On the thermodynamics of Kaluza-Klein black holes

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Iberian Strings 2023, Murcia

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Introduction

The 4 laws of BH mechanics (Bardeen, Carter & Hawking 1973)

- 0th law : κ constant on $\mathcal{H}$ (Racz & Wald)
- 1st law : $\delta M = \frac{\kappa \delta A}{8\pi} + \Omega \delta J$ (in GR)
- 2nd law : $\delta A > 0$ (in GR)
- 3rd law : $\kappa \longrightarrow 0$ impossible (?)

Hawking (1974) $T_H = \frac{\kappa}{2\pi} \Rightarrow S_{BH} = \frac{A}{4}$ (Bekenstein-Hawking)

What happens to these laws beyond GR?

"S is the Noether charge" (Wald 1993)

Noether's 2nd th.

d-form

$$S[\phi] = \int L(\phi) = \int d^dx\, \mathcal{L}(\phi)\,;$$

$$\delta S = \int \left\{ E_\phi \wedge \delta\phi + d\Theta(\phi, \delta\phi) \right\};$$

$$\frac{\delta S}{\delta \phi} = 0 :$$ Euler-Lagrange eqs. $$\frac{\delta S}{\delta\phi} \equiv E_\phi\,;$$

$$\delta_\Lambda \phi \quad \rightarrow \quad \delta_\Lambda S = \int dB(\Lambda, \phi)\,;$$

$$= \int \left\{ E_\phi \wedge \delta_\Lambda\phi + d\Theta(\phi, \delta_\Lambda\phi) \right\}$$

Noether identities $\longrightarrow$

$$= \int d\Theta'(\phi, \delta_\Lambda \phi)\,;$$

Noether's 2nd th.

$$\int d \left\{ \underbrace{\Theta'(\phi, \delta_\Lambda \phi) - B(\phi, \delta_\Lambda \phi)}_{J[\Lambda]} \right\} = 0$$

(d-2)-form

$$\Rightarrow \; d\, J[\Lambda] = 0 ; \quad \Rightarrow \; J[\Lambda] = d\, Q[\Lambda] ;$$

$$\delta_K \phi = 0 \quad \Rightarrow \quad J[K] \doteq 0 ; \quad \Rightarrow \quad d\, Q[K] \doteq 0$$

$$\Rightarrow \quad q(K) \equiv \int_{\Sigma^{d-2}} Q(K) ; \quad \text{conserved charge}$$

cobordism

$$\int_{\Sigma_1^{d-2}} Q(K) - \int_{\Sigma_2^{d-2}} Q(K) = \int_{\Sigma^{d-1}} d\, Q(K) \doteq 0 ;$$

Noether's 2nd th.

Example: Maxwell field $A = A_\mu dx^\mu$; $F = dA$;

$$S[A] = (-1)^d \int \frac{1}{2} F \wedge * dF;$$

$$\delta S = \int \Big\{ \underbrace{-d*F}_{E_A} \wedge \delta A + d\big(\underbrace{*F \wedge \delta A}_{\Theta(A,\delta A)}\big) \Big\}$$

$$\delta_\alpha A = d\alpha; \quad \delta_\alpha S = 0$$

$$= \int \Big[-d*F \wedge d\alpha + d(*F \wedge d\alpha) \Big]$$

$$= \int \Big\{ (-1)^{d-1} \underbrace{dE_A}_{=0} \alpha + d\Big(\overbrace{(-1)^{d-2} d*F \alpha + *F \wedge d\alpha}^{\Theta'(A,\alpha) \,=\, J[\alpha]} \Big) \Big\}$$

Noether's 2nd th.

$$J[\alpha] = d\big[\underbrace{(-1)^{d-2} * F\,\alpha}_{Q[\alpha]}\big] \;;$$

$$\delta_\alpha A = 0 \quad \Rightarrow \quad \alpha \text{ constant} \quad (\alpha = 1)$$

$$\boxed{q = \int_{\Sigma^{d-2}} * F \;;} \qquad (d * F \doteq 0)$$

Diff invariance ? $\rightarrow$ Noether-Wald charge

NW charge and 1st law

1) Diff-invariant theory $\qquad \delta_\xi S = -\int d\, i_\xi L$

2) $\delta_\xi S = \int d\,\Theta'(\phi, \delta_\xi \phi) \Rightarrow d\Big(\underbrace{\Theta'(\phi, \delta_\xi\phi) + i_\xi L}_{J[\xi] \,=\, dQ[\xi]}\Big) = 0$

3) $\xi = k \;/\; \delta_k \phi = 0 \;\&\; \dfrac{\delta S}{\delta \phi} = 0$

$$\boxed{d\left(\delta \mathbf{Q}[k] + \imath_k \Theta'\right) \doteq 0} \longrightarrow \text{1st law}$$

Komar charge

$$\boxed{d\left\{\mathbf{Q}[k] - \omega_k\right\} \doteq 0} \longrightarrow \text{Smarr formulae}$$

