Unitarization of infinite range forces, resonances in gravitational scattering of gravitons: The graviball

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[1] D. Blas,J. Martín-Camalich,JAO,Phys.Lett.B827,136991(2022) 2009.07817

[2] JHEP 08,266(2020) 2010.12459

[3] JAO,Phys.Lett.B835,137568(2022) 2207.08784

Iberian Strings 2023

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Outline

EFT Paradigm

Ultraviolet (UV) Completion
String Theory?

Regularization Group (RG) Flow

EFT for gravitons = General Relativity
+ higher derivative corrections

$E \ll \Lambda$

$\Lambda \lesssim G^{-1/2} = 10^{19}$ GeV

$$S[g]_{\text{gravity}} = \int d^4x \sqrt{-g} \left\{ \frac{2}{\kappa^2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \ldots \right\}$$

$$\kappa^2 = 32\pi G \qquad R \sim R_{\mu\nu} \sim R_{\mu\nu\alpha\beta} \sim \partial^2 \sim p^2 \ll \Lambda^2$$

Introduction

$$S_{\rm grav} = \int d^4x\sqrt{-g}\left\{\frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} \right.$$
$$\left. + c_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \ldots\right\} \quad \kappa^2 = 32\pi G$$

- EFT for $E \ll \Lambda \lesssim G^{-1/2} = 10^{19}$ GeV

Donoghue, PRL72,2996(1994); arXiv:1702.00319[hep-ph]

- For pure gravity: $c_1 = c_2 = c_3 = 0$

Scattering amplitudes in the low-energy EFT satisfy

- Lorentz invariance
- (Perturbative) Unitarity
- Analyticity
- Crossing

QCD

EFT - ChPT

QCD

KG

EFT Chiral Perturbation Theory (ChPT)

$E \ll 4\pi f_\pi \approx 1.2$ GeV

f_π → Pion-decay constant

Massless quark limit: $m_u = m_d = 0$, neglect EM interactions

Chiral symmetry and its spontaneous breaking (SχSB)

SχSB: $SU(2)_L \otimes SU(2)_R \longrightarrow SU(2)_{L+R}$ Isospin symmetry

Pions (π) are the Goldstone bosons

Chiral Lagrangian $U(x) = \exp\left[\frac{i}{f_\pi}\vec{\sigma}\cdot\vec{\pi}\right]$

Weinberg (1979)

Gasser, Leutwyler (1984)

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \ell_1\left(\mathrm{tr}\,\partial_\mu U \partial^\mu U^\dagger\right)^2 + \ell_2\,\mathrm{tr}\left(\partial_\mu U^\dagger \partial_\nu U\right)\mathrm{tr}\left(\partial^\mu U^\dagger \partial^\nu U\right) + \ldots$$

→ low-energy constants

Similarities with Chiral Perturbation Theory (ChPT), pion physics (QCD)

Expansion in powers of derivatives:

$i\partial_\mu \sim p_\mu$ Momentum expansion

EFT valid for $E \ll \Lambda$. Unitarity cutoff $\Lambda_U = 4\pi f_\pi \approx 1.2$ GeV
$\Lambda = M_\rho = 0.77$ GeV

Pion scattering amplitudes are expected to be perturbative
BUT

Surprise

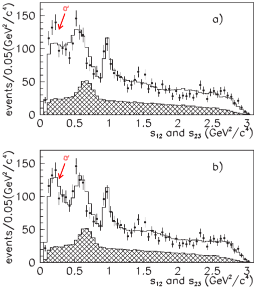


FIG. 2. s_{12} and s_{13} projections for data (error bars) and fast MC (solid line). The shaded area is the background distribution, (a) solution with the Fit 1, and (b) solution with Fit 2.

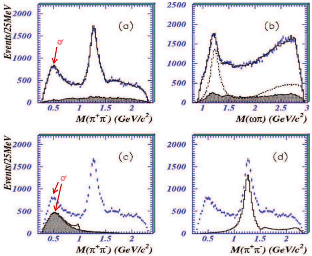


Fig. 2. Mass projections of data compared with the fit (histograms) using Eqs. (10)–(12) for the σ; the shaded region shows background estimated from sidebands. (a) and (b): $\pi\pi$ and $\omega\pi$ mass; the dashed curve in (b) shows the fitted $b_1(1235)$ signal (two charge combinations). (c) and (d): mass projections of 0^{++} and 2^{++} contributions to $\pi^+\pi^-$ from the fit; in (c), the shaded area shows the σ contribution alone, and the full histogram shows the coherent sum of σ and $f_0(980)$.

E791 PRD86,770(2001)

$D^+ \to \pi^-\pi^+\pi^+$

BES PLB598,149(2004)

$J/\Psi \to \omega\pi^+\pi^+$

The σ affects prominently low-energy scalar dynamics in QCD Dobado,Peláez,PRD56,3057(1997) hep-ph/9604416; Oset,JAO,NPA620,438(1997) hep-ph/9702314

σ of $f_0(500)$: Isoscalar Scalar $\pi\pi$ scattering is resonant

Pole positions of the resonance

GKPY Equation García-Martín, Kaminski, Peláez, Ruiz de Elvira, PRL107,072001(2011)
$\sqrt{s_\sigma} = 457^{+14}_{-13} - i\,(297^{+11}_{-7})$ MeV

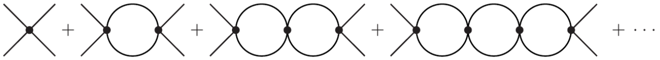
NLO Unitarized ChPT, Albaladejo, JAO, PRD86,034003(2012)
$\sqrt{s_\sigma} = 458 \pm 14 - i\,(261 \pm 17)$ MeV

In an EFT the light degrees of freedom must be accounted for

$$\left|\frac{s_\sigma}{(4\pi f_\pi)^2}\right| = 0.22 \ll 1$$

Enhanced string of diagrams

General $\pi\pi$ interaction



Unitarization: Resumming these infinite string of unitarity-loop diagrams

The σ stems from the unitarization of the $I = J = 0$ $\pi\pi$ ChPT scattering amplitude

This is an example of a Parametric enhancement JAO, Oset, PRD60,074023(1999)

LO 0^{++} $\pi\pi$ ChPT partial-wave amplitude (PWA), σ

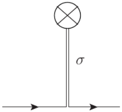
$$T_{00}(s) = \frac{s - m_\pi^2/2}{f_\pi^2}$$

LO 1^{--} $\pi\pi$ ChPT PWA, $\rho(770)$

$$T_{11}(s) = \frac{s - 4m_\pi^2}{6f_\pi^2}$$

⋆ Vacuum, excitations of quark condensate: Scalar form factors of π

$$\langle \pi | \bar{q}q | \pi \rangle$$

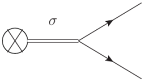


⋆ π-nucleon σ term

$$\langle N | \bar{q}q | N \rangle$$

Great impact in looking for hypothetical supersymmetric dark-matter particles Ellis,Olive, Savage, Phys.Rev.D77,065026(2008)

⋆ Two-pion event distributions from heavy-meson decays



⋆ Large corrections to the current-algebra prediction of 0^{++} $\pi\pi$ scattering lengths, and phase shifts in general

Key Question

Is there a graviball (gravi-σ) in the QG EFT that could affect so much relatively low-energy gravitational physics?

