{P\gamma\gamma}}{Q_1^2-Q_2^2}\frac{\left[\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right)-\psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right]}{\psi^{(1)}\left(\frac{M^2}{a}\right)}$$

	C_1^0	C_2^1	C_3^2	C_4^3
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—Logarithmic Model—

$$F^{\log}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2) = \frac{F_{P\gamma\gamma}M^2}{Q_1^2-Q_2^2}\ln\left(\frac{1+Q_1^2/M^2}{1+Q_2^2/M^2}\right)$$

	C_1^0	C_2^1	C_3^2	C_4^3
LE	56.7	64.4	66.1	66.8
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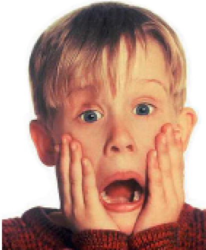
Fit^{OPE}	79.6	71.9	69.3	68.4
Exact		67.6		

- The convergence result is excellent!
- The OPE choice seems the best $\to$ high energy matters
- Still, low energies provide a good performance
- Error $\sim$ difference among elements $\to$ Systematics!



Pseudoscalar-pole contribution: Final results

$$—C_1^0(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact $(a_{P;1,1} = b_P^2)$	OPE $(a_{P;1,1} = 2b_P^2)$
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

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η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

$$—C_2^1(Q_1^2, Q_2^2)—$$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t$	$62.9(1.2)_L(0.3)_\delta[1.2]_t$
η	$16.6(0.8)_L(0)_\delta[0.8]_t$	$16.2(0.8)_L(0.5)_\delta[0.9]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
Total	$95.2[1.7]_t$	$93.4[1.7]_t$

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact $(a_{P;1,1} = b_P^2)$	OPE $(a_{P;1,1} = 2b_P^2)$
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.9(1.2)_L(0.3)_\delta[1.2]_t\{2.0\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[0.8]_t$	$16.2(0.8)_L(0.5)_\delta[0.9]_t$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
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Pseudoscalar-pole contribution: Final results

$—C_1^0(Q_1^2, Q_2^2)—$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact $(a_{P;1,1} = b_P^2)$	OPE $(a_{P;1,1} = 2b_P^2)$
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

$—C_2^1(Q_1^2, Q_2^2)—$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.9(1.2)_L(0.3)_\delta[1.2]_t\{2.0\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[0.8]_t\{0.4\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.5)_\delta[0.9]_t\{0.8\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t$	$14.3(0.5)_L(0.5)_\delta[0.7]_t$
Total	$95.2[1.7]_t$	$93.4[1.7]_t$

Pseudoscalar-pole contribution: Final results

$—C_1^0(Q_1^2, Q_2^2)—$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact $(a_{P;1,1} = b_P^2)$	OPE $(a_{P;1,1} = 2b_P^2)$
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

$—C_2^1(Q_1^2, Q_2^2)—$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.9(1.2)_L(0.3)_\delta[1.2]_t\{2.0\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[1.0]_t\{0.4\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.5)_\delta[0.9]_t\{0.8\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.2[1.7]_t$	$93.4[1.7]_t$

Pseudoscalar-pole contribution: Final results

$—C_1^0(Q_1^2, Q_2^2)—$

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η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

$—C_2^1(Q_1^2, Q_2^2)—$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.9(1.2)_L(0.3)_\delta[1.2]_t\{2.0\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[1.0]_t\{0.4\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.5)_\delta[0.9]_t\{0.7\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.2[1.7]_t\{2.7\}_{\mathrm{sys}}$	$93.4[1.7]_t\{4.5\}_{\mathrm{sys}}$

Pseudoscalar-pole contribution: Final results

$—C_1^0(Q_1^2, Q_2^2)—$

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact $(a_{P;1,1} = b_P^2)$	OPE $(a_{P;1,1} = 2b_P^2)$
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
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$—C_2^1(Q_1^2, Q_2^2)—$

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η	$16.6(0.8)_L(0)_\delta[1.0]_t\{0.4\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.5)_\delta[0.9]_t\{0.7\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.2[1.7]_t\{2.7\}_{\mathrm{sys}}$	$93.4[1.7]_t\{4.5\}_{\mathrm{sys}}$

—Final Result (preliminary)

$$a_\mu^{\pi,\eta,\eta'} = (63.4[1.3]\{2.0\} + 16.4[0.9]\{0.7\} + 14.5[0.7]\{1.7\}) \times 10^{-11} = 94.3[1.7]\{4.5\} \times 10^{-11}$$

Pseudoscalar-pole contribution: Final results

—$C_1^0(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Fact ($a_{P;1,1} = b_P^2$)	OPE ($a_{P;1,1} = 2b_P^2$)
π^0	$54.0(1.1)_F(2.5)_{b_\pi}[2.7]_t$	$64.9(1.4)_F(2.8)_{b_\pi}[3.1]_t$
η	$13.0(0.4)_F(0.4)_{b_\eta}[0.6]_t$	$17.0(0.6)_F(0.4)_{b_\eta}[7]_t$
η'	$12.0(0.4)_F(0.3)_{b_{\eta'}}[0.5]_t$	$16.0(0.5)_F(0.3)_{b_{\eta'}}[6]_t$
Total	$79.0[2.8]_t$	$97.9[3.2]_t$