$\left(d\omega_k \doteq \imath_k \mathbf{L}\right)$

(Iyer & Wald 1994, Liberati & Pacilio 2015, Mitsios, O., Pereñiguez 2021)

NW charge and 1st law

Special setup:

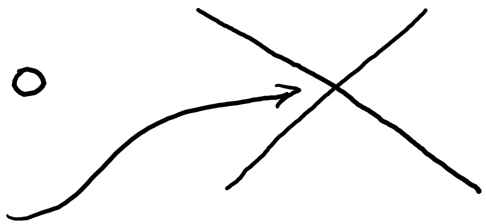
i) A stationary BH spacetime

$\to \{\partial_t, \partial_{\phi^\mu}\}$ Killing

$\to \ell \equiv \partial_t - \Omega^\mu \partial_{\phi^\mu}$ / $\ell^2 \overset{\mathcal{H}}{=} 0$ (Killing horizon)

ii) $\mathcal{H}$ bifurcate horizon $\overset{\text{Boyer}}{\Longrightarrow}$ $\kappa \neq 0$

Bifurcation surface ($\mathcal{BH}$) $\quad \ell \overset{\mathcal{BH}}{=} 0$



$d\ell \overset{\mathcal{BH}}{=} 2\kappa n$; $n_{\mu\nu} n^{\mu\nu} = -2$

iii) Spacelike hypersurface Σ^{d-1} / $\partial \Sigma^{d-1} = \mathcal{BH} \cup S^{d-2}_\infty$

NW charge and 1st law

Integrate $\boxed{d\left(\delta \mathbf{Q}[k] + \imath_k \Theta'\right) \doteq 0}$ over Σ^{d-1} and apply Stokes' theorem

or $\boxed{d\left\{\mathbf{Q}[k] - \omega_k\right\} \doteq 0}$

$$\int_{BH} = \int_{S^{d-2}_\infty}$$

Iyer-Wald

$$\int_{S^{d-2}_\infty} \delta Q[k] + \imath_k \Theta' = \delta(\text{conserved charge associated to } k)$$

$$= \delta M - \Omega^{m} \delta J_{m}$$

GR Iyer-Wald

$$\int_{BH} \delta Q[k] + \cancel{\imath_k \Theta'} = \int_{BH} \delta Q[k] = \int_{BH} \delta \frac{*}{16\pi} d\hat{k} = \frac{\kappa}{8\pi} \delta \int_{BH} \hat{n} = \frac{\kappa}{8\pi} \delta A;$$

Coupling to matter

Iyer-Wald (1994) : for matter transforming under diffs as tensor fields

$$S = -2\pi \int_{BH} d^{d-2} \Sigma_{\mu\nu} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} n_{\rho\sigma} \; ;$$

$\Rightarrow$ the presence of matter doesn't matter

Problems :

i) How do the work terms arise?

ii) Incorrect results in some theories!

$$\left(\frac{\partial S}{\partial M} \neq \frac{1}{T} \right)$$

Coupling to matter

MOST FIELDS
ARE **NOT**
TENSORS

They are sections of fiber bundles or
more complicated mathematical
structures (gerbes ...)

Gauge fields and diffs

A more pedestrian point of view:

Most fields have gauge freedoms $\delta_\Lambda \phi$

Diffeomorphisms induce gauge transformations

(Are spinors scalars under diffs?)

It is necessary to take into account the induced gauge transformations

$$\delta_\xi \phi = -\pounds_\xi \phi \longrightarrow \delta_\xi \phi = -\mathbb{L}_\xi \phi$$

$$[\pounds_\xi, \delta_\Lambda] \neq 0 \longrightarrow [\mathbb{L}_\xi, \delta_\Lambda] = 0$$

Gauge fields and diffs

If k generates a symmetry $\delta_k \phi = 0$, and ϕ has a gauge freedom, $\delta_k \phi \neq -\pounds_k \phi$ because $[\pounds_k, \delta_\Lambda] \neq 0$ and $\pounds_k \phi = 0$ is not gauge-invariant (or meaningful)

Demand $\quad \pounds_k \phi = \delta_{\Lambda_k} \phi \quad \Longrightarrow \quad \mathbb{L}_k \phi \equiv \underbrace{\pounds_k \phi - \delta_{\Lambda_k} \phi}_{\text{gauge-covariant}}$

We have to determine Λ_k (Λ_ξ)

$\Longrightarrow$ Additional terms in $Q[\xi]$ $\Longrightarrow$ 1st law and Smarr f.

