Precise determination of the low-energy hadronic contribution to the muon $g-2$ from analyticity and unitarity

Irinel Caprini
National Institute for Physics and Nuclear Engineering
Bucharest, Romania

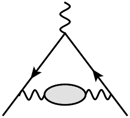
Phys Rev D **93**, 116007 (2016), with B. Ananthanarayan, D. Das and I.S. Imsong

HC2NP, 25-30 September 2016, Tenerife, Spain

Outline

1. Aim and strategy
2. Mathematical formalism
3. Phenomenological input
4. Error evaluation
5. Low-energy contribution to a_μ
6. Summary

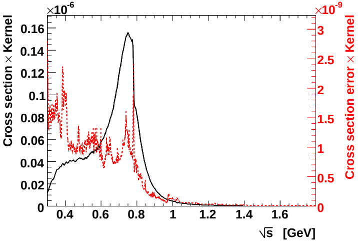
- New generation experiments at Fermilab and J-PARC: error on a_μ at the level of $\delta a_\mu^{exp} = 1.6 \times 10^{-10}$
- Largest theoretical error, $\delta a_\mu^{th} \sim 4.3 \times 10^{-10}$, from LO hadronic vacuum polarization (HVP)

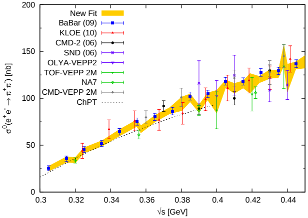


- A large part of it comes from the $\pi^+\pi^-$ contribution from low-energies
  - Compilation of e^+e^- data, including *BaBar* : Davier et al. (2010)

$$\boxed{a_\mu^{\pi\pi,\,\mathrm{LO}}[2m_\pi,\, 0.63\,\mathrm{GeV}] = (133.2 \pm 1.3) \times 10^{-10}}$$

  - Integration of *BaBar* data alone: error $\sim 1.5 \times 10^{-10}$ Malaescu (2013)
  - Inclusion of KLOE 11: modest improvement, due to tension between *BaBar* and KLOE Hagiwara et al. (2011)
  - More recent experiments (KLOE 13, BESIII 16, CMD-3 preliminary) do not report data at low energies

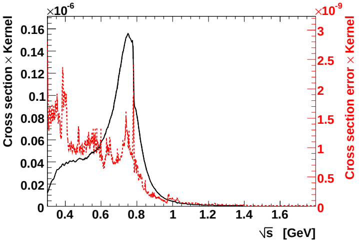


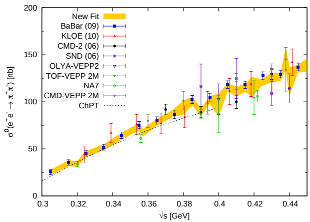


Davier et al. EPJ C**66**, 1 (2010) Hagiwara et al. J.Phys. G**38**, 085003 (2011)

- Left, black: combined data on the $e^+e^- \to \pi^+\pi^-$ cross section multiplied by the kernel function $K(s)$ in the integral for a_μ
- Left: red: corresponding error contribution, with statistical and systematic errors added in quadrature
- Right: low-energy data on the $e^+e^- \to \pi^+\pi^-$ cross section

Large experimental errors on the cross-section amplified by the QED kernel





Davier et al. EPJ C**66**, 1 (2010) Hagiwara et al. J.Phys. G**38**, 085003 (2011)

- Left, black: combined data on the $e^+e^- \to \pi^+\pi^-$ cross section multiplied by the kernel function $K(s)$ in the integral for a_μ
- Left: red: corresponding error contribution, with statistical and systematic errors added in quadrature
- Right: low-energy data on the $e^+e^- \to \pi^+\pi^-$ cross section

Large experimental errors on the cross-section amplified by the QED kernel

$\Rightarrow$ More convenient to use the pion electromagnetic form factor

$$\langle \pi^+(p')|J_\mu^{\mathrm{elm}}|\pi^+(p)\rangle = (p+p')_\mu F(t), \;\; t = (p-p')^2$$

$$a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[\sqrt{t_l},\sqrt{t_u}] = \frac{\alpha^2 m_\mu^2}{12\pi^2}\int_{t_l}^{t_u}\frac{dt}{t}\,|F(t)|^2\,K(t)\,\beta_\pi^3(t)\,|F_\omega(t)|^2\left(1+\frac{\alpha}{\pi}\,\eta_\pi(t)\right)$$