$I = J = 0$ $\pi\pi$ are attractive
$J = 0$ graviton-graviton interactions are also attractive

The Graviball

At Unitarized LO-EFT of QG the relative pole positions of the graviball and σ are very similar

[1] D. Blas, J. Martín-Camalich,JAO, Phys.Lett.B827,136991(2022)

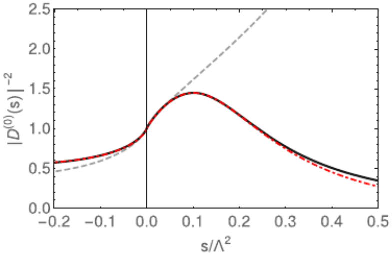
$$\left|\frac{s_P}{\Lambda_{\rm G}^2}\right| \approx \left|\frac{s_\sigma}{\Lambda_{\rm Hadron}^2}\right| = 0.22 \ll 1$$

$$\Lambda_{\rm G} = \pi(G\log a)^{-1} \ , \ \ \Lambda_{\rm Hadron} = 4\pi f_\pi \ ,$$
$$\log a = \mathcal{O}(1)$$

$\frac{s_P}{\Lambda_G^2} = 0.07 - i\,0.20 \to$ Resonant shape peaks at surprisingly low values of s

Omnès function $\Omega^{(J)}(s) = 1/D^{(J)}(s)$

$$\Omega^{(J)}(s) = \left[1 + V^{(J)}(s)g(s)\right]^{-1} = T^{(J)}(s)/V^{(J)}(s)$$



At $s \simeq 0.2$ then $1 \sim V^{(0)}g$ (graviball)

Dashed line: Perturbative $|1 - V^{(0)}(s)g(s)|^2$

Solid line: $|D^{(0)}(s)|^{-2}$

Analogous to $|\Omega^{(0)}_{\pi\pi}(s)|^2$ driving e.g. final-state interactions in $D^+ \to \pi^+\pi^+\pi^-$ [E791]

Making lighter the graviball

By increasing the number $\boldsymbol{N}$ of fields with $m^2 \ll G^{-2}$
The number of channels in the intermediate increases as $\sim N$
The unitarity loop function $g(s) \to Ng(s)$

$$T^{(0)}_{22,22}(s) \approx \left[\frac{\pi}{8Gs\log a} + \frac{\boldsymbol{N}}{8}\log\frac{-s}{\Lambda^2}\right]^{-1}$$

Secular equation

$$\frac{1}{x_N} + N(\log(-x_N) - i2\pi) = 0$$

<u>Solution</u>

$$x_N \approx -i\frac{2}{3\pi N} + i\frac{\log N}{10\pi N}$$

Gravity interactions between fields are attractive

Scenarios with a large number N of light fields:

Dvali,Fortschr.Phys.58,528(2010); Dvali,Redi,PRD77,045027(2008);

Arkani-Hamed,Cohen,D'Agnolo,Hook,Kim,Pinner,PRL117,251801(2016);

Extra Dimensions Arkani-Hamed,Dimopoulos,Dvali,PLB429,263(1998);

Antoniadis,Arkani-Hamed,Dimopoulos,Dvali,PLB436,257(1998);

Dvali,arXiv:0806.3801[hep-th]

Later works on $d = 10$ supergravity

Guerrieri, Penedones, Vieira, PRL127,081601(2021) 2102.02847 *Where Is String Theory in the Space of Scattering Amplitudes?*

Maximal supergravity EFT:

$$S = \frac{1}{(2\pi)^7 \ell_P^8} \int d^{10}x \sqrt{-g} \left[R + \alpha \, \ell_P^6 \, R^4 + \ldots \right]$$

$$16\pi G = (2\pi)^{d-3} \ell_P^{d-2}$$

α is the leading Wilson coefficient sensitive to UV completion of gravity

$d = 10$ supergravity

Guerrieri, Penedones, Vieira, PRL127,081601(2021)

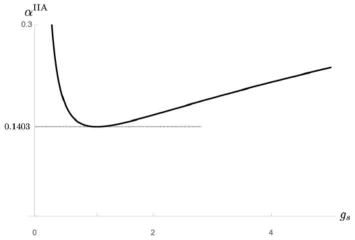
Maximal supergravity EFT:

$$S = \frac{1}{(2\pi)^7 \ell_P^8} \int d^{10}x \sqrt{-g}\left[R + \alpha\, \ell_P^6\, R^4 + \dots\right]$$

α is the leading Wilson coefficient sensitive to UV completion of gravity

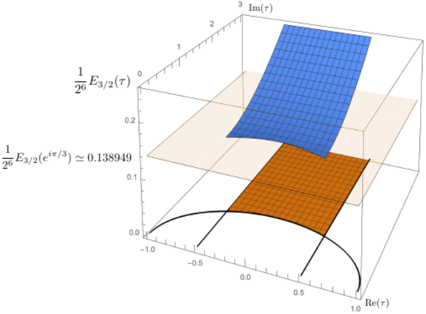
Type IIA superstring theory:

$$\alpha^{II\mathrm{A}} = \frac{\zeta(3)}{32 g_s^{3/2}} + g_s^{1/2}\frac{\pi^2}{96} \geq \frac{\pi^{3/2}\zeta(3)^{1/4}}{24\sqrt{3}} \approx 0.1403$$



Type IIB superstring theory:

$$\alpha^{IIB} = 2^{-6} E_{\frac{3}{2}}(\tau, \bar{\tau}) \geq 2^{-6} E_{\frac{3}{2}}(e^{i\pi/3}, e^{-i\pi/3}) \approx 0.1389$$
$$\tau = \chi_s + (i/g_s)$$



Guerrieri, Penedones, Vieira, PRL127,081601(2021)