$$S \neq \int Q[k]$$

Maxwell field

$$\delta_\alpha A = d\alpha\,; \qquad \Rightarrow \quad \delta_k A = -\mathcal{L}_k A + \delta_{\alpha_k} A = 0\,; \qquad \alpha_k\,?$$

$$\delta_\alpha F = 0\,; \qquad \Rightarrow \quad \delta_k F = -\mathcal{L}_k F = -(\cancel{\iota_k d} + d\iota_k)F = 0$$

$$\Rightarrow \quad \exists\, P_k \;/\; \iota_k F = -dP_k\,; \quad \text{locally}$$

momentum map   m-map equation

$$-\mathcal{L}_k A + \delta_{\alpha_k} A = -(\iota_k d + d\iota_k)A + d\alpha_k = -\iota_k F + d(\alpha_k - \iota_k A)$$

$$= d(\alpha_k - \iota_k A + P_k) = 0\,; \quad \Rightarrow \boxed{\alpha_k = \iota_k A - P_k}$$

$$\Rightarrow \quad \boxed{\delta_\xi A = -\underbrace{(\iota_\xi F + dP_\xi)}_{\mathbb{L}_\xi A}}\,; \qquad \boxed{\begin{array}{c}\mathbb{L}_k A = 0 \\ \text{(gauge-invariant)}\end{array}}$$

The Vielbein

(or any other Lorentz field)

(Lichnerowicz (1963), Kosmann (1966, 1972) Hurley & Vandyke (1988, 1994), Figueroa-O'Farrill (1999), O. (2002))

$$e^a = e^a{}_\mu \, dx^\mu ;$$

$$\delta_\sigma e^a = \sigma^a{}_b \, e^b ; \quad \Rightarrow \quad \delta_\ell \, e^a = -\pounds_\ell \, e^a + \delta_{\sigma_\ell} e^a ;$$

$$\sigma_\ell{}^a{}_b = \iota_\ell \, \omega^a{}_b - P_\ell{}^a{}_b ;$$

$$P_\ell{}^{ab} = \nabla^{[a} \ell^{b]} ;$$

(Killing bivector - Lorentz m-map)

$$\iota_\ell R^a{}_b = -\mathcal{D} P_\ell{}^a{}_b$$

(Lorentz m-map equation)

$$\delta_\ell \, e^a = -\pounds_\ell \, e^a + \delta_{\sigma_\ell} e^a = -(\iota_\ell d + d \iota_\ell) e^a + (\iota_\ell \, \omega^a{}_b - P_\ell{}^a{}_b) e^b$$

$$= (\nabla^a \ell_b + \nabla_b \ell^a) e^b = 0$$ for Killing vectors ℓ!

GR 0th laws

(Prabhu (2015), Elgood, Meessen, O. (2018))

In our setup $\quad dP_\ell = -\iota_\ell F \overset{\mathcal{BH}}{=} 0 \quad \Rightarrow \quad P_\ell \overset{\mathcal{BH}}{=} \text{constant}$

$P_\ell = \Phi$ (electrostatic potential)

$\Phi \overset{\mathcal{BH}}{=}$ constant : generalized 0th law restricted to $\mathcal{BH}$

In this case, we can easily extend it to $\mathcal{H}$

$$\iota_\ell dP_\ell = -\iota_\ell \iota_\ell F = 0 \Rightarrow \Phi = P_\ell \overset{\mathcal{H}}{=} \text{constant}$$

generalized 0th law

Another example $\quad \mathcal{D} P_\ell{}^{ab} = -\iota_\ell R^{ab} \overset{\mathcal{BH}}{=} 0 \Rightarrow P_\ell{}^{ab} \overset{\mathcal{BH}}{=}$ covariantly constant

The binormal $\quad \mathcal{D} n^{ab} \overset{\mathcal{BH}}{=} 0$

If there are no more covariantly constant T^{ab}, $\quad P_\ell{}^{ab} \overset{\mathcal{BH}}{=} c\, n^{ab}$

$c \overset{\mathcal{BH}}{=}$ constant $\quad (c = \kappa \ldots)$

Einstein-Maxwell

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$$Q[\ell] = *(e^a \wedge e^b) \wedge P_{\ell\, ab} - P_\ell *F$$

Einstein-Maxwell

$$Q[\ell] = \underbrace{*(e^a \wedge e^b) \wedge P_\ell}_{\text{gravitational}} \text{ or } -P_\ell * F$$

Einstein-Maxwell

$$Q[\ell] = \overbrace{*(e^a \wedge e^b) \wedge P_\ell}^{\text{gravitational}} \text{ or } \overbrace{-P_\ell *F}^{\text{electric}}$$

Einstein-Maxwell

$$Q[\ell] - \omega_\ell \doteq \Big\{ \overbrace{*(e^a \wedge e^b) \wedge P_\ell\ ab}^{\text{gravitational}} \overbrace{- P_\ell * F}^{\text{electric}} \Big\}$$

$$+ \frac{1}{2} \Big\{ P_\ell * F + \tilde{P}_\ell F \Big\}$$

Einstein-Maxwell

$$Q[\ell] - \omega_\ell \doteq \Big\{ \overbrace{\star(e^a \wedge e^b) \wedge P_\ell}^{\text{gravitational}} \ \text{or}\ \overbrace{-P_\ell \star F}^{\text{electric}} \Big\} + \frac{1}{2}\Big\{ P_\ell \star F + \tilde{P}_\ell F \Big\}$$

magnetic momentum map

Einstein-Maxwell

$$Q[\ell] - \omega_\ell \doteq \Big\{ \overbrace{*(e^a \wedge e^b)\wedge P_\ell{}_{ab}}^{\text{gravitational}} \; \overbrace{-P_\ell *F}^{\text{electric}} \Big\}$$

$$+ \frac{1}{2}\Big\{ P_\ell *F + \tilde{P}_\ell F \Big\}$$

magnetic momentum map

$$\doteq \Big\{ *(e^a \wedge e^b)\wedge P_\ell{}_{ab} - \frac{1}{2}\Big(P_\ell *F - \tilde{P}_\ell F \Big) \Big\} ;$$