- $F(t)$: the pion electromagnetic form factor in the isospin limit
- $K(t) = \int_0^1 du(1-u)u^2(t-u+m_\mu^2 u^2)^{-1}$ the QED kernel
- $\beta_\pi(t) = (1-4m_\pi/t)^{1/2}$ two-pion phase space
- $F_\omega(t) = 1+\epsilon\frac{t}{(m_\omega - i\Gamma_\omega/2)^2 - t}$ isospin-breaking correction ($\omega - \rho$ mixing)
- $1+\frac{\alpha}{\pi}\,\eta_\pi(t)$: FSR correction (scalar QED)

$$a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[\sqrt{t_l},\sqrt{t_u}] = \frac{\alpha^2 m_\mu^2}{12\pi^2}\int_{t_l}^{t_u}\frac{dt}{t}\,|F(t)|^2\,K(t)\,\beta_\pi^3(t)\,|F_\omega(t)|^2\left(1+\frac{\alpha}{\pi}\,\eta_\pi(t)\right)$$

- $F(t)$: the pion electromagnetic form factor in the isospin limit
- $K(t)=\int_0^1 du(1-u)u^2(t-u+m_\mu^2u^2)^{-1}$ the QED kernel
- $\beta_\pi(t)=(1-4m_\pi/t)^{1/2}$ two-pion phase space
- $F_\omega(t)=1+\epsilon\frac{t}{(m_\omega-i\Gamma_\omega/2)^2-t}$ isospin-breaking correction ($\omega-\rho$ mixing)
- $1+\frac{\alpha}{\pi}\,\eta_\pi(t)$: FSR correction (scalar QED)

Consider the low-energy contribution: $a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[2m_\pi, 0.63\,\mathrm{GeV}]\equiv a_\mu$

Aim: reduce the error on a_μ by exploiting analyticity, unitarity and more precise phenomenological information on $F(t)$ available at other energies

Basic idea:

- Use as input, instead of the modulus, the phase $\arg[F(t)]$, known with precision in the elastic region of the unitarity cut from Fermi-Watson theorem and Roy equations for $\pi\pi$ scattering
- Use additional, more precise, values of $F(t)$, available at other energies

Strategy

Basic idea:

- Use as input, instead of the modulus, the phase $\arg[F(t)]$, known with precision in the elastic region of the unitarity cut from Fermi-Watson theorem and Roy equations for $\pi\pi$ scattering
- Use additional, more precise, values of $F(t)$, available at other energies

Requirements on the method:

- No specific parametrization
- Independence on the unknown phase of $F(t)$ above the inelastic threshold
- Reliable evaluation of the errors

Basic idea:

- Use as input, instead of the modulus, the phase $\arg[F(t)]$, known with precision in the elastic region of the unitarity cut from Fermi-Watson theorem and Roy equations for $\pi\pi$ scattering
- Use additional, more precise, values of $F(t)$, available at other energies

Requirements on the method:

- No specific parametrization
- Independence on the unknown phase of $F(t)$ above the inelastic threshold
- Reliable evaluation of the errors

Achieved by using:

- Analyticity and unitarity of the form factor
- Adequate mathematical methods: extremal problems for analytic functions
- Statistical simulations to account for the uncertainties

Extremal problem

Find optimal upper and lower bounds on $|F(t)|$ on the elastic unitarity cut, $4m_\pi^2 < t < t_{in}$, for $F(t)$ in the class of functions real analytic in the t-plane cut along the real axis for $t \geq 4m_\pi^2$, which satisfy the following conditions:

- Phase known in the elastic region (from δ_1^1 phase-shift of $\pi\pi$ scattering):

$$\mathrm{Arg}[F(t+i\epsilon)] = \delta_1^1(t), \qquad 4m_\pi^2 \leq t \leq t_{in}$$

- An integral condition on the modulus squared above the inelastic threshold:

$$\frac{1}{\pi}\int_{t_{in}}^{\infty} dt\, w(t)\, |F(t)|^2 \leq I$$

- Given values for the first two Taylor coefficients at $t = 0$:

$$F(0) = 1, \qquad \left[\frac{dF(t)}{dt}\right]_{t=0} = \frac{1}{6}\langle r_\pi^2\rangle$$