Lower bound from string theory

$$\alpha \geq \alpha^{ST}_{\min} \equiv 2^{-6} E_{\frac{3}{2}}(e^{i\pi/3}, e^{-i\pi/3}) \approx 0.1389$$

$d = 10$ supergravity

Maximal supergravity EFT:

$$S = \frac{1}{(2\pi)^7 \ell_P^8} \int d^{10}x \sqrt{-g} \left[R + \alpha\, \ell_P^6\, R^4 + \ldots \right]$$

α is the leading Wilson coefficient sensitive to UV completion of gravity

S-matrix **Bootstrap**

Amplitude ansatz fulfilling

- Lorentz invariance
- Unitarity
- Analyticity
- Crossing
- Low-energy EFT

Guerrieri, Penedones, Vieira, PRL127,081601(2021) carve out the allowed set of parameters (counterterms) compatible with those requirements

Dispersive formula

$$\alpha = \frac{1}{32\pi^8 \ell_P^{14}} \int_0^\infty \frac{ds}{s^5} \mathrm{Im}\, T(s+i\epsilon, t=0) > 0$$

From the supersymmetric action of supergravity

$$\frac{T(s,t,u)}{8\pi G} = s^4 \left(\frac{1}{stu} + \alpha \ell_P^6 + \mathcal{O}(s) \right)$$

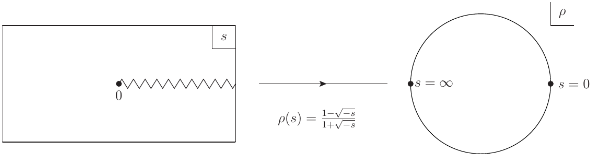
Mandelstam variables

$$s = (p_1 + p_2)^2 \ , \ t = (p_1 - p_3)^2 \ , \ u = (p_1 - p_4)^2 \ , \ s + t + u = 0$$

Ansatz

$[\ell_P = 1]$

$$\frac{T(s,t,u)}{8\pi G} = s^4\left(\frac{1}{stu} + \prod_{A=s,t,u}(\rho_A+1)^2 \sum_{a+b+c\leq N}{}' \alpha_{abc}\rho_s^a\rho_t^b\rho_u^c\right)$$

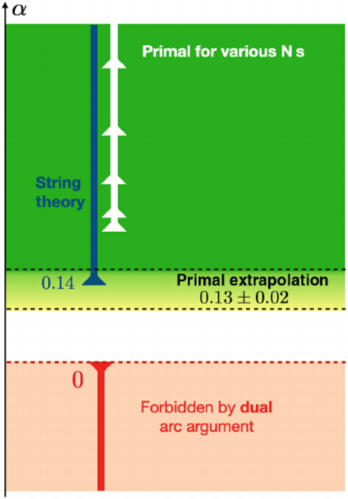


Unitarity bound in partial-wave amplitudes

$$|S_\ell(s)| \leq 1 \;\;,\;\; s \geq 0$$

Parameters are constrained

Results from S-matrix bootstrap



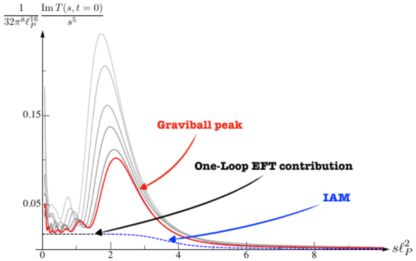
Guerrieri, Penedones, Vieira, PRL127,081601(2021)

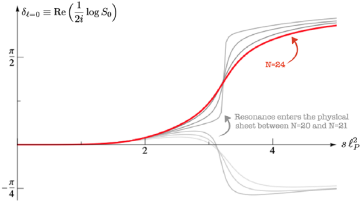
Updated arXiv:2212.00151[hep-th]

Dimension	Bootstrap	String/M-theory
9	0.223 ± 0.002	0.241752
10	0.124 ± 0.003	0.138949
11	0.101 ± 0.005	0.102808

Dominance of the graviball in the DR for α

$$\alpha = \frac{1}{32\pi^8 \ell_P^{14}} \int_0^\infty \frac{ds}{s^5} \mathrm{Im}\, T(s + i\epsilon, t = 0) > 0$$





$$s_P \simeq (3.2 + 0.3\,i)\ \ell_P^{-2}\ (d = 10)$$

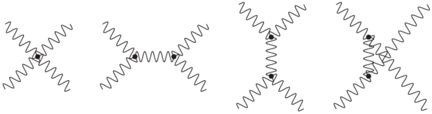
Guerrieri, Penedones, Vieira, PRL127,081601(2021) *"It is therefore tempting to identify the graviball as the first excited string state."*

§3 graviton-graviton scattering

Graviton-graviton Born terms

$$|\mathbf{p}_1,\lambda_1\rangle|\mathbf{p}_2,\lambda_2\rangle \to |\mathbf{p}_3,\lambda_3\rangle|\mathbf{p}_4,\lambda_4\rangle$$

$$\lambda_i = \pm 2\,, \quad \mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2\,, \quad \mathbf{p}' = \mathbf{p}_3 = -\mathbf{p}_4$$



Born terms. Grisaru,van Nieuwenhuizen,Wu,PRD12,397(1975)

$$F_{22,22}(s,t,u) = F_{-2-2,-2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{s^4}{stu}\,,$$

$$F_{-22,-22}(s,t,u) = F_{2-2,2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{u^4}{stu}\,,$$

$$F_{2-2,-22}(s,t,u) = F_{2-2,2-2}(s,u,t) = F_{-22,2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{t^4}{stu}$$

Related by parity and Bose-Einstein symmetry

Infrared (IR) divergences

graviton-graviton PWAs are IR divergent

because of the infinite range of gravity interactions

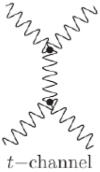
$J = 0$ partial-wave projection of the Born term

$$-\int_{-1}^{+1} \frac{dcos\theta}{t} = \frac{1}{2p^2}\left[\log 2 - \lim_{\theta\to 0}\log(1-\cos\theta)\right]$$

$$t = (\mathbf{p}-\mathbf{p}')^2 = -2\mathbf{p}^2(1-\cos\theta)$$

This is due to the exchange of a *virtual soft* photon(graviton) ($t \to 0$) in between two *external on-shell* lines

Classification of Weinberg in PR140,B516(1965)



soft: $|t| << s$

Phase divergences

This phase was demonstrated by Weinberg in *Infrared Photons and Gravitons*, PR140,B516(1965)

"*the full effect of virtual infrared photons is to contribute to the S matrix for any process $\alpha \to \beta$ a factor*"

$$\frac{S_{\beta\alpha}}{S^0_{\beta\alpha}(\mathcal{L})} = \exp\left\{\frac{1}{2}\int_\mu^{\mathcal{L}} A(q)\right\}$$