—$C_2^1(Q_1^2, Q_2^2)$—

$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	$a_{P;1,1}^{\min}$	$a_{P;1,1}^{\max}$
π^0	$63.9(1.3)_L(0)_\delta[1.3]_t\{1.0\}_{\mathrm{sys}}$	$62.9(1.2)_L(0.3)_\delta[1.2]_t\{2.0\}_{\mathrm{sys}}$
η	$16.6(0.8)_L(0)_\delta[1.0]_t\{0.4\}_{\mathrm{sys}}$	$16.2(0.8)_L(0.5)_\delta[0.9]_t\{0.7\}_{\mathrm{sys}}$
η'	$14.7(0.7)_L(0)_\delta[0.7]_t\{1.3\}_{\mathrm{sys}}$	$14.3(0.5)_L(0.5)_\delta[0.7]_t\{1.7\}_{\mathrm{sys}}$
Total	$95.2[1.7]_t\{2.7\}_{\mathrm{sys}}$	$93.4[1.7]_t\{4.5\}_{\mathrm{sys}}$

—Final Result (preliminary)

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

What has been achieved?

Final Updated Result $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

Final Updated Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\rm HVP}$, then 11×10^{-11})
- η and η' relevant, of the order of $\delta a_\mu^{\rm exp}$

Final Updated Result — $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\text{HVP}}$, then 11×10^{-11})
- η and η' relevant, of the order of $\delta a_\mu^{\text{exp}}$

Previous KN Result — $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (58(10) + 13(1) + 12(1)) \times 10^{-11} = 83(12) \times 10^{-11}$$

- Intended for $\delta a_\mu = 63 \times 10^{-11}$; no systematics ($N_c \to 30\%$?)

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\rm HVP}$, then 11×10^{-11})
- η and η' relevant, of the order of $\delta a_\mu^{\rm exp}$
- Full use of current data with systematics and good data description

Previous KN Result $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (58(10) + 13(1) + 12(1)) \times 10^{-11} = 83(12) \times 10^{-11}$$

- Intended for $\delta a_\mu = 63 \times 10^{-11}$; no systematics ($N_c \to 30\%$?)
- Old data-base: new preciser data exists

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Final Updated Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

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- Updated value meeting future exp. precision (if $\delta a_\mu^{\rm HVP}$, then 11×10^{-11})
- η and η' relevant, of the order of $\delta a_\mu^{\rm exp}$
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'

Previous KN Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (58(10) + 13(1) + 12(1)) \times 10^{-11} = 83(12) \times 10^{-11}$$

- Intended for $\delta a_\mu = 63 \times 10^{-11}$; no systematics ($N_c \to 30\%$?)
- Old data-base: new preciser data exists
- η η' factorized: roughly 6×10^{-11} shift

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result — $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\mathrm{HVP}}$, then 11×10^{-11})
- η and η' relevant, of the order of $\delta a_\mu^{\mathrm{exp}}$
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'

Recent GLCR Result — $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (57.5(0.6) + 14.4(2.6) + 10.8(0.9)) \times 10^{-11} = 82.7(2.8) \times 10^{-11}$$

- There are no systematic errors included above ($N_c \to 30\%$?)
- No data used for the η, η' but $SU(3)$-symmetry

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Final Updated Result — $\delta a_\mu^{\rm exp} = 16 \times 10^{-11}$

$$a_\mu^{\pi,\eta,\eta'} = (63.4(2.4) + 16.4(1.1) + 14.5(1.8)) \times 10^{-11} = 94.3(4.8) \times 10^{-11}$$

- Updated value meeting future exp. precision (if $\delta a_\mu^{\rm HVP}$, then 11×10^{-11})
- η and η' relevant, of the order of $\delta a_\mu^{\rm exp}$
- Full use of current data with systematics and good data description
- Full QCD constraints, also for the η and η'

Possible improvements

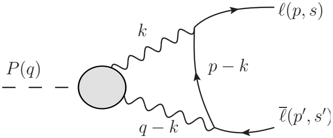
- Double virtuality measurements $(a_{P;1,1}, \delta^2)$: BESIII
- Lattice results
- π^0: SL at BESIII & KLOE-2; TL at NA62, A2
- η': SL at BESIII, Belle II & *GlueX*; TL at NA60 & A2
- η': SL at BESIII, Belle II & *GlueX*; TL A2

KN: Phys.Rev., D65, 073034 (2002); **GLCR**: Phys.Rev., D89, 073016 (2014)

Section 4

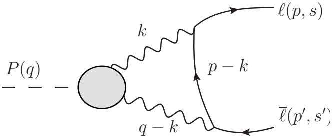
$P \to \bar{\ell}\ell$ decays: further information and new physics