- Given values at several spacelike and timelike energies:

$$F(t_n) = F_n \pm \epsilon_n, \qquad n = 1, 2\ldots$$

Find optimal upper and lower bounds on $|F(t)|$ on the elastic unitarity cut, $4m_\pi^2 < t < t_{in}$, for $F(t)$ in the class of functions real analytic in the t-plane cut along the real axis for $t \geq 4m_\pi^2$, which satisfy the following conditions:

- Phase known in the elastic region (from δ_1^1 phase-shift of $\pi\pi$ scattering):

$$\mathrm{Arg}[F(t+i\epsilon)] = \delta_1^1(t), \qquad 4m_\pi^2 \leq t \leq t_{in}$$

- An integral condition on the modulus squared above the inelastic threshold:

$$\frac{1}{\pi}\int_{t_{in}}^{\infty} dt\, w(t)\, |F(t)|^2 \leq I$$

- Given values for the first two Taylor coefficients at $t = 0$:

$$F(0) = 1, \qquad \left[\frac{dF(t)}{dt}\right]_{t=0} = \frac{1}{6}\langle r_\pi^2\rangle$$

- Given values at several spacelike and timelike energies:

$$F(t_n) = F_n \pm \epsilon_n, \qquad n = 1, 2 \ldots$$

Combined phase-modulus problem. Can be reduced to a standard Schur-Carathéodory and Pick-Nevanlinna interpolation problem

Caprini, EPJC(2000), Abbas, Ananthanarayan, Caprini, Imsong and Ramanan, EPJA(2010)

Solution of the extremal problem

The solution is expressed in terms of several auxiliary quantities:

- Conformal mapping of the t-plane cut for $t > t_{in}$ onto the unit disc $|z| < 1$:

$$z \equiv \tilde{z}(t) = \frac{\sqrt{t_{in}} - \sqrt{t_{in} - t}}{\sqrt{t_{in}} + \sqrt{t_{in} - t}}, \qquad \tilde{z}(0) = 0; \qquad \tilde{t}(z) = t_{in} \frac{4z}{(1+z)^2}$$

- An Omnès function:

$$\mathcal{O}(t) = \exp\left(\frac{t}{\pi} \int_{4m_\pi^2}^{\infty} dt \frac{\delta(t')}{t'(t'-t)} \right)$$

where $\delta(t) = \delta_1^1(t)$ for $t \leq t_{in}$ and an arbitrary smooth function for $t > t_{in}$

- A function analytic without zeros in $|z| < 1$ ("outer function") with modulus on $|z| = 1$ equal to $\sqrt{w(t)\,|\mathrm{d}t/\mathrm{d}\tilde{z}(t)|}$:

$$\mathcal{C}_1(z) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \frac{e^{i\theta} + z}{e^{i\theta} - z} \ln[w(\tilde{t}(e^{i\theta}))|\frac{d\tilde{t}}{dz}|] \right]$$

- Another outer function in $|z| < 1$ with modulus on the boundary equal to $|\mathcal{O}(\tilde{t}(z))|$:

$$\mathcal{C}_2(z) = \exp\left(\frac{\sqrt{t_{in} - \tilde{t}(z)}}{\pi} \int_{t_{in}}^{\infty} \mathrm{d}t' \frac{\ln|\mathcal{O}(t')|}{\sqrt{t' - t_{in}}(t' - \tilde{t}(z))} \right)$$

- Define the function: $g(z) \equiv F(\tilde{t}(z))\,[\mathcal{O}(\tilde{t}(z))]^{-1}\,\mathcal{C}_1(z)\,\mathcal{C}_2(z)$ analytic in $|z|<1$
  - Define $g_k \equiv \left[\frac{1}{k!}\frac{d^k g(z)}{dz^k}\right]_{z=0}, \qquad 0 \le k \le K-1$
  - For $z_n \in (-1,1)$, define $\bar{\xi}_n = g(z_n) - \sum_{k=0}^{K-1} g_k z_n^k, \quad 1 \le n \le N$
  - Let $\bar{I} = I - \sum_{k=0}^{K-1} g_k^2$
- Construct the determinant $\mathcal{D}$:

$$\mathcal{D} = \begin{vmatrix} \bar{I} & \bar{\xi}_1 & \bar{\xi}_2 & \cdots & \bar{\xi}_N \\ \bar{\xi}_1 & \frac{z_1^{2K}}{1-z_1^2} & \frac{(z_1 z_2)^K}{1-z_1 z_2} & \cdots & \frac{(z_1 z_N)^K}{1-z_1 z_N} \\ \bar{\xi}_2 & \frac{(z_1 z_2)^K}{1-z_1 z_2} & \frac{(z_2)^{2K}}{1-z_2^2} & \cdots & \frac{(z_2 z_N)^K}{1-z_2 z_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\xi}_N & \frac{(z_1 z_N)^K}{1-z_1 z_N} & \frac{(z_2 z_N)^K}{1-z_2 z_N} & \cdots & \frac{z_N^{2K}}{1-z_N^2} \end{vmatrix}$$

$\Rightarrow$ **the determinant $\mathcal{D}$ and its minors are nonnegative**

Application of the solution

- $K = 2$
  - By this we implement the condition $F(0) = 1$ and the charge radius $\langle r_\pi^2 \rangle$
- $N = 3$
  - 2 points used as input, one at a spacelike energy and the other at a timelike energy
  - One point where we want to calculate bounds on $|F(t)|$

- The condition $\mathcal{D} \geq 0$ gives a quadratic inequality with coefficients known from the input for the unknown modulus $|F(t)|$, from which we obtain upper and lower bounds

$$m \leq |F(t)| \leq M, \qquad t < t_{in}$$

- The positivity of the minors provide consistency constraints on the quantities that enter as input, which ensures that the quadratic equations for the bounds have real solutions

Input: the phase

- First inelastic threshold $t_{in} = (m_\pi + m_\omega)^2 = (0.92\,\mathrm{GeV})^2$

  Eidelman and Lukaszuk (2004)

- The phase shift $\delta_1^1(t)$ determined from Roy equations for the $\pi\pi$ amplitudes

  Ananthanarayan, Colangelo, Gasser and Leutwyler (2001),
  Caprini, Colangelo and Leutwyler (2013), Garcia-Martin et al. (2011)

  $\Rightarrow$ two phases denoted as Bern and Madrid

- For $t > t_{in}$, $\delta(t)$ taken as an arbitrary smooth function
  - The results are not affected by this arbitrariness
  - Rigorous proof based on theory of analytic functions

    Abbas, Ananthanarayan, Caprini, Imsong and Ramanan (2010)
  - Checked numerically for sufficiently smooth functions $\delta(t)$

Input: the phase

- First inelastic threshold $t_{in} = (m_\pi + m_\omega)^2 = (0.92\,\mathrm{GeV})^2$

  Eidelman and Lukaszuk (2004)

- The phase shift $\delta_1^1(t)$ determined from Roy equations for the $\pi\pi$ amplitudes

  Ananthanarayan, Colangelo, Gasser and Leutwyler (2001), Caprini, Colangelo and Leutwyler (2013), Garcia-Martin et al. (2011)

  $\Rightarrow$ two phases denoted as Bern and Madrid

- For $t > t_{in}$, $\delta(t)$ taken as an arbitrary smooth function
  - The results are not affected by this arbitrariness
  - Rigorous proof based on theory of analytic functions

    Abbas, Ananthanarayan, Caprini, Imsong and Ramanan (2010)
  - Checked numerically for sufficiently smooth functions $\delta(t)$

$\Rightarrow$ The results do not depend on the phase of $F(t)$ in the inelastic region

Input: high-energy integral

- Input data on $|F(t)|$:
  - From t_{in} to $\sqrt{t} = 3$ GeV Babar data   Aubert et al. (2009)
  - For 3 GeV$\leq \sqrt{t} \leq 20$ GeV a constant value
  - Above 20 GeV we impose a $1/t$ decrease according to QCD scaling

- Choice of the weight $w(t)$
  - The weights with a rapid decrease allow a precise calculation of the integral, but lead to weaker bounds
  - The weights with a slower decrease lead to stronger bounds, but do not suppress the unknown high energy part

- Suitable choice: $w(t) = \frac{1}{t}$ Ananthanarayan, Caprini, Das and Imsong (2012, 2013)
  - The range above 3 GeV contributes with $\sim 1\%$
  - Numerical evaluation: $\boxed{I = 0.578 \pm 0.022}$