The real part of $A(q)$ generates the IR divergence $(\mu/\mathcal{L})^{A/2}$. This is cancelled by real soft-photon emission

Its imaginary part generates the Dalitz phase: *"so each different pair of particles in the initial or final state contributes to the S matrix a phase factor which for $\mu \ll \mathcal{L}$ may be written"*

$$\exp\left\{\frac{i}{2\pi}\frac{e_n e_m}{\beta_{nm}}\log\frac{\mu^2}{\mathcal{L}^2}\right\}\ , \quad \mathcal{L}^2 = \frac{4p^2}{a^2} \to \text{Left-hand cut (LC)}$$

For gravity one also has the same rule

Every pair of particles in the initial or final state contributes with

$$\exp\left\{-i\frac{Gm_n m_m(1+\beta_{nm}^2)}{\beta_{nm}(1-\beta_{nm}^2)^{1/2}}\log\frac{\mu}{\mathcal{L}}\right\}$$

For graviton-graviton scattering (one pair in initial and final states)

$$\begin{aligned} S_c(s) &= \lim_{m\to 0}\exp\left\{-i2\frac{Gm^2(1+\beta^2)}{\beta(1-\beta^2)^{1/2}}\log\frac{\mu}{\mathcal{L}}\right\} \\ &= \exp\left\{-i2Gs\log\frac{\mu}{\mathcal{L}}\right\} \end{aligned}$$

This is our first resummation taken from Weinberg, $\mathcal{L} \ll \sqrt{s}$

The phase does not depend on angle.

It is the same for all PWAs

Redefinition of the S matrix in PWAs

Comparing with Weinberg:

$$S_{\alpha\beta} \to \bar{S}_J \text{ and } S^0_{\alpha\beta} \to S_J$$

$$S_J = S_c^{-1}\bar{S}_J = \exp\left\{2iGs\log\frac{\mu}{\mathcal{L}}\right\}\bar{S}_J$$
$$S_J = S_c^{-1}\bar{S}_J = \exp\left\{\frac{2i\alpha}{\beta}\log\frac{\mu}{\mathcal{L}}\right\}\bar{S}_J$$

Note: S_J is unitary because only a phase factor has been introduced

$$S_J S_J^\dagger = S_J^\dagger S_J = I$$

$$\bar{S}_J = 1 + i\frac{m_r p}{\pi}\bar{T}_J \qquad\qquad S_J = 1 + i\frac{m_r p}{\pi}T_J$$

IR divergent — IR finite

Redefinition of PW-projected Born terms

Up to $\mathcal{O}(G)$

$$S^{(J)} = S_c^{-1}\left(1 + i\frac{\pi 2^{|\lambda|/4}}{4}\bar{T}^{(J)}\right)$$

$$= 1 + i\frac{\pi 2^{|\lambda|/4}}{4}\underbrace{\left(\frac{8Gs}{\pi 2^{|\lambda|/4}}\log\frac{\mu}{\mathcal{L}} + F^{(J)}\right)}_{V^{(J)}} + \mathcal{O}(G^2)$$

Example: $J=0$, $\lambda=0$

$$F^{(0)}_{22,22} = -\frac{\kappa^2 s^2}{16\pi^2}\int_{-1}^{+1}\frac{d\cos\theta}{t-\mu^2} = \frac{\kappa^2 s}{8\pi^2}\log\left(1+\frac{4p^2}{\mu^2}\right)$$
$$\to \frac{8Gs}{\pi}\log\frac{2p}{\mu}$$

Thus

$$V^{(0)}_{22,22}(s) = \frac{8Gs}{\pi}\log\frac{2p}{\mathcal{L}}$$

We know about $\mathcal{L}$

1. By dimensional analysis $\mathcal{L} \propto p$. It stems from

$$\int_{-1}^{+1} \frac{d\cos\theta}{-2\mathbf{p}^2(1-\cos\theta)-\mu^2}$$

2. $\mathcal{L} \ll p$

$$\mathcal{L} = \frac{\sqrt{s}}{a} = \frac{2p}{a}\ , \quad a \gg 1$$

$$V^{(0)}_{22,22} = \frac{8Gs}{\pi}\log a\ , \quad \log a = \mathcal{O}(1)$$

Coulomb scattering: $\log a = \gamma_E$

The removal of the Weinberg phase is analogous to redefining the **asymptotically unperturbed** Hamiltonian.

Kulish and Faddeev *Asymptotic conditions and infrared divergences in quantum electrodynamics*, Theor.Math.Phys.4,745(1970)

Simple quantum argument. Phase shifts

$$\int dr \left(\sqrt{2m(E - \frac{\alpha e^{-\mu r'}}{r'})} - \sqrt{2mE} \right) = \gamma \Gamma(0, \mu r_-)$$
$$= \gamma(-\gamma_E - \log(\mu r_-)) + \mathcal{O}(\mu r_-)$$

There is an accumulation of phase because $1/r$ persists for $r \to \infty$.

This is no longer the case for $1/r^{1+\epsilon}$, $\epsilon > 0$.

Removal of this extra contribution. Coulomb wave functions.

This extra $\log \mu r$ terms do not contribute to the outgoing flux for $r \to \infty$. Cross sections are left invariant

§3 Unitarized PWAs

Unitarity of $S^{(J)}(s)$

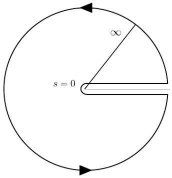
$$\Im \frac{1}{T^{(J)}} = -\frac{\pi 2^{|\lambda|/4}}{8}\theta(s)$$

IR-safe Born terms $V^{(J)}(s)$ do not have left-hand cut (LC)

Unitarized Born terms $\rightarrow T(s)$ only having right-hand cut (RC)

Our second resummation

JAO,Oller,PRD60,074023(1999); JAO,Meißner,PLB500,263(2001)



Contour of integration used for the dispersion of $T^{(J)}(s)^{-1}$

$$g(s) = c(s_0) - \frac{s-s_0}{8}\int_0^{\Lambda^2} \frac{ds'}{(s'-s)(s'-s)}$$
$$= c(s_0) + \frac{1}{8}\log\frac{-s}{s_0}$$

s_0: Subtraction point

$$T^{(J)}_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(s) = \left[\frac{1}{R^{(J)}_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(s)} + 2^{|\lambda|/4} g(s)\right]^{-1}$$