$P \to \bar{\ell}\ell$ decays: a brief introduction



- Probes the (double virtual) TFF
- Clean check assuming no NP
- Alternatively, deviation $\to$ NP

$P\to\bar{\ell}\ell$ decays: a brief introduction

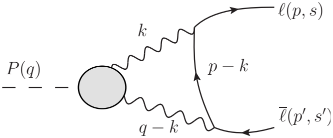


$$\frac{\mathrm{BR}(P\to\bar{\ell}\ell)}{\mathrm{BR}(P\to\gamma\gamma)} = 2\left(\frac{\alpha m_\ell}{m_P}\right)^2 |\mathcal{A}(m_P^2)|^2$$

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2}\int d^4k\,\frac{\left(k^2q^2-(k\cdot q)^2\right)\tilde{F}_{P\gamma^*\gamma^*}(k^2,(q-k)^2)}{k^2(q-k)^2\left((p-k)^2-m_\ell^2\right)}.$$

- The process is low-energy dominated
- UV divergent for a constant TFF

$P\to\bar{\ell}\ell$ decays: a brief introduction



$$\frac{\mathrm{BR}(P\to\bar{\ell}\ell)}{\mathrm{BR}(P\to\gamma\gamma)} = 2\left(\frac{\alpha m_\ell}{m_P}\right)^2 |\mathcal{A}(m_P^2)|^2$$

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Ideal case for our approach

Previous comments apply to this case, but novelties ...

- $-m_P^2 \leq Q^2 \leq \infty$: care with η and η'
- Loop integral approximations: not admissible for the η, η'

Systematics errors: toy models (I)

Prediction for the reconstructed C_1^0

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}; a_{P;1,1} \in (b_P^2, 2b_P^2)$$

Systematics errors: toy models (I)

Prediction for the reconstructed C_1^0

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}; \; a_{P;1,1} \in (b_P^2, 2b_P^2)$$

	Regge			Log		
BR($P \to \bar{\ell}\ell$)	Fact	OPE	**Exact**	Fact	OPE	**Exact**
$\pi^0 \to e^+e^- \times 10^8$	6.218	6.080	**6.138**	5.996	5.869	**5.869**

Systematics errors: toy models (I)

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		Regge			Log	
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$\eta \to e^+e^- \times 10^9$	4.950	5.064	**5.012**	4.614	4.717	**4.626**
$\eta \to \mu^+\mu^- \times 10^6$	4.844	5.151	**4.992**	5.461	5.889	**5.859**

Ok for the desired 5% precision we are aiming for π^0, η

Systematics errors: toy models (I)

Prediction for the reconstructed C_1^0

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{P;1,1})Q_1^2 Q_2^2}; \, a_{P;1,1} \in (b_P^2, 2b_P^2)$$

	Regge			Log		
BR($P \to \bar{\ell}\ell$)	Fact	OPE	**Exact**	Fact	OPE	**Exact**
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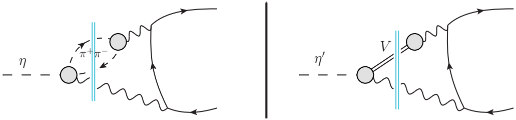
$\eta \to \mu^+\mu^- \times 10^6$	4.844	5.151	**4.992**	5.461	5.889	**5.859**
$\eta' \to e^+e^- \times 10^{10}$	1.825	1.781	**1.754**	1.469	1.437	**1.472**
$\eta' \to \mu^+\mu^- \times 10^7$	1.518	1.407	**1.266**	1.419	1.405	**1.319**

Ok for the desired 5% precision we are aiming for π^0, η
Does not seems to apply for the η' ... recall $-m_P^2 \leq Q^2 \leq \infty$!

Systematics errors: toy models (II)

We have been neglecting a new feature: hadronic thresholds

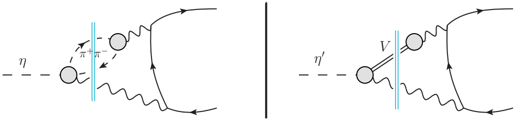


Never considered in previous calculations:

(1) Can our approach deal with it?
(2) Associated C_1^0 systematic error?

Systematics errors: toy models (II)

We have been neglecting a new feature: hadronic thresholds



Never considered in previous calculations:

(1) Can our approach deal with it?
(2) Associated C_1^0 systematic error?