Input: high-energy integral

- Input data on $|F(t)|$:
  - From t_{in} to $\sqrt{t} = 3$ GeV Babar data   Aubert et al. (2009)
  - For 3 GeV$\leq \sqrt{t} \leq 20$ GeV a constant value
  - Above 20 GeV we impose a $1/t$ decrease according to QCD scaling

- Choice of the weight $w(t)$
  - The weights with a rapid decrease allow a precise calculation of the integral, but lead to weaker bounds
  - The weights with a slower decrease lead to stronger bounds, but do not suppress the unknown high energy part

- Suitable choice: $w(t) = \frac{1}{t}$ Ananthanarayan, Caprini, Das and Imsong (2012, 2013)
  - The range above 3 GeV contributes with $\sim 1\%$
  - Numerical evaluation: $\boxed{I = 0.578 \pm 0.022}$

- **Remarkable property:** for a fixed weight $w(t)$, the bounds depend in a monotonous way on I, becoming stronger/weaker when I is decreased/increased

  $\Rightarrow$ The most conservative bounds are obtained with $I = 0.578 + 0.022$

Input: values from the holomorphy domain

1. Normalization condition $F(0) = 1$

2. Charge radius in a rather large range $\langle r_\pi^2 \rangle \in (0.41,\, 0.45)\,\mathrm{fm}^2$ inferred from previous studies   Ananthanarayan, Caprini, Das, Imsong (2013)

3. Most recent data at spacelike energies   Horn et al. (2006), Huber et al. (2008)

$$F(-1.60\,\mathrm{GeV}^2) = 0.243 \pm 0.012^{+0.019}_{-0.008}$$
$$F(-2.45\,\mathrm{GeV}^2) = 0.167 \pm 0.010^{+0.013}_{-0.007}$$

- Included in order to constrain the high-energy behaviour of the form factor

Input: modulus from the elastic region of the cut

Optimal region: $0.65\,\mathrm{GeV} \leq \sqrt{t} \leq 0.71\,\mathrm{GeV}$ determined from previous studies
Ananthanarayan, Caprini, Das, Imsong (2013)

- Close to the low-energy region of interest $\Rightarrow$ strong bounds
- Data are more precise and more consistent among them

Experiment	Number of points
CMD2	2
SND	2
BABAR	26
KLOE 2011	8
KLOE 2013	8
BESIII	10
CLEO	3
ALEPH	3
OPAL	3
Belle	2

Number of measurements in the region $0.65\,\mathrm{GeV} \leq \sqrt{t} \leq 0.71\,\mathrm{GeV}$ in e^+e^- and τ-decay experiments

Extraction of $|F(t)|$ from data

- The formalism requires $F(t)$ in the isospin limit
- $|F(t)|$ extracted from measurements by applying suitable corrections

Extraction of $|F(t)|$ from data

- The formalism requires $F(t)$ in the isospin limit
- $|F(t)|$ extracted from measurements by applying suitable corrections

- e^+e^- experiments:

$$|F(t)|^2 = \frac{3t}{\alpha^2 \pi \beta_\pi(t)^3} \frac{\sigma^0_{\pi\pi(\gamma)}(t)}{1 + \frac{\alpha}{\pi}\eta_\pi(t)}$$

- $\sigma^0_{\pi\pi(\gamma)}$: bare cross-section (without VP) corrected for $\rho - \omega$ mixing

Extraction of $|F(t)|$ from data

- The formalism requires $F(t)$ in the isospin limit
- $|F(t)|$ extracted from measurements by applying suitable corrections

- e^+e^- experiments:

$$|F(t)|^2 = \frac{3t}{\alpha^2 \pi \beta_\pi(t)^3} \frac{\sigma^0_{\pi\pi(\gamma)}(t)}{1 + \frac{\alpha}{\pi}\eta_\pi(t)}$$