- $R^{(J)}(s)$ has no two-body RC, $\Im g(s) = -\frac{\pi}{8}\theta(s)$
- $R^{(J)}(s)$ is treated in perturbation theory
- $R^{(J)}(s)$ is obtained by matching with the EFT for graviton-graviton scattering

At LO

$$R^{(J)} = V^{(J)} + \mathcal{O}((Gs)^2)$$

Naturalness to estimate $c(s_0)$

Compare $g(s)$ with $g_\Lambda(s)$ calculated with a cutoff

$$g_c(s) = -\frac{1}{8}\int_0^{\Lambda^2} \frac{ds'}{s'-s} = \frac{1}{8}\log\frac{-s}{\Lambda^2} + \mathcal{O}(s/\Lambda^2)$$

by taking $s_0 = \Lambda^2$ then

$$c(\Lambda^2) = 0$$

We will later isolate the dependence on Λ in s_P

Unitarity cutoff Λ_U

Expanding up to one loop $T^{(J)}(s)$

$$T^{(J)} = V^{(J)}\left(1 - \frac{V^{(J)}}{8}\log\frac{-s}{\Lambda^2}\right) + \mathcal{O}\left((Gs)^3\right)$$

If NLO/LO $\sim s/\Lambda_U^2$, from $J = 0$

$$\Lambda_U^2 = \pi(G\log a)^{-1} \sim G^{-1} \ .$$

Blas,Martin Camalich,JAO,PLB827,136991(2022)

$$V^{(0)}_{22,22}(s) = \frac{8Gs}{\pi}\log a$$

$$T^{(0)}_{22,22;II}(s) = \left[\frac{\pi}{8Gs\log a} + \frac{1}{8}\log\frac{-s}{\Lambda^2} - i\frac{\pi}{4}\right]^{-1}$$

$$\omega = \frac{\Lambda^2}{\Lambda_U^2} = \Lambda^2\frac{G\log a}{\pi}~,~\Lambda_U = \frac{\pi}{G\log a}~,~\log a \simeq 1$$

Secular equation

$$\frac{1}{x} + \log(-x) - i2\pi = 0~,~x = \frac{s_P}{\Lambda^2}$$

$$x \simeq -i\frac{2}{3\pi} = -i0.20 \qquad\qquad x \sim \frac{1}{\omega}$$

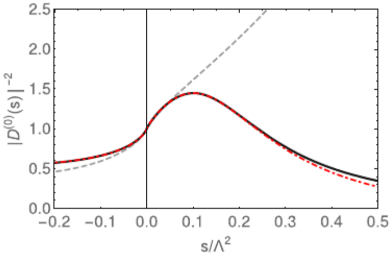
$$x = 0.07 - i\,0.20~,~s_P = 0.22 - i\,0.63~~G^{-1}$$

Estimated 20% uncertainty

$\frac{s_P}{\Lambda_G^2} = 0.07 - i\,0.20 \rightarrow$ Resonant shape peaks at surprisingly low values of s

Omnès function $\Omega^{(J)}(s) = 1/D^{(J)}(s)$

$$\Omega^{(J)}(s) = \left[1 + V^{(J)}(s)g(s)\right]^{-1} = T^{(J)}(s)/V^{(J)}(s)$$



At $s \simeq 0.2$ then $1 \sim V^{(0)}g$ (graviball)

Dashed line: Perturbative $|1 - V^{(0)}(s)g(s)|^2$

Solid line: $|D^{(0)}(s)|^{-2}$

Analogous to $|\Omega^{(0)}_{\pi\pi}(s)|^2$ driving e.g. final-state interactions in $D^+ \rightarrow \pi^+\pi^+\pi^-$ [E791]

Making lighter the graviball

By increasing the number $\boldsymbol{N}$ of fields with $m^2 \ll G^{-2}$
The number of channels in the intermediate increases as $\sim N$
The unitarity loop function $g(s) \to Ng(s)$

$$T^{(0)}_{22,22}(s) \approx \left[\frac{\pi}{8Gs\log a} + \frac{\boldsymbol{N}}{8}\log\frac{-s}{\Lambda^2}\right]^{-1}$$

Secular equation

$$\frac{1}{x_N} + N(\log(-x_N) - i2\pi) = 0$$

Solution

$$x_N \approx -i\frac{2}{3\pi N} + i\frac{\log N}{10\pi N}$$

Gravity interactions between fields are attractive

Include light degrees of freedom in EFT

We have a parametric enhancement in the EFT both for the σ and graviball

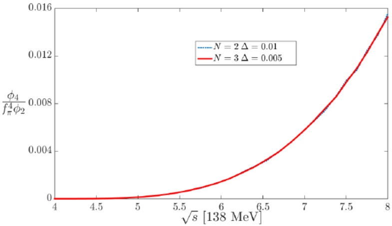
One has to account for such light resonances $|s_P/\Lambda^2| \approx 0.2$

Unitarized EFT is a way to accomplish this by resumming $(s/\Lambda^2)^n$ to account for two-body unitarity along the RC: Unitarity and analyticity

Suppression of phase space for massless multi-particle states Salas-Bernández, Llanes-Estrada, JAO, Escudero-Pedrosa,SciPost Phys.11,020(2021)

Phase space of n massless particles pions

$$\phi_n = \frac{s^{n-2}}{2(4\pi)^{2n-3}(n-1)!(n-2)!}$$



$\sqrt{s} \in [0.55, 1.1]$ GeV , 4π & 2π

Monomials involving three or four Riemannian tensors

$\{R^3\}$: Six derivatives; $\{R^4\}$: Eight derivatives

$\{R^3\}$ gives vanishing contributions to $F_{22,22}$

van Nieuwenhuizen,Wu, J.Math.Phys.18,182(1977)

$\{R^4\}$ contributions to $F_{22,22}$ evaluated in Huber, Brandhuber, De Angelis, Travaglini, PRD102,046014(2020) with spinor formalism

$$F_{R^4;22,22}(s,t,u) = \frac{\widetilde{\beta}\kappa^2}{\pi} s^4$$

$$V^{(0)}_{R^4;22,22}(s) = \frac{8\widetilde{\beta}}{\pi^2} s^4 \;, \quad \widetilde{\beta} \sim \Lambda^{-6}$$

$$V^{(0)}_{R^4;22,22}(s) = \frac{32 s^4}{\pi \Lambda^8 \log a}$$

Interaction kernel in the unitarization formula

$$R^{(0)}_{22,22}(s) = \frac{8s}{\Lambda^2} + \frac{32s^4}{\pi\Lambda^8}\ , \quad \log a = 1$$