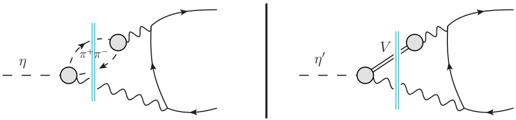
$$\text{Factorized ansatz } \tilde{F}_{P\gamma^*\gamma}(q_1^2, q_2^2) = \tilde{F}_{P\gamma^*\gamma}(q_1^2) \times \tilde{F}_{P\gamma^*\gamma}(q_2^2)$$

$$\tilde{F}_{P\gamma^*\gamma}(s) = c_{P\rho} G_\rho(s) + c_{P\omega} G_\omega(s) + c_{P\phi} G_\phi(s)$$

With $G_V(s)$ fulfilling appropriate analytic and unitary constraints

Systematics errors: toy models (II)

We have been neglecting a new feature: hadronic thresholds



Never considered in previous calculations:

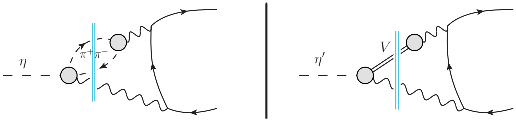
(1) Can our approach deal with it?
(2) Associated C_1^0 systematic error?

$$G_\rho(s) = \frac{M_\rho^2}{M_\rho^2 - s + \frac{sM_\rho^2}{96\pi^2 F_\pi^2}\left(\ln\left(\frac{m_\pi^2}{\mu^2}\right) + \frac{8m_\pi^2}{s} - \frac{5}{3} - \sigma(s)^3 \ln\left(\frac{\sigma(s)-1}{\sigma(s)+1}\right)\right)}$$

D. Gomez Dumm, A. Pich, J. Portoles '00

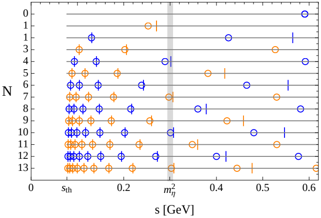
Systematics errors: toy models (II)

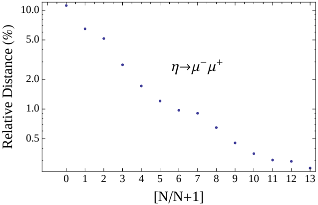
We have been neglecting a new feature: hadronic thresholds



Never considered in previous calculations:

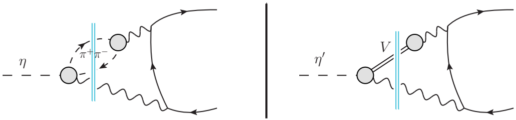
(1) Can our approach deal with it? **It works for the loop integral**
(2) Associated C_1^0 systematic error?





Systematics errors: toy models (II)

We have been neglecting a new feature: hadronic thresholds



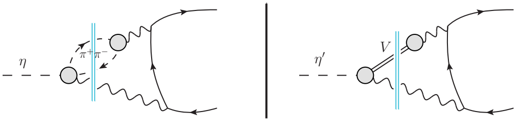
Never considered in previous calculations:

(1) Can our approach deal with it? **It works for the loop integral**
(2) Associated C_1^0 systematic error? **From realistic unitary model**

$BR(P \to \ell\ell)$	toy model	C_1^0	Error (%)
$(\eta \to ee) \times 10^{-9}$	5.410	5.418	0.16
$(\eta \to \mu\mu) \times 10^{-6}$	4.494	4.527	0.74
$(\eta' \to ee) \times 10^{-10}$	1.705	1.883	9
$(\eta' \to \mu\mu) \times 10^{-7}$	1.195	1.461	18

Systematics errors: toy models (II)

We have been neglecting a new feature: hadronic thresholds



Never considered in previous calculations:

(1) Can our approach deal with it? **It works for the loop integral**
(2) Associated C_1^0 systematic error? **From realistic unitary model**

$BR(P \to \ell\ell)$	toy model	C_1^0	Error (%)
$(\eta \to ee) \times 10^{-9}$	5.410	5.418	0.16
$(\eta \to \mu\mu) \times 10^{-6}$	4.494	4.527	0.74
$(\eta' \to ee) \times 10^{-10}$	1.705	1.883	9
$(\eta' \to \mu\mu) \times 10^{-7}$	1.195	1.461	18

(3) Final systematic eror: $(\text{Fact}\div\text{OPE})|_{\text{band}} \oplus \text{Threshold}|_{\%}$

Final Results

BR	Our result (OPE÷Fact)	Approx	Exp
$\pi^0 \to e^+e^- \times 10^8$	$(6.20 \div 6.35)(4)$	$(6.17 \div 6.31)$	$7.48(38)$
$\eta \to e^+e^- \times 10^9$	$(5.31 \div 5.44)(4)$	$(4.58 \div 4.68)$	$\leq 2.3 \times 10^3$
$\eta \to \mu^+\mu^- \times 10^6$	$(4.72 \div 4.52)(5)$	$(5.16 \div 4.88)$	$5.8(8)$
$\eta' \to e^+e^- \times 10^{10}$	$(1.82 \div 1.87)(18)$	$(1.22 \div 1.24)$	≤ 56
$\eta' \to \mu^+\mu^- \times 10^7$	$(1.36 \div 1.49)(26)$	$(1.42 \div 1.41)$	-

P. Masjuan, P. Sanchez, JHEP 1608 (2016) 108

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- Approximate results $\Rightarrow$ large systematics; similar for LO χPT