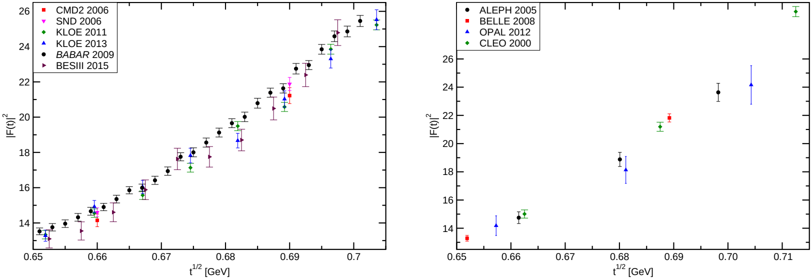
  - $\sigma^0_{\pi\pi(\gamma)}$: bare cross-section (without VP) corrected for $\rho - \omega$ mixing

- τ-decay experiments

$$|F^-(t)|^2 = \frac{2m_\tau^2}{|V_{ud}|^2} \frac{1}{S_{EW}} \left(1 - \frac{t}{m_\tau^2}\right)^{-2} \left(1 + \frac{2t}{m_\tau^2}\right)^{-1} \frac{\mathcal{B}_{\pi\pi}}{\mathcal{B}} \left(\frac{1}{N_{\pi\pi}} \frac{dN_{\pi\pi}}{dt}\right) \frac{1}{\beta_-^3(t)} \frac{1}{G_{EM}}$$

  - $dN_{\pi\pi}/N_{\pi\pi}dt$: normalized invariant mass spectrum of the two-pion final state
  - S_{EW}: short distance correction
  - $\beta_-(t)$: two-pion phase space relevant for τ decay
  - G_{EM}: long-distance radiative correction
  - $\rho - \gamma$ mixing advocated recently Jegerlehner (2011) negligible in the input range

Modulus of $F(t)$ used as input

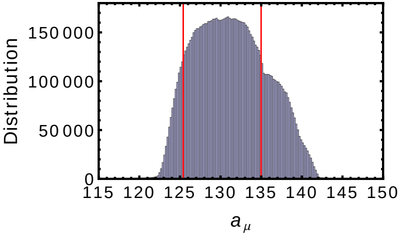


Modulus squared $|F(t)|^2$ in the region 0.65-0.71 GeV extracted from $e^+e^- \to \pi^+\pi^-$ (left) and τ-decay (right) experiments

Statistical simulations

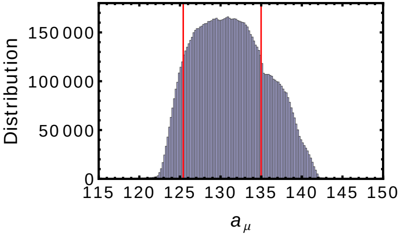
- The formalism provides upper and lower bounds on $|F(t)|$ at low energies for definite values of the input quantities (phase, charge radius, spacelike value, timelike modulus)
- To account for the uncertainties, we have generated a large number of pseudodata for each of the input quantities, using *a priori* given distributions (uniform or gaussian)
- For each set of inputs in the sample, upper and lower bounds on the modulus squared $|F(t)|^2$ at all energies below 0.63 GeV have been calculated using the mathematical algorithm
- A number of random admissible values for $|F(t)|^2$ between the upper and lower bounds have been generated at each energy and used in the integral giving a_μ
- We obtained a large sample ($\sim 10^6$) of values for the quantity a_μ for each timelike input
- The entire sample was binned to obtain a mean value and a 68.3% confidence level interval

- Distribution of a_μ ($\times 10^{10}$) obtained without input from the timelike region

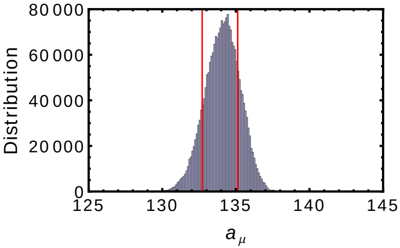


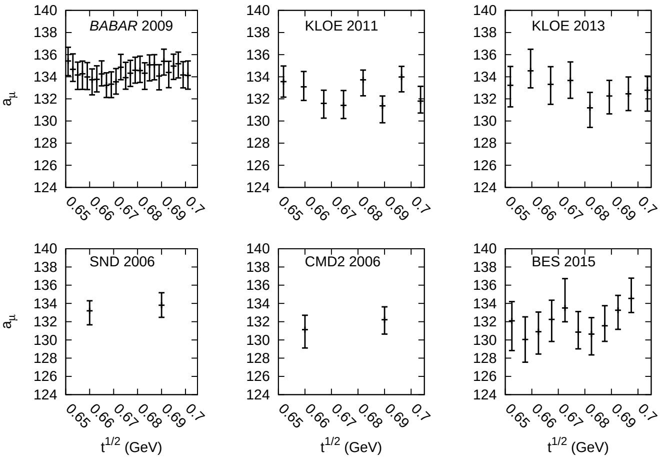
Distributions of a_μ values (Bern phase)