Secular equation $s = s_P/\Lambda^2$

$$(x + 4x^4/\pi)^{-1} + \log(-x) - i\,2\pi = 0$$
$$x = 0.07 - i\,0.21\ , \quad 3\% \text{ of deviation}$$

In agreement with the estimate $|x|^3 \sim 1\%$ for a N^3LO correction ($s^4\&s$)

Expected leading corrections are NLO loop ones $|x| \sim 0.2$ (there are no counterterms at NLO in pure gravity)

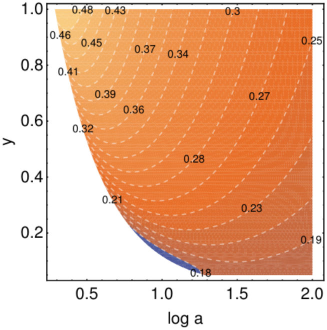
The graviball persists in $d > 4$

Maximal-stability estimate of $\log a$

$$r(\log a, y) = \frac{|\Lambda_{d_c}^2 - \Lambda^2|}{\Lambda^2}$$

which should be minimized to enhance smoothness in the transition from $d = 5$ to $d = 4$

Graviball: $\log a \approx 1$



Postdiction of the σ

Chiral limit

$$V^{(0)}_{\pi\pi} = \frac{s}{f_\pi^2}$$

$$T^{(0)}_{\pi\pi;II} = \left[\frac{f_\pi^2}{s} + \frac{1}{(4\pi)^2}\log\frac{-s}{\Lambda^2} - i\frac{1}{8\pi}\right]^{-1}$$

$$\Lambda = 4\pi f_\pi$$

$$x_\sigma = \frac{s_\sigma}{\Lambda^2}$$

Secular equation

$$\frac{1}{x_\sigma} + \log(-x_\sigma) - i2\pi = 0$$

$$x_\sigma \simeq -i\frac{2}{3\pi} = -i\,0.20$$

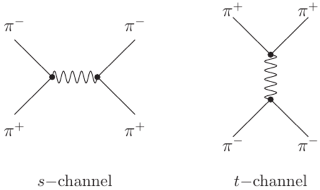
Numerically, $x_\sigma = 0.07 - i\,0.20$

Numerically, $x_\sigma = 0.07 - i\,0.20$

Physical π mass, GKPY: $x_\sigma = 0.09 - i\,0.20$

Not bad! Back envelope calculation 

§2 Coulomb scattering. E.g. $\pi^+\pi^-$



One-photon exchange $\pi^+\pi^- \to \pi^+\pi^-$ electromagnetic (EM) scattering amplitude

$$F(\pi^+\pi^- \to \pi^+\pi^-) = -\frac{e^2}{s}(s+2t-4m^2) - e^2\frac{s-u}{t}$$

Partial-wave amplitudes (PWAs)

$$\bar{T}_J(s) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta P_J(\cos\theta)\bar{T}(\cos\theta)$$

Unitarity in PWAs:

$$\Im \bar{T}_J(s) = \frac{p}{8\pi\sqrt{s}}|\bar{T}_J(s)|^2$$

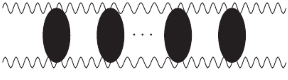
Phase divergences

Dalitz,Proc.Roy.Soc.(London)206,509(1951)
Phase conjectured by Dalitz when studying the Born series up to 2nd order (e^2) for the scattering of a Dirac electron by a Yukawa/Coulomb potential

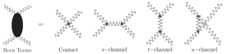
$$V(r) = \frac{e_1 e_2}{4\pi r} e^{-\mu r} \qquad \text{Phase} - \text{factor} = \exp\left\{ \frac{i e_1 e_2}{2\pi \beta_{12}} \log \mu \right\}$$

Lorentz invariant relative velocity between particles a and b

$$\beta_{ab} = \frac{\left[(p_a p_b)^2 - (m_a m_b)^2\right]^{1/2}}{p_a p_b}$$



Born Series



Extended up to 3rd order (e^3) for the non-relativistic case

$$f(\mathbf{p}') = \frac{e^2}{4p^2 \sin^2 \frac{1}{2}\theta} \left\{ 1 - i\frac{m\alpha}{p}\left(\log \sin^2 \frac{1}{2}\theta + \log \frac{4p^2}{\mu^2} \right) + \left(\frac{m\alpha}{p}\right)^2 \left(-\frac{3}{4}\left(\log \frac{4p^2}{\mu^2} \right)^2 + [\log(\alpha\mu)]^2 \right) + \mathcal{O}(\alpha^3) \right\}$$

One-loop calculation, JAO arXiv:2207.08784

$$\bar{T}^{(2)}(\mathbf{p}',\mathbf{p}) = \frac{me^4}{4\pi^3}\int \frac{d^3q}{[\mu^2+(\mathbf{p}'-\mathbf{q})^2][\mu^2+(\mathbf{p}-\mathbf{q})^2][q^2-p^2-i\epsilon]}\,.$$

S-wave projection

$$F_0^{(2)}(p) = \frac{me^4}{16\pi^2p^2}\int_0^\infty \frac{dq}{q^2-p^2}\left[\log\frac{\mu^2+(p+q)^2}{\mu^2+(p-q)^2}\right]^2$$

It satisfies perturbative unitarity

$$\Im F_0^{(2)}(p) = \frac{mp}{2\pi}{F_0^{(1)}}^2 = \frac{me^2}{8\pi p^3}\left(\log\frac{2p}{\mu}\right)^2$$

Its real part is zero

$$\Re F_0^{(2)}(p) = \lim_{\mu\to 0}\frac{me^4}{16\pi^2p^2}\mathrm{PV}\int_0^\infty \frac{dq}{q^2-p^2}\left[\log\frac{\mu^2+(p+q)^2}{\mu^2+(p-q)^2}\right]^2 = 0$$

Up to and including one loop

$$F_0(p) = \frac{e^2}{2p^2}\log\frac{2p}{\mu} + i\frac{me^4}{8\pi p^3}\left(\log\frac{2p}{\mu}\right)^2 + \mathcal{O}(\alpha^3)$$