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- Approximate results $\Rightarrow$ large systematics; similar for LO χPT
- Recent RC studies imply lower BR for π^0 [T. Husek, K. Kampf, J. Novotny, '14; P. Vasko, J. Novotny '11]

P. Masjuan, P. Sanchez, JHEP 1608 (2016) 108

Experimental implications (briefly)

Implications on HLbL

$\mathrm{BR}(\pi^0 \to e^+e^-)|^{RC}_{Exp} \Rightarrow$ 60% reduction on $a_\mu^{\mathrm{HLbL};\pi^0}$

Requires $a_{P;1,1} < 0$ and $\delta^2 \gg$ OPE ... $\eta \to \mu^+\mu^-$ strongly opposed

Double virtual measurement $\Rightarrow$ Smoking Gun! (BESIII)

P. Masjuan, P. Sanchez, arXiv:1504.07001 & JHEP 1608 (2016) 108

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Double virtual measurement $\Rightarrow$ Smoking Gun! (BESIII)

Implications on New Physics: Pseudoscalar/Axial

- Results for generic couplings obtained
- Many constraints exist, but could be (eg. $^8\mathrm{Be}^* \to {}^8\mathrm{Be}\ e^+e^-$ decays)
- LFV tests, but avoid LO χPT or approx. not suitable

π^0@NA62 ? $\eta^{(\prime)} \to \mu^+\mu^-$@LHCb ? $K_L \to \bar{\ell}\ell$@NA62 ?

P. Masjuan, P. Sanchez, arXiv:1504.07001 & JHEP 1608 (2016) 108

Section 5

Summary & Outlook

Summary & Outlook

- Systematic data-driven TFF description [Canterbury approximants]
- Full use of SL and low-energy TL data + theory constraints
- New value $a_\mu^{HLbL;\pi,\eta,\eta'} = 94.2(5.4) \times 10^{-11}$ including systematics
- Error meets future experiments $\delta a_\mu \sim 16 \times 10^{-11}$ requirements
- Reanalysis of $P \to \bar{\ell}\ell$ decays: interesting experimental results

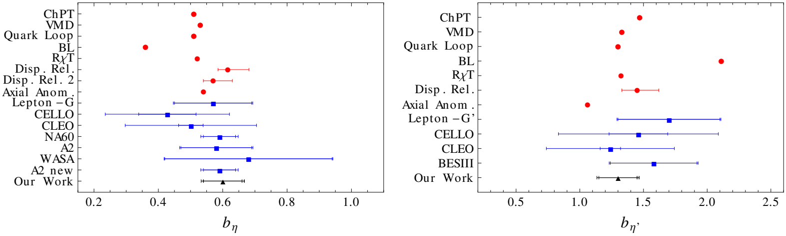
- Improvement: double-virtual measurements $\gamma^*\gamma^* \to P$ BESIII
- User friendly and potential tool for experimentalists/lattice

Section 6

Backup

Inputs for reconstructing the TFFs

	$F_{P\gamma\gamma}$ (GeV^{-1})	b_P	c_P	d_P	P_∞ (GeV)
π^0	0.2725(29)	0.0324(12)(19)	0.00106(9)(25)	—	$2F_\pi$
η	0.2738(47)	0.576(11)(4)	0.339(15)(5)	0.200(14)(18)	0.177(15)
η'	0.3437(55)	1.31(3)(1)	1.74(9)(2)	2.30(20)(12)	0.255(4)
η^{SL}	0.2738(47)	0.60(6)(3)	0.37(10)(7)	—	0.160(24)
η'^{SL}	0.3437(55)	1.30(15)(7)	1.72(47)(34)	—	0.255(4)



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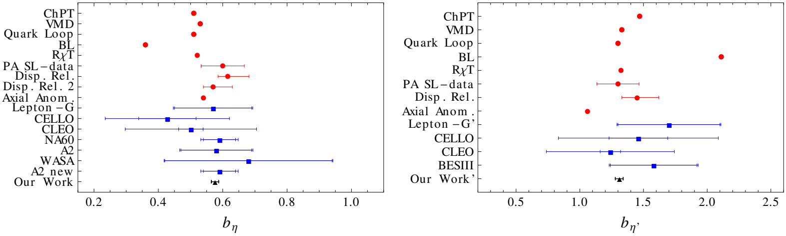

Step I: $F_{P\gamma^*\gamma}(Q^2)$ and Padé Approximants

Series expansion from data-fitting: robustness

Accuracy test: compare to the low-energy time-like region

- Convergence expected below threshold at $\sqrt{q^2} = 2m_\pi = 0.280$ GeV

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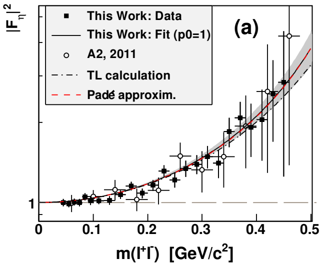
- Convergence expected below threshold at $\sqrt{q^2} = 2m_\pi = 0.280$ GeV
- Compare to later released A2@MAMI data $(\eta \to \gamma e^+ e^-)$