Einstein-Maxwell

magnetic momentum map

$$Q[\ell] - \omega_\ell \doteq \Big\{ \overbrace{\star(e^a\wedge e^b)\wedge P_\ell\, \sigma_{ab}}^{\text{gravitational}}\ \overbrace{-P_\ell \star F}^{\text{electric}} \Big\} + \frac{1}{2}\Big\{ P_\ell \star F + \tilde{P}_\ell\, F \Big\}$$

$$\doteq \Big\{ \star(e^a\wedge e^b)\wedge P_\ell\, \sigma_{ab} - \frac{1}{2}\underbrace{\left(P_\ell \star F - \tilde{P}_\ell\, F\right)}_{\text{symplectic-invariant}} \Big\}\,;$$

Einstein-Maxwell

$$\underbrace{Q[\ell] - \omega_\ell}_{\text{Komar charge}} \doteq \Big\{ \overbrace{\star(e^a \wedge e^b) \wedge P_{\ell\, ab}}^{\text{gravitational}} \overbrace{- P_\ell \star F}^{\text{electric}} \Big\}$$

$$+ \frac{1}{2} \Big\{ P_\ell \star F + \tilde{P}_\ell F \Big\}$$

magnetic momentum map

$$\doteq \Big\{ \star(e^a \wedge e^b) \wedge P_{\ell\, ab} - \frac{1}{2} \underbrace{\big(P_\ell \star F - \tilde{P}_\ell F \big)}_{\text{symplectic-invariant}} \Big\} ;$$

KK theories

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Fiber bundles are geometric objects (spaces)

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KK spaces are spacetime realizations of fiber bundles

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The metric "sees" the fiber.

KK theories

Fiber bundles are geometric objects (spaces)

KK spaces are spacetime realisations of fiber bundles.

The metric "sees" the fiber.

We should recover our results in a "spacetime geometric" form.

Basic KK theory

$$ds^2_{(5)} = \hat{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} x^{\hat{\nu}} . \qquad S[\hat{g}] = \frac{1}{16\pi G_N^{(5)}} \int d^5x \sqrt{|\hat{g}|}\, \hat{R} ,$$

$$\hat{k} = \hat{k}^{\hat{\mu}} \partial_{\hat{\mu}} \qquad (x^{\hat{\mu}}) = (x^{\mu}, x^4 \equiv z) \qquad z \in [0, 2\pi\ell]$$

$$\left\{ \begin{array}{l} \hat{k} = \partial_{\underline{z}} \\ \partial_{\underline{z}} \hat{g}_{\hat{\mu}\hat{\nu}} = 0. \end{array} \right.$$

$$ds^2_{(5)} = ds^2_{(4)} - k^2 (dz + A)^2 \qquad \left\{ \begin{array}{l} \hat{k}^2 = \hat{g}_{\underline{zz}} \equiv -k^2 \\ A_{\mu} \equiv \hat{g}_{\mu\underline{z}} / \hat{g}_{\underline{zz}} , \qquad A \equiv A_{\mu} dx^{\mu} , \end{array} \right.$$

Basic KK theory

$$\delta_{\hat{\xi}} \hat{g}_{\hat{\mu}\hat{\nu}} = -\pounds_{\hat{\xi}} \hat{g}_{\hat{\mu}\hat{\nu}} = -\left(\hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{\hat{\mu}\hat{\nu}} + 2\partial_{(\hat{\mu}} \hat{\xi}^{\hat{\rho}} \hat{g}_{\hat{\nu})\hat{\rho}} \right)$$

Basic KK theory

$$\delta_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = -\pounds_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = -\left(\hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} + 2\partial_{(\hat{\mu}}\hat{\xi}^{\hat{\rho}}\hat{g}_{\hat{\nu})\hat{\rho}}\right) \Rightarrow \begin{cases} \delta_{\hat{\xi}}k = -\hat{\xi}^{\rho}\partial_{\rho}k\,, \\ \delta_{\hat{\xi}}A_{\mu} = -\left(\hat{\xi}^{\rho}\partial_{\rho}A_{\mu} + \partial_{\mu}\hat{\xi}^{\rho}A_{\rho}\right) - \partial_{\mu}\hat{\xi}^{z}\,, \\ \delta_{\hat{\xi}}g_{\mu\nu} = -\left(\hat{\xi}^{\rho}\partial_{\rho}g_{\mu\nu} + 2\partial_{(\mu}\hat{\xi}^{\rho}g_{\nu)\rho}\right)\,. \end{cases}$$