IR-safe PWAs up to $\mathcal{O}(\alpha^2)$

JAO Phys.Lett.B835,137568 2207.08784

Including the Weinberg phase $S_c = \exp\left(2i\gamma\log\frac{\mathcal{L}}{\lambda}\right)$

$$S_J = \left[1 + i\frac{mp}{\pi}(F_J^{(1)} + F_J^{(2)})\right]\left[1 - 2i\gamma\log\frac{\mathcal{L}}{\lambda} - 2\gamma^2(\log\frac{\mathcal{L}}{\mu})^2\right] + \mathcal{O}(\alpha^3)$$

$$= 1 - 2i\gamma\log\frac{\mathcal{L}}{\mu} - 2\gamma^2(\log\frac{\mathcal{L}}{\mu})^2 + i\frac{mp}{\pi}\left[F_J^{(1)}\left[1 - 2i\gamma\log\frac{\mathcal{L}}{\mu}\right] + F_J^{(2)}\right]$$

$$+ \mathcal{O}(\alpha^3)\ , \qquad \gamma = \frac{m\alpha}{p}$$

$$S_J = 1 + \frac{mp}{4\pi}T_J$$

$$T_J^{(1)}(p) = F_J^{(1)}(p) - \frac{e^2}{2p^2}\log\frac{\mathcal{L}}{\mu} = \frac{e^2}{2p^2}\log\frac{2p}{\mu} - \frac{e^2}{2p^2}\log\frac{\mathcal{L}}{\mu} = \frac{e^2}{2p^2}\log a$$

$$T_J^{(2)}(p) = F_J^{(2)}(p) - iF_J^{(1)}(p)\frac{me^2}{2\pi p}\log\frac{\mathcal{L}}{\mu} + i\frac{me^4}{8\pi p^3}\left(\log\frac{\mathcal{L}}{\mu}\right)^2$$

IR-safe PWAs

$$T_0^{(1)}(p) = \frac{e^2}{2p^2} \log \frac{2p}{\mathcal{L}} = \frac{e^2}{2p^2} \log a \; ,$$

$$T_0^{(2)}(p) = i \frac{me^4}{8\pi p^3} \left(\log \frac{2p}{\mathcal{L}} \right)^2 = i \frac{me^4}{8\pi p^3} (\log a)^2$$

Matching with the unitarization formula

$$T_0^{(1)} + T_0^{(2)} = \left[\frac{1}{V_0^{(1)} + V_0^{(2)}} - i \frac{mp}{2\pi} \right]^{-1} = V_0^{(1)} + V_0^{(2)} + i \frac{mp}{2\pi} {V_0^{(1)}}^2 + \mathcal{O}(\alpha^3)$$

$$V_0^{(1)}(p) = T_0^{(1)}(p) = \frac{e^2}{2p^2} \log a$$

$$V_0^{(2)}(p) = 0$$

In agreement with the exact solution

Considering higher orders

$$S_J = \frac{\Gamma\left(1+J-i\gamma\right)}{\Gamma\left(1+J+i\gamma\right)} = 1 + i\frac{m_r p}{\pi} T_J$$

$$T_J = \left[V_J^{-1} - i\frac{m_r p}{2\pi}\right]^{-1}$$

$$V_J = \frac{2i\pi}{m_r p}\frac{\Gamma(1+J+i\gamma)+\Gamma(1+J-i\gamma)}{\Gamma(1+J+i\gamma)-\Gamma(1+J-i\gamma)}$$

$$V_J = -\frac{2\pi}{m_r p} v_J$$

$$v_J(p) = \psi_0(1+J)\gamma + \frac{1}{6}[2\psi_0(1+J)^3 - \psi_2(1+J)]\gamma^3$$

$$+ \frac{1}{120}[16\psi_0(1+J)^5 - 20\psi_0(1+J)^2\psi_2(1+J) + \psi_4(1+J)]\gamma^5$$

$$+ \mathcal{O}(\alpha^7)$$

$$V_0 = \underbrace{-\frac{2\pi}{mp}\psi_0(1)\gamma}_{V_0^{(1)}} \underbrace{+\mathcal{O}(\alpha^3)}_{V_0^{(2)}=0} = \frac{2\pi\alpha\gamma_E}{p^2} + \mathcal{O}(\alpha^3)$$

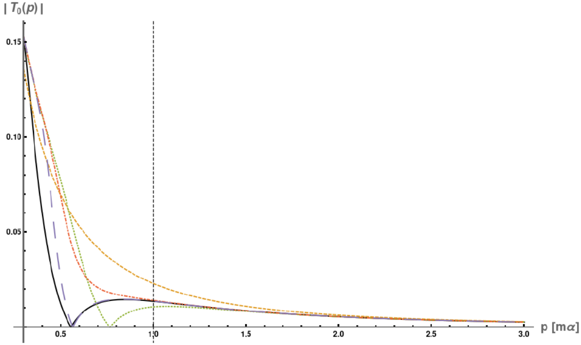
$$\psi_0(1) = -\gamma_E$$

$$\log a = \gamma_E = 0.577$$

Pole position of the ground state, $p_{\rm exact} = i m_r \alpha$

n	1	3	5	7
$p^{(n)}/p_{\rm exact}$	0.58	0.95	1.00	1.00

Comparison with the exact Coulomb PWA (black line)



n = 1(orange), 3(green), 5(red), 7(magenta)

Screened Coulomb potential

$$\frac{\alpha}{r} \to \frac{\alpha}{r}\theta(R-r)\ , \quad R \to \infty$$

IR-safe $V_0^C(s)$ up to $\mathcal{O}(\gamma)$ from the partial-wave projected Born term

$$F = \frac{4\pi\alpha}{q^2}(1-\cos qR)$$

$$F_0 = \frac{2\pi\alpha}{p^2}(\gamma_E + \log 2pR) + \mathcal{O}(R^{-2})$$

$$V_0 = F_0 - \underbrace{\delta F_c}_{\frac{2\pi\alpha}{p^2}\log 2pR} = \frac{2\pi\alpha\gamma_E}{p^2}$$

It is the same as the LO term in the expansion of $v_0(p)$ $\quad V_0 = -\frac{2\pi}{mp}\psi_0(1)\gamma + \mathcal{O}(\alpha^3) = \frac{2\pi\alpha\gamma_E}{p^2} + \mathcal{O}(\alpha^3), \psi_0(1) = -\gamma_E$