Excellent results even above threshold

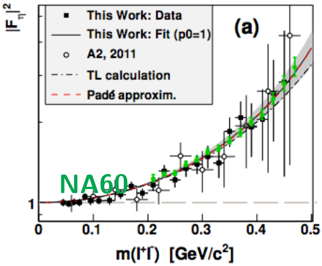
- Understood due to $\pi\pi$ P-wave smooth discontinuity $(q^2 - 4m_\pi^2)^{3/2}$

Step I: $F_{P\gamma^*\gamma}(Q^2)$ and Padé Approximants

Series expansion from data-fitting: robustness

Accuracy test: compare to the low-energy time-like region

- Convergence expected below threshold at $\sqrt{q^2} = 2m_\pi = 0.280$ GeV
- Confirmed by brand new NA60 data $(\eta \to \gamma\mu^+\mu^-)$



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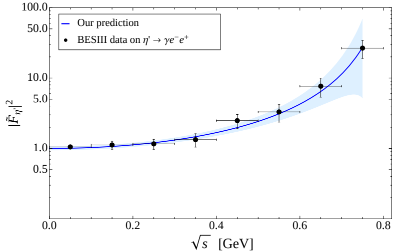
- Compare to later released BESIII data $(\eta' \to \gamma e^+ e^-)$



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Accuracy test: compare to the low-energy time-like region

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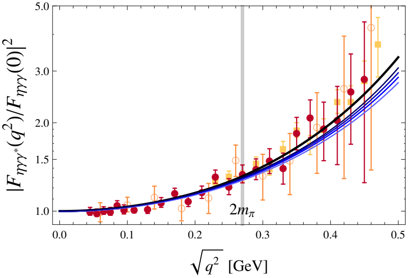
- Compare to P_1^N fits to DR-like



Excellent results even above threshold

- Understood due to $\pi\pi$ P-wave smooth discontinuity $(q^2 - 4m_\pi^2)^{3/2}$

Incorporating further OPE terms

$$F_{\pi\gamma^*\gamma^*} = \underbrace{\frac{1}{Q^2}}_{\mathrm{OPE}^1} \underbrace{\frac{2F_\pi}{3}}_{\mathrm{OPE}^2} \left[1 - \underbrace{\frac{8}{9}\frac{\delta^2}{Q^2}}_{\mathrm{OPE}^3} + \mathcal{O}(\alpha_s(Q^2))\right]$$

	Regge Model				Log Model			
	C_1^0	C_2^1	C_3^2	C_4^3	C_1^0	C_2^1	C_3^2	C_4^3
LE	55.2	59.7	60.4	60.6	56.7	64.4	66.1	66.8
OPE^1	65.7	60.8	60.7	60.7	65.7	67.3	67.5	67.6
OPE^2	–	60.6	60.7	60.7	65.7	67.3	67.5	67.6
OPE^3	–	60.8	60.7	60.7	65.7	67.3	67.5	67.6
Fact	54.6	57.3	57.4	57.5	54.6	60.3	61.3	61.6
$\mathrm{Fit}^{\mathrm{OPE}}$	66.3	62.7	61.1	60.8	79.6	71.9	69.3	68.4
Exact	60.7				67.6			

$\delta^2 = 0.20(2)$ [sum-rules] ; $\eta, \eta' \Rightarrow 30\%$ for $SU(3)_F$ and large-N_c

$F_{P\gamma\gamma} = 1 : 1.00 : 1.26 \quad b_P m_P^{-2} = 1 : 1.08 : 0.80 \quad P_\infty = 1 : 0.96 : 1.38$

A baby problem: Light pseudoscalars

Given the relevant QCD scale $M_V \gg m_P, m_\ell$, approximations are possible

$$\mathcal{A}(m_{\pi^0}^2) \simeq \frac{i\pi}{2\beta_\ell} L + \frac{1}{\beta_\ell}\left(\frac{1}{4}L^2 + \frac{\pi^2}{12} + Li_2\left(\frac{\beta_\ell - 1}{1+\beta_\ell}\right)\right) - \frac{5}{4} + \int_0^\infty dQ \frac{3}{Q}\left(\frac{m_\ell^2}{m_\ell^2 + Q^2} - \tilde{F}_{\pi^0\gamma^*\gamma^*}(Q^2, Q^2)\right)$$

P. Masjuan, P. Sanchez, arXiv:15040.07001 [hep-ph]

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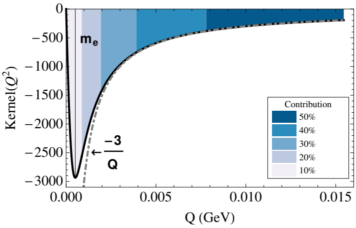

- Singularity from $\gamma\gamma$ $\frac{1}{Q}$ suppression
  Low space-like energies peak
- Peak at lepton mass
  IR regulator $\sim ln(m_e^2)$
- High energies, $F_{\pi\gamma\gamma}$ dominates
  UV regulator $\sim -ln(\Lambda^2)$