Basic KK theory

$$\delta_{\hat\xi}\hat g_{\hat\mu\hat\nu} = -\pounds_{\hat\xi}\hat g_{\hat\mu\hat\nu} = -\left(\hat\xi^{\hat\rho}\partial_{\hat\rho}\hat g_{\hat\mu\hat\nu} + 2\partial_{(\hat\mu}\hat\xi^{\hat\rho}\hat g_{\hat\nu)\hat\rho}\right)$$

$$\left\{\begin{aligned}
\delta_{\hat\xi}k &= -\hat\xi^{\rho}\partial_{\rho}k\,,\\
\delta_{\hat\xi}A_{\mu} &= -\left(\hat\xi^{\rho}\partial_{\rho}A_{\mu} + \partial_{\mu}\hat\xi^{\rho}A_{\rho}\right) - \partial_{\mu}\hat\xi^{z}\,,\\
\delta_{\hat\xi}g_{\mu\nu} &= -\left(\hat\xi^{\rho}\partial_{\rho}g_{\mu\nu} + 2\partial_{(\mu}\hat\xi^{\rho}g_{\nu)\rho}\right)\,.
\end{aligned}\right.$$

$$\left.\begin{aligned}
\chi &\equiv -\hat\xi^{z}\\
\delta_{\chi}A &= d\chi
\end{aligned}\right\}$$

Basic KK theory

$$\delta_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = -\pounds_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = -\left(\hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} + 2\partial_{(\hat{\mu}}\hat{\xi}^{\hat{\rho}}\hat{g}_{\hat{\nu})\hat{\rho}}\right) \Rightarrow \left\{ \begin{aligned} \delta_{\hat{\xi}}k &= -\hat{\xi}^{\rho}\partial_{\rho}k\,, \\ \delta_{\hat{\xi}}A_{\mu} &= -\left(\hat{\xi}^{\rho}\partial_{\rho}A_{\mu} + \partial_{\mu}\hat{\xi}^{\rho}A_{\rho}\right) - \partial_{\mu}\hat{\xi}^{\underline{z}}\,, \\ \delta_{\hat{\xi}}g_{\mu\nu} &= -\left(\hat{\xi}^{\rho}\partial_{\rho}g_{\mu\nu} + 2\partial_{(\mu}\hat{\xi}^{\rho}g_{\nu)\rho}\right)\,. \end{aligned} \right.$$

$$\left. \begin{aligned} \chi &\equiv -\hat{\xi}^{\underline{z}} \\ \delta_{\chi}A &= d\chi \end{aligned} \right\}$$

Why compensating gauge transformations?

Basic KK theory

$$S[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

Basic KK theory

$$S[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

$$\hat{e}^a = e^a \,,$$

$$\hat{e}^z = k(dz + A) \,,$$

Basic KK theory

$$\hat{e}^{a} = e^{a}\,,$$

$$S[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

$$\hat{e}^{z} = k(dz + A)\,,$$

$$S[e, A, k] = \frac{1}{16\pi G_N^{(5)}} \int \left\{k\left[-\star(e^{a} \wedge e^{b}) \wedge R_{ab} + \tfrac{1}{2}k^{2}F \wedge \star F\right] + d\left[2 \star dk\right]\right\} \wedge dz$$

Basic KK theory

$$\hat{e}^a = e^a \,,$$

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$$S[e, A, k] = \frac{2\pi\ell}{16\pi G_N^{(5)}} \int \left\{ k \left[-\star (e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} k^2 F \wedge \star F \right] + d \left[2 \star dk \right] \right\}$$

Basic KK theory

$$S[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

$$\hat{e}^a = e^a \,,$$

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$$S[e, A, k] = \frac{1}{16\pi G_N^{(5)}} \int \left\{ k \left[-\star\,(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2}k^2 F \wedge \star F \right] + d\,[2 \star dk] \right\} \wedge dz$$

$$S[e, A, k] = \frac{2\pi\ell\, k_\infty}{16\pi G_N^{(5)}} \int \left\{ \frac{k}{k_\infty} \left[-\star\,(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2}k^2 F \wedge \star F \right] + d\,[2 \star dk] \right\}$$

$$k_\infty = R/\ell$$

Basic KK theory

$$\hat{e}^a = e^a\,,$$

$$S[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

$$\hat{e}^z = k(dz + A)\,,$$

$$S[e, A, k] = \frac{1}{16\pi G_N^{(5)}} \int \left\{ k \left[-\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2}k^2 F \wedge \star F \right] + d\left[2 \star dk\right] \right\} \wedge dz$$

$$S[e, A, k] = \frac{2\pi\ell\, k_\infty}{16\pi G_N^{(5)}} \int \left\{ \underbrace{k}_{k_\infty} \left[-\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2}k^2 F \wedge \star F \right] + d\left[2 \star dk\right] \right\}$$