QED

Bazhanov *et al* Theor.Math.Phys.33,982(1977) up to 2nd order in the expansion of small t (one order more than in Eikonal approximation)

$$T(s,t)=\frac{8\pi\alpha Z_1Z_2(s-m_1^2-m_2^2)}{t+i0}\left(\frac{-t-i0}{\lambda^2}\right)^{\gamma/2}e^{-\gamma C}\frac{\Gamma(1-\gamma/2)}{\Gamma(1+\gamma/2)}\left\{1-\frac{\pi\alpha Z_1Z_2(m_1+m_2)}{2(s-m_1^2-m_2^2)}D(s)\,(-t-i0)^{1/2}+o(\sqrt{t})\right\},$$

$$D(s)=\frac{\Gamma(1+\gamma/2)\,\Gamma(1/2-\gamma/2)}{\Gamma(1-\gamma/2)\,\Gamma(1/2+\gamma/2)},$$

$$\gamma=-2i\alpha Z_1Z_2/v,$$

The extra phase ($\lambda\equiv\mu$, $C\equiv\gamma_E$)

$$\exp\left[2i\alpha\frac{Z_1Z_2}{v}\left(\gamma_E+\log\frac{\lambda}{2p}\right)\right]$$

$$\mathcal{L}=\frac{2p}{a}\to\log a=\gamma_E$$

and this is a calculation to all orders in α

In the static limit (scattering in an external Coulomb field)

$$T = T_B \left(\frac{-t - i0}{\lambda^2} \right)^{\gamma/2} e^{-\gamma C} \frac{\Gamma(1-\gamma/2)}{\Gamma(1+\gamma/2)} \left[1 - \frac{\pi \alpha Z_1 Z_2 v}{2} D \sin \frac{\theta}{2} \right].$$

Versus the nonrelativistic amplitude for Coulomb

$$f_{\mathrm{N.R.}}(\theta) = \frac{\gamma}{2p} \left(\sin \frac{\theta}{2} \right)^{-2+2i\gamma} \frac{\Gamma(1 - i\gamma)}{\Gamma(1 + i\gamma)}$$

The difference is the phase factor $\exp\left(2i\gamma \frac{\mathcal{L}}{\lambda}\right)$ with $\mathcal{L} = 2p/e^{\gamma_E}$, or $\log a = \gamma_E$

Delgado,Dobado,Espriu,arXiv:2207.06070: Keeping $\mu \to 0$

Talk by Delgado on Monday, 16h, Session G.

They keep the IR divergent $F_J^{(n)}$. E.g. for non-relativistic Coulomb scattering

They do not take into account the Weinberg phase

$$F_0^{(1)}(p) = \frac{e^2}{2p^2} \log \frac{2p}{\mu}$$

$$F_0^{(2)}(p) = i\,\frac{me^4}{8\pi p^3} \left(\log \frac{2p}{\mu}\right)^2$$

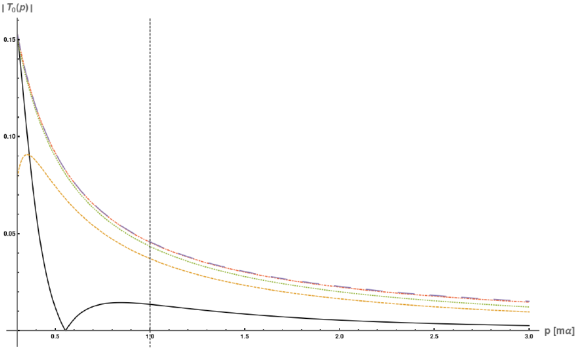
The Inverse Amplitude Method (IAM) is applied with the IR divergent perturbative PWAs $F_J^{(1)}$, $F_J^{(2)}$

$$T_0(p) = \frac{F_0^{(1)}(p)^2}{F_0^{(1)}(p) - F_0^{(2)}(p)} \xrightarrow[\mu \to 0]{} \frac{i2\pi}{mp}$$

The method of arXiv:2207.06070 clearly fails to reproduce Coulomb PWAs

μ [$2m\alpha$]	0.5	10^{-1}	10^{-2}	10^{-4}
p/μ	$0.79 - i\,0.32$	$0.99 - i\,0.10$	$1.00 - i0.01$	$1.00 - i0.00$

Pole position of the ground state, $p \to 0$ for $\mu \to 0$



$\mu = 0.5$(orange),10^{-1}(green),10^{-4}(red),10^{-12}(magenta)

Delgado,Dobado,Espriu,arXiv:2207.06070 is not a suitable unitarization method for infinite-range interactions

§4 Summary and outlook

1. Formalism for unitarizing forces of infinite range
2. Removing of a global phase factor S_c in the S-matrix. Dalitz-Weinberg phase
3. IR-safe PWAs
4. Standard unitarization techniques from hadron physics satisfying two-body unitarity and analyticity
5. Prediction of the graviball (or gravi-σ) $s_P = (\varkappa - i\,\frac{2}{3\pi})\Lambda^2$, $\Lambda^2 \simeq \pi G^{-1}$ and $\varkappa \ll 1$
6. Its resonance effects peak at $s \ll \Lambda^2$

7. Large corrections to S-wave graviton-graviton scattering calculated in perturbation theory from EFT
8. Suppression of multi-graviton[pion] intermediate states for $s < G^{-1}[(4\pi f_\pi)^2]$
9. Close analogy with the σ of $f_0(500)$ resonance
10. The exactly solvable Coulomb scattering is a check and good example for our formalism
11. Coulomb PWAs cannot be reproduced by the method of arXiv:2207.06070

Outlook

1. Unitarization of the known one-loop graviton-graviton scattering amplitude
2. More applications should be pursued in gravity and hadron physics

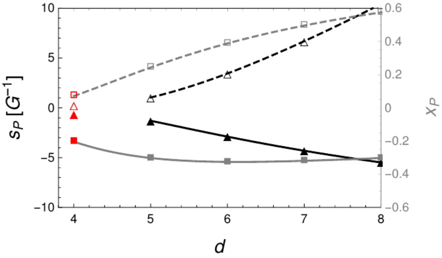


Figure: Graviball pole s_p and x for $d \geq 4$

Independence of J

$\lambda = 0$:

$$F^{(J)}_{22,22}(s) = -\frac{2Gs^2}{\pi}\int_{-1}^{+1} d\cos\theta \frac{P_J(\cos\theta)}{t-\mu^2}$$
$$= -\frac{2Gs^2}{\pi}\int_{-1}^{+1} d\cos\theta \frac{P_J(\cos\theta)-1}{t} + \frac{8Gs}{\pi}\log\frac{2p}{\mu}$$

$\lambda = 4$

$$F^{(J)}_{2-2,2-2}(s) = \frac{\kappa^2}{32\pi^2 s}\int_{-1}^{+1} d\cos\theta \frac{d^{(J)}_{44}(\theta)u^3}{t}$$
$$= \frac{G}{\pi s}\int_{-1}^{+1}\frac{d\cos\theta}{t}\left[d^{(J)}_{44}(\theta)u^3 + s^3\right] + \frac{4Gs}{\pi}\log\frac{2p}{\mu}$$

$$V^{(J)} = \frac{8Gs}{\pi 2^{|\lambda|/4}}\log\frac{\mu}{\mathcal{L}} + F^{(J)}$$