- Distribution of a_μ ($\times 10^{10}$) obtained without input from the timelike region

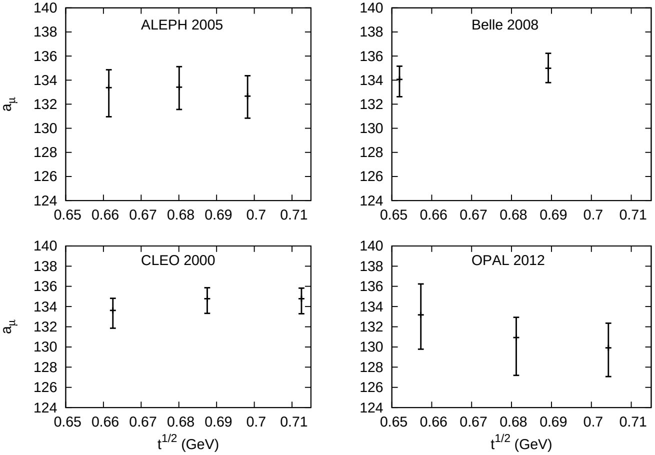


- Distribution of a_μ ($\times 10^{10}$) obtained using as input a timelike modulus from *BABAR*





Allowed intervals at 68.3% C.L. for $a_\mu \equiv a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[2m_\pi,\, 0.63\,\mathrm{GeV}] \times 10^{10}$, as a function of the energy where the timelike modulus was used as input



Allowed intervals at 68.3% C.L. for $a_\mu \equiv a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[2m_\pi,\, 0.63\,\mathrm{GeV}] \times 10^{10}$, as a function of the energy where the timelike modulus was used as input

Combining results with input from different timelike energies

Averaging prescription where the effective size of the correlations is estimated from the data themselves Schmelling (1995), PDG

- Average: given n values a_i with errors δ_i, the most robust prescription is

$$\bar{a} = \sum_{i=1}^{n} w_i a_i, \qquad w_i = \frac{1/\delta a_i^2}{\sum_{j=1}^{n} 1/\delta a_j^2}$$

Averaging prescription where the effective size of the correlations is estimated from the data themselves Schmelling (1995), PDG

- Average: given n values a_i with errors δ_i, the most robust prescription is

$$\bar{a} = \sum_{i=1}^{n} w_i a_i, \qquad w_i = \frac{1/\delta a_i^2}{\sum_{j=1}^{n} 1/\delta a_j^2}$$

- Standard deviation $\sigma(\bar{a})$:
  - Define $\chi^2(f) = \sum_{i,j=1}^{n} (a_i - \bar{a})(C(f)^{-1})_{ij}(a_j - \bar{a})$

$$C_{ij} = \begin{cases} \delta a_i \delta a_i & \text{if } i = j, \\ f \delta a_i \delta a_j & \text{if } i \neq j, \quad f \in [0,1] \end{cases}$$

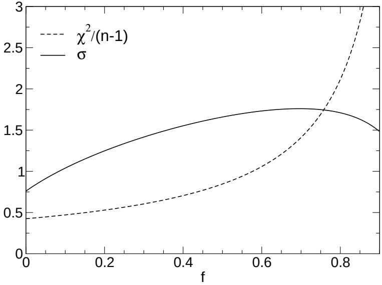
  - If $\chi^2(0) < n - 1$: the data might indicate the existence of a positive correlation. Increase f until $\chi^2(f) = n - 1$ and adopt the variance

$$\sigma^2(\bar{a}) = \left(\sum_{i,j=1}^{n} (C(f)^{-1})_{ij} \right)^{-1}$$

  - If $\chi^2(0) > n - 1$: indication that the individual errors are underestimated. Rescale $\sigma^2(\bar{a})$ by the factor $\chi^2(0)/(n-1)$.