$$k_\infty = R/\ell$$

$$G_N^{(4)} = \frac{G_N^{(5)}}{2\pi R}$$

Basic KK theory

$$\hat{e}^a = e^a \,,$$

$$S[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

$$\hat{e}^z = k(dz + A) \,,$$

$$S[e, A, k] = \frac{1}{16\pi G_N^{(5)}} \int \left\{ k \left[-\star (e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} k^2 F \wedge \star F \right] + d\left[2 \star dk \right] \right\} \wedge dz$$

$$S[e, A, k] = \frac{2\pi\ell\, k_\infty}{16\pi G_N^{(5)}} \int \left\{ \frac{k}{k_\infty} \left[-\star (e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} k^2 F \wedge \star F \right] + d\left[2 \star dk \right] \right\}$$

$$k_\infty = R/\ell$$

$$G_N^{(4)} = \frac{G_N^{(5)}}{2\pi R}$$

$$k_E \equiv k/k_\infty$$

Basic KK theory

$$k_E = e^{\phi/\sqrt{3}}$$

Basic KK theory

$$k_E = e^{\phi/\sqrt{3}}$$

$$S[e_E, A_E, \phi] = \frac{1}{16\pi G_N^{(4)}} \int \left\{ - \star_E \left({e_E}^a \wedge {e_E}^b \right) \wedge R_{E\,ab} - \tfrac{1}{2} d\phi \wedge \star_E d\phi + \tfrac{1}{2} e^{\sqrt{3}\phi} F_E \wedge \star_E F_E \right\}$$

$$+ \frac{1}{16\pi G_N^{(4)}} \int d\left(-\tfrac{1}{\sqrt{3}} \star_E d\phi \right) ,$$

Basic KK theory

$$k_E = e^{\phi/\sqrt{3}}$$

$$S[e_E, A_E, \phi] = \frac{1}{16\pi G_N^{(4)}} \int \left\{ - \star_E \left({e_E}^a \wedge {e_E}^b \right) \wedge R_{E\,ab} - \tfrac{1}{2} d\phi \wedge \star_E d\phi + \tfrac{1}{2} e^{\sqrt{3}\phi} F_E \wedge \star_E F_E \right\}$$

$$+ \frac{1}{16\pi G_N^{(4)}} \int d\left(-\tfrac{1}{\sqrt{3}} \star_E d\phi \right),$$

$$ds^2_{(5)} = e^{-\phi/\sqrt{3}} ds^2_{E\,(4)} - e^{2\phi/\sqrt{3}} \left[d(Rz/\ell) + A_E \right]^2$$

Electric KK BH

Electric KK BH

$$\left\{\begin{aligned} ds^2_{E\,(4)} &= H^{-1/2} W dt^2 - H^{1/2}\left(W^{-1} dr^2 + r^2 d\Omega^2_{(2)}\right), \\ A_E &= \alpha e^{-\sqrt{3}\phi_\infty/2}\left(H^{-1} - 1\right) dt, \\ e^{\sqrt{3}\phi} &= e^{\sqrt{3}\phi_\infty} H^{3/2}, \end{aligned}\right.$$

$$\left\{\begin{aligned} &H = 1 + \frac{h}{r}, \qquad W = 1 + \frac{w}{r}, \\ &w = h(1 - \alpha^2). \end{aligned}\right.$$

Electric KK BH

$$\begin{cases} ds^2_{E\,(4)} = H^{-1/2} W dt^2 - H^{1/2}\left(W^{-1}dr^2 + r^2 d\Omega^2_{(2)}\right), \\ A_E = \alpha e^{-\sqrt{3}\phi_\infty/2}\left(H^{-1} - 1\right) dt, \\ e^{\sqrt{3}\phi} = e^{\sqrt{3}\phi_\infty} H^{3/2}, \end{cases}$$

$$\begin{cases} H = 1 + \frac{h}{r}, \qquad W = 1 + \frac{w}{r}, \\ w = h(1-\alpha^2). \end{cases}$$

$$u = Rz/\ell - \alpha t, \qquad v = \alpha t,$$

$$\boxed{ds^2_{(5)} = \frac{w}{h}\alpha^{-2}dv^2 - 2du\left(dv + \alpha^{-1}H du\right) - W^{-1}dr^2 - r^2 d\Omega^2_{(2)}.}$$

Basic KK th.: motion

Basic KK th.: motion

$$\left.\begin{aligned} \ddot{x}^{\hat{\mu}} + \hat{\Gamma}_{\hat{\nu}\hat{\rho}}{}^{\hat{\mu}}\dot{x}^{\hat{\nu}}\dot{x}^{\hat{\rho}} &= 0\,, \\ \hat{g}_{\hat{\mu}\hat{\nu}}\dot{x}^{\hat{\mu}}\dot{x}^{\hat{\nu}} &= \alpha\,, \end{aligned}\right\}$$

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$$P_z = -k^2(\dot{z} + A_\rho \dot{x}^\rho)$$