P. Masjuan, P. Sanchez, arXiv:15040.07001 [hep-ph]

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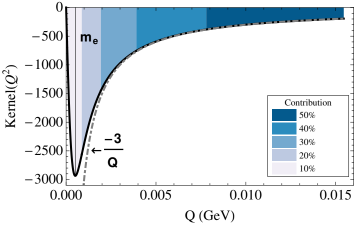

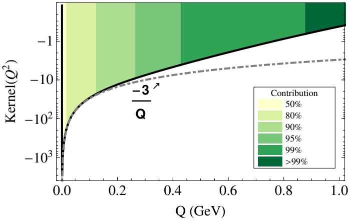

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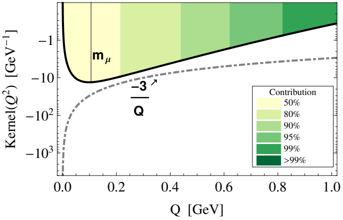
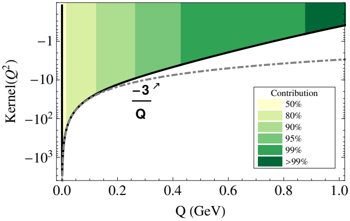




—Calculation Requires

$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ description precise at low space-like energies

P. Masjuan, P. Sanchez, arXiv:15040.07001 [hep-ph]

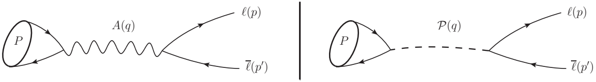
New Physics contributions

$$\mathcal{L} = \frac{g}{4m_W} \sum_f m_A c_f^A \left(\overline{f} \not{A} \gamma_5 f \right) + 2 m_f c_f^{\mathcal{P}} \left(\overline{f} i \gamma_5 f \right) \mathcal{P}$$

$$—\mathcal{A}(q^2) \to \mathcal{A}(q^2) + \frac{\sqrt{2} G_F F_\pi}{4\alpha_{em}^2 F_{P\gamma\gamma}} (\lambda_P^A + \lambda_P^{\mathcal{P}})—$$

$$\lambda_P^A = c_\ell^A \left[\frac{F_P^3}{F_\pi} (c_u^A - c_d^A) + \frac{F_P^q}{F_\pi} (c_u^A + c_d^A) + \frac{F_P^s}{F_\pi} \sqrt{2} c_s^A \right],$$

$$\lambda_P^{\mathcal{P}} = \frac{c_\ell^{\mathcal{P}}}{1 - \frac{M_{\mathcal{P}}^2}{m_P^2}} \left[\frac{F_P^3}{F_\pi} (c_u^{\mathcal{P}} - c_d^{\mathcal{P}}) + \frac{F_P^q}{F_\pi} (-c_s^{\mathcal{P}}) + \frac{F_P^s}{F_\pi} \sqrt{2} c_s^{\mathcal{P}} \right].$$



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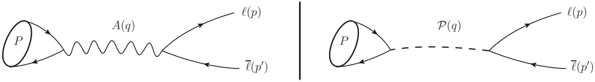
$$BR(\pi^0 \to e^+e^-) \left(1 + 0.001 \left[c_\ell^A (c_u^A - c_d^A) + c_\ell^{\mathcal{P}} \frac{c_u^{\mathcal{P}} - c_d^{\mathcal{P}}}{1 - M_{\mathcal{P}}^2/m_P^2}\right]\right),$$

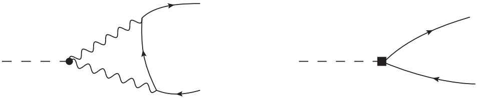
$$BR(\eta \to^{\mu^+\mu^-}_{e^+e^-}) \left(1 + \left(\begin{smallmatrix}-0.002\\+0.001\end{smallmatrix}\right) \left[0.84 c_\ell^A (c_u^A + c_d^A) - 1.27 c_\ell^A c_s^A - \frac{2.11 c_\ell^{\mathcal{P}} c_s^{\mathcal{P}}}{1 - M_{\mathcal{P}}^2/m_P^2}\right]\right),$$

$$BR(\eta' \to^{\mu^+\mu^-}_{e^+e^-}) \left(1 + \left(\begin{smallmatrix}+0.003\\+0.001\end{smallmatrix}\right) \left[0.72 c_\ell^A (c_u^A + c_d^A) + 1.61 c_\ell^A c_s^A + \frac{0.89 c_\ell^{\mathcal{P}} c_s^{\mathcal{P}}}{1 - M_{\mathcal{P}}^2/m_P^2}\right]\right).$$



χPT at higher orders: leading logs

Results from χPT at LO



$$\mathcal{A} = \frac{i\pi}{2\beta_\ell} L + \frac{1}{\beta_\ell}\left[\frac{1}{4}L^2 + \frac{\pi^2}{12} + Li_2\left(\frac{\beta_\ell - 1}{1+\beta_\ell}\right)\right] - \frac{5}{2} + 3\ln\left(\frac{m_\ell}{\mu}\right) + \chi(\mu)$$