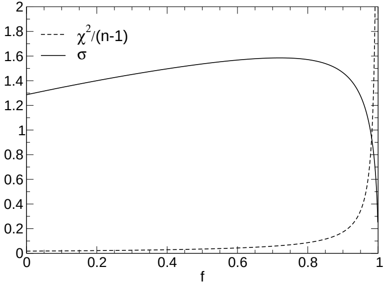
Typical situation

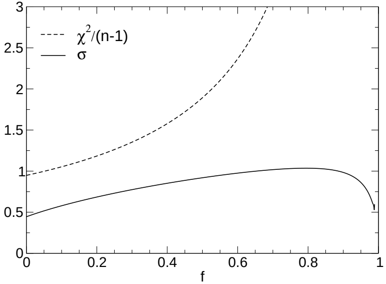
- The prescription indicated a positive correlation in all cases
- Illustration for BESIII and Madrid phase:



Dependence on f of the ratio $\chi^2(f)/(n-1)$ and of the standard deviation $\sigma \equiv \sqrt{\sigma^2(f)}$. The error is obtained with f determined from the equation $\chi^2(f)/(n-1) = 1$.

Pathologies





Left (CMD2 and Madrid phase): The equality $\chi^2(f)/(n-1) = 1$ holds for f very close to 1, because the individual values are much closer than expected from the ascribed errors. For f close to 1, $\sigma^2(f)$ starts to decrease, so the blind application of the prescription would lead to an unreliably small error.

Right (KLOE 11 and Madrid phase): The equality $\chi^2(f)/(n-1) = 1$ holds for f very close to 0, because the individual values are rather different and their errors are too small. A further error reduction by their combination is not reliable.

Conservative approach: take the maximum error for f in the range (0, 1)

	Bern phase	Madrid phase
CMD2	131.804 ± 1.563	131.396 ± 1.585
SND	133.535 ± 1.371	133.102 ± 1.306
BABAR	134.338 ± 0.939	134.086 ± 0.862
KLOE 11	132.560 ± 1.220	132.017 ± 1.035
KLOE 13	132.864 ± 1.413	132.343 ± 1.224
BESIII	131.958 ± 1.725	132.753 ± 1.719
CLEO	134.478 ± 1.389	133.897 ± 1.183
OPAL	131.176 ± 2.803	129.910 ± 2.970
ALEPH	133.114 ± 1.703	132.298 ± 1.783
Belle	134.588 ± 1.227	134.280 ± 1.136

Central values and standard deviations for $a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[2m_\pi,\, 0.63\,\mathrm{GeV}] \times 10^{10}$, obtained by combining the results from different energies for each experiment

Average of different experiments

- The prescription indicated a positive correlation between the values from different experiments
- The results from the two phases have been combined in a simple average
- The data from e^+e^- and τ-decay experiments are consistent in the region $0.65 - 0.71$ GeV $\Rightarrow$ the results from all 10 experiments can be combined into a single average:

$$a_\mu^{\pi\pi(\gamma),\,\mathrm{LO}}[2m_\pi,\, 0.63\,\mathrm{GeV}] = (133.258 \pm 0.723) \times 10^{-10}$$

Direct determination: $(133.2 \pm 1.3) \times 10^{-10}$ Davier et al. (2010)

Summary

- The low-energy hadronic VP contribution to a_μ has a relatively large error due to the low experimental accuracy amplified by the QED kernel
- I presented an attempt to reduce this error based on the analyticity and unitarity properties of the pion electromagnetic form factor
  - The strategy was to use, instead of the modulus at low energies, the phase in the elastic region and measurements of the modulus outside the low-energy region
  - By solving a suitable extremal problem, upper and lower bounds on $|F(t)|$ at low energies have been obtained in a parametrization-free approach
  - The bounds are optimal and independent on the unknown phase of $F(t)$ above the inelastic threshold
  - The uncertainties of the input have been included by statistical simulations
- The result for the contribution to $a_\mu^{\pi\pi,\,\mathrm{LO}}$ of energies below 0.63 GeV is consistent with the direct determination from combined e^+e^- data
- The error has been reduced by about 0.6×10^{-10} (a factor of 2)
- Inclusion of forthcoming data from CMD-3 and SND at VEPP-2000 collider in Novosibirsk expected to improve the precision