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$$P_z = -k^2 (\dot{z} + A_{\rho} \dot{x}^{\rho})$$

Basic KK th.: motion

electric charge

$$\ddot{x}^{\hat{\mu}} + \hat{\Gamma}_{\hat{\nu}\hat{\rho}}{}^{\hat{\mu}}\dot{x}^{\hat{\nu}}\dot{x}^{\hat{\rho}} = 0\,, \qquad \ddot{x}^{\mu} + \Gamma_{\nu\rho}{}^{\mu}\dot{x}^{\nu}\dot{x}^{\rho} = P_z F^{\mu}{}_{\nu}\dot{x}^{\nu} - \tfrac{1}{2}P_z^2\partial^{\mu}k^{-2}\,,$$

$$\hat{g}_{\hat{\mu}\hat{\nu}}\dot{x}^{\hat{\mu}}\dot{x}^{\hat{\nu}} = \alpha\,, \qquad g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \alpha + k^{-2}P_z^2\,,$$

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mass

$$P_z = -k^2(\dot{z} + A_\rho \dot{x}^\rho)$$

Basic KK th.: motion

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4d lightcones $\longrightarrow$ 5 lightcones

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4d lightcones $\longrightarrow$ 5 lightcones

4d event horizons $\longrightarrow$ 5d event horizons

NW charges

NW charges

$$\mathbf{Q}[\xi] = \frac{1}{16\pi G_N^{(4)}} \left\{ \star_E (e_E{}^a \wedge e_E{}^b) P_{E\,\xi\,ab} - e^{\sqrt{3}\phi} \star_E F_E P_{E\,\xi} + \tfrac{1}{\sqrt{3}} \imath_\xi \star_E d\phi \right\}$$

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$$\mathbf{K}[l] = \mathbf{Q}[l] - \omega_l = \frac{1}{16\pi G_N^{(4)}} \left\{ \star_E (e_E{}^a \wedge e_E{}^b) P_{E\,l\,ab} - \tfrac{1}{2} \left[e^{\sqrt{3}\phi} \star_E F_E P_{E\,l} - F_E \tilde{P}_{E\,l} \right] \right\}$$

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$$\hat{\mathbf{Q}}[\hat{\xi}] = -\frac{1}{16\pi G_N^{(5)}} \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \hat{P}_{\hat{\xi}\,\hat{a}\hat{b}}\,,$$

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Where do the electric and magnetic contributions come from?

The 5d story: KV

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The $d=5$ BH lies at the same place as the $d=4$ BH

The 5d story: KV

The $d=5$ $\mathcal{H}$ lies at the same place as the $d=4$ $\mathcal{H}$

$$l = \partial_t$$

The 5d story: KV

The $d=5$ $\mathcal{H}$ lies at the same place as the $d=4$ $\mathcal{H}$

$l = \partial_t$. $l^2 = l^\mu g_{\mu\nu} l^\nu \stackrel{\mathcal{H}}{=} 0$.

The 5d story: KV

The $d=5$ $\mathcal{H}$ lies at the same place as the $d=4$ $\mathcal{H}$

$$l = \partial_t \qquad\qquad l^2 = l^\mu g_{\mu\nu} l^\nu \stackrel{\mathcal{H}}{=} 0$$

$$l^\mu \hat{g}_{\mu\nu} l^\nu = l^2 - k^2 (\imath_l A)^2 \stackrel{\mathcal{H}}{=} -k^2 (\imath_l A)^2$$

The 5d story: KV

The d = 5 $\mathcal{H}$ lies at the same place as the d=4 $\mathcal{H}$

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$$\hat{l} = l + f\hat{k},$$

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$\hat{l} = l + f\hat{k}\,,$ $\qquad$ $\hat{k} = \partial_{\underline{z}}$

gauge transf.!

The 5d story: KV

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i) $\hat{l}^2 = \hat{l}^{\hat{\mu}} \hat{g}_{\hat{\mu}\hat{\nu}} \hat{l}^{\hat{\nu}} \overset{\mathcal{H}}{=} 0$

gauge transf.!

The 5d story: KV

The d=5 $\mathcal{H}$ lies at the same place as the d=4 $\mathcal{H}$

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$$l^\mu \hat{g}_{\mu\nu} l^\nu = l^2 - k^2 (\imath_l A)^2 \stackrel{\mathcal{H}}{=} -k^2 (\imath_l A)^2$$

$\hat{l} = l + f\hat{k}$, $\qquad \hat{k} = \partial_{\underline{z}}$

i) $\hat{l}^2 = \hat{l}^{\hat{\mu}} \hat{g}_{\hat{\mu}\hat{\nu}} \hat{l}^{\hat{\nu}} \stackrel{\mathcal{H}}{=} 0$

ii) Killing vector of the 5-dimensional metric

gauge transf.!