	$\pi^0 \to e^+e^-$	$\eta \to e^+e^-$	$\eta \to \mu^+\mu^-$	$\eta' \to e^+e^-$	$\eta' \to \mu^+\mu^-$
$\chi(\mu)$	$(2.53 \div 2.99)$	$(5.90 \div 6.46)$	$(3.29 \div 3.82)$	$(14.2 \div 14.9) + 2.52i$	$(5.61 \div 6.31) + 0.75i$
$\chi(\mu)_{m_\pi}$	$(2.53 \div 2.99)$	$(2.66 \div 3.12)$	–	$(2.16 \div 2.62)$	–
$\chi(\mu)_{UV}$	$(2.53 \div 2.99)$	$(5.50 \div 6.05)$	$(3.11 \div 3.64)$	$(16.8 \div 17.7) + 7.09i$	$(6.56 \div 7.35) + 2.12i$

Higher order corrections are clearly required

χPT at higher orders: leading logs

$$\tilde{F}_{P\gamma^*\gamma^*}(q_1^2, q_2^2) = \underbrace{1}_{\text{LO}} + \underbrace{\frac{1}{\Lambda^2}(q_1^2 + q_2^2)}_{\text{NLO}} + \underbrace{\frac{1}{\Lambda^4}(q_1^4 + q_2^4) + \frac{1}{\Lambda^4}(q_1^2 q_2^2)}_{\text{NNLO}} + \mathcal{O}\left(\frac{q^6}{\Lambda^6}\right).$$

$$\mathcal{A}(q^2, m_\ell^2) = \frac{2i}{\pi^2 q^2} \int d^4k \frac{\left(k^2 q^2 - (k\cdot q)^2\right)}{k^2 (q-k)^2 \left((p-k)^2 - m_\ell^2\right)} \left[1 + \frac{(...)}{\Lambda^2} + \frac{(...)}{\Lambda^4} + ...\right]$$
$$\equiv \mathcal{A}^{\text{LO}}(q^2, m_\ell^2) + \mathcal{A}^{\text{NLO}}(q^2, m_\ell^2) + \mathcal{A}^{\text{NNLO}}(q^2, m_\ell^2) + ... ,$$

$$\mathcal{A}^{\text{NLO}}(q^2, m_\ell^2) = \frac{1}{3\Lambda^2}(q^2 - 10 m_\ell^2)(1 - L_\ell) + \frac{1}{9\Lambda^2}(4 m_\ell^2 - q^2); \quad L_\ell = \ln(m_\ell^2/\Lambda^2)$$

$$\mathcal{A}^{\text{NNLO}}(q^2, m_\ell^2) = \left[\frac{126 m\ell^4 - q^4 - 8 m_\ell^2 q^2}{12\Lambda^4} L_\ell + \frac{26 m_\ell^2 q^2 + 7 q^4 - 702 m_\ell^4}{72\Lambda^4}\right],$$

χPT at higher orders: leading logs

$$\tilde{F}_{P\gamma^*\gamma^*}(q_1^2, q_2^2) = \underbrace{1}_{\text{LO}} + \underbrace{\frac{1}{\Lambda^2}(q_1^2 + q_2^2)}_{\text{NLO}} + \underbrace{\frac{1}{\Lambda^4}(q_1^4 + q_2^4) + \frac{1}{\Lambda^4}(q_1^2 q_2^2)}_{\text{NNLO}} + \mathcal{O}\left(\frac{q^6}{\Lambda^6}\right).$$

$$\mathcal{A}(q^2, m_\ell^2) = \frac{2i}{\pi^2 q^2} \int d^4k \frac{\left(k^2 q^2 - (k\cdot q)^2\right)}{k^2 (q-k)^2 \left((p-k)^2 - m_\ell^2\right)} \left[1 + \frac{(...)}{\Lambda^2} + \frac{(...)}{\Lambda^4} + ...\right]$$
$$\equiv \mathcal{A}^{\rm LO}(q^2, m_\ell^2) + \mathcal{A}^{\rm NLO}(q^2, m_\ell^2) + \mathcal{A}^{\rm NNLO}(q^2, m_\ell^2) + \ldots\ ,$$

$$\mathcal{A}(q^2, m_e^2) - \mathcal{A}(q^2, m_\mu^2) \simeq \mathcal{A}^{\rm LO}(q^2, m_e^2) - \mathcal{A}^{\rm LO}(q^2, m_\mu^2)$$
$$+ \frac{q^2}{3\Lambda^2}\left(1 + \frac{q^2}{4\Lambda^2}\right) \ln\left(\frac{m_\mu^2}{m_e^2}\right) + \frac{10 m_\mu^2}{3\Lambda^2} \ln\left(\frac{\Lambda^2}{m_\mu^2}\right).$$