The 5d story: KV

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ii) Killing vector of the 5-dimensional metric

gauge transf.!

i) $\longrightarrow$ $f = -\imath_l A + g\,,$ where $g|_{\mathcal{H}} = 0$

The 5d story: KV

ii)

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$$\pounds_{\hat{l}}\hat{g}_{\underline{zz}} = -2k\pounds_l k\,, \quad \Rightarrow \pounds_l k = 0$$

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The 5d story: KV

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$$\downarrow$$

$$\imath_l F_E + d(k_\infty g) = 0$$

The 5d story: KV

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$$\imath_l F_E + d(k_\infty g) = 0$$

$$k_\infty g = P_{E\,l} - P_{E\,l}|_{\mathcal{H}} \equiv \overline{P}_{E\,l}$$

The 5d story: KV

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The 5d story: KV

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The 5d story: KV

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$$\downarrow$$

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$$\downarrow$$

$$f = \underbrace{-k_\infty^{-1}\left(\imath_l A_E - \overline{P}_{E\,l}\right)}_{\chi_\ell} \quad \Vert \quad \leftarrow \quad k_\infty g = P_{E\,l} - P_{E\,l}|_{\mathcal{H}} \equiv \overline{P}_{E\,l}$$

$$\pounds_{\hat{\ell}}\hat{g}_{\mu\nu} \rightarrow \pounds_\ell g_{\mu\nu} = 0$$

The 5d story: KV

$$\hat{l} \stackrel{\mathcal{H}}{=} l - k_\infty^{-1} \Omega \hat{k} \, ,$$

The 5d story: KV

$$\hat{l} \overset{\mathcal{H}}{=} l - k_{\infty}^{-1}\Omega\hat{k}\,, \qquad \text{w/} \qquad \Omega = \imath_l A_E|_{\mathcal{H}} = \Phi\,.$$

Electrostatic potential on the horizon

The 5d story: KV

$\hat{l} \stackrel{\mathcal{H}}{=} l - k_{\infty}^{-1}\Omega\hat{k}$, w/ $\Omega = \iota_l A_E|_{\mathcal{H}} = \Phi$. Electrostatic potential on the horizon

Free-falling particles $P_z \equiv \hat{g}_{z\hat{\mu}}\dot{x}^{\mu}$ conserved

The 5d story: KV

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Electrostatic potential on the horizon

Free-falling particles $\qquad P_z \equiv \hat{g}_{z\hat{\mu}}\dot{x}^\mu$ conserved

$$P_z = 0,$$

The 5d story: KV

$$\hat{l} \stackrel{\mathcal{H}}{=} l - k_\infty^{-1}\Omega\hat{k}\,,$$ w/ $$\Omega = \imath_l A_E|_{\mathcal{H}} = \Phi\,.$$ Electrostatic potential on the horizon

Free-falling particles $$P_z \equiv \hat{g}_{\underline{z}\hat{\mu}}\dot{x}^\mu$$ conserved

$$P_z = 0, \longrightarrow \frac{dz}{dt} = -\imath_l A_E\,,$$

The 5d story: KV

$$\hat{\imath} \stackrel{\mathcal{H}}{=} l - k_\infty^{-1}\Omega\hat{k}\,, \qquad \text{w/} \qquad \Omega = \imath_l A_E|_{\mathcal{H}} = \Phi\,.$$

Electrostatic potential on the horizon

Free-falling particles $\quad P_z \equiv \hat{g}_{z\hat{\mu}}\dot{x}^\mu$ conserved

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Frame-dragging

The 5d story: KV

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Electrostatic potential on the horizon

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Frame-dragging

Analog to ZAMOs in Kerr

The 5d story: KV

$$\hat{l} \overset{\mathcal{H}}{=} l - k_\infty^{-1} \Omega \hat{k} \,, \qquad \text{w/} \qquad \Omega = \imath_l A_E|_{\mathcal{H}} = \Phi \,.$$

Electrostic potential on the horizon

Free-falling particles $\quad P_z \equiv \hat{g}_{z\hat{\mu}} \dot{x}^{\mu}$ conserved

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Frame-dragging

Analog to ZAMOs in Kerr

$$\left(\frac{dz}{d\varphi} = (R/\ell)^{-1} \imath_{\partial_\varphi} A_E \,. \right)$$

The 5d story: NW charge

$$\hat{Q}[\hat{\ell}] \overset{\mathcal{J}\ell}{=} \hat{Q}[\ell] - \Phi\, \hat{Q}[\hat{h}]$$

$$P_z \sim q$$

The 5d story: NW charge

$$\hat{Q}[\hat{\ell}] \stackrel{\mathcal{H}}{=} \hat{Q}[\ell] - \Phi \, \hat{Q}[\hat{h}]$$

$P_z \sim q$

Magnetic & scalar terms?

Conclusions/Questions

- Wald's approach gives an interesting point of on BH thermodynamics, conserved charges, hair ...
- The KK framework has allowed us to test the ideas we have proposed to deal with matter couplings (covariant Lie derivatives, momentum maps ...)
- Many things yet to be understood: scalar and magnetic contributions, supersymmetry...
- M-theory derivation of all lower-dimensional laws of BH thermodynamics?

Thanks!