On the Running of Gauge Couplings in String Theory

Luca Armando Nutricati

Department of Mathematical Sciences,
Durham University

Based on work in progress with **Steven A. Abel** and **Keith R. Dienes**,
arXiv: 2301.nnnnn (to appear)

12/01/2023

Motivations

Facts

- Closed string theories are **UV/IR mixing** theories (*modular invariance*);
- They may have **divergences**.

Questions:

- How do we **treat** these **divergences** without breaking modular invariance?
- Is it sensible to introduce a **spacetime energy scale** and a **renormalisation procedure** which respects the **UV/IR mixed structure**?

Outline

- **One-loop corrected gauge couplings** in closed String Theory;
- How to deal with **divergences**;
- A ***modular invariant* regulator**;
- An example of how to **use** it .

One-loop (threshold) correction to gauge couplings in closed String Theory

Internal part

$$\Delta_\alpha = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left\{ \frac{i}{\pi} \frac{1}{|\eta(\tau)|^4} \sum_{(a,b)\neq(1,1)} \partial_\tau \left(\frac{\theta\left[{\alpha \atop \beta}\right]}{\eta(\tau)} \right) \left(Q_\alpha^2 - \frac{k_\alpha}{4\pi\tau_2} \right) C\begin{bmatrix} a \\ b \end{bmatrix} - b_a \right\}$$

Spacetime part

[V. Kaplunovsky, 1987]

Subtraction of the contribution of massless states

$$b_a = \mathrm{Str}\left[Q_\alpha^2 \left(\frac{1}{12} - S^2 \right) \right] \Big|_{\text{massless}}$$

helicity

$$\eta(\tau) \equiv e^{\frac{\pi}{12} i\tau} \prod_{\ell=1}^{\infty} (1 - e^{2\pi\ell\tau})$$

$$\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) \equiv \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\alpha/2)^2} e^{2\pi i (n+\alpha/2)(z+\beta/2)}$$

- k_α — *Kac-Moody level at which the group factor* G_α *is realised;*
- Q_α — *Cartan charge.*

One-loop (threshold) correction to gauge couplings in closed String Theory

$$\Delta_\alpha = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left\{ \frac{i}{\pi} \frac{1}{|\eta(\tau)|} \sum_{(a,b)\neq(1,1)} \partial_\tau \left(\frac{\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}{\eta(\tau)} \right) \left(Q_\alpha^2 - \frac{k_\alpha}{4\pi\tau_2} \right) C\begin{bmatrix} a \\ b \end{bmatrix} - b_a \right\}$$

[V. Kaplunovsky, 1987]

$$b_a = \mathrm{Str}\left[Q_\alpha^2 \left(\frac{1}{12} - S^2 \right) \right]\Big|_{\text{massless}}$$

This subtraction is needed to <u>make the integral finite</u>
BUT
it <u>breaks worldsheet modular invariance</u>

Many different regulators are possible

- A simple **subtraction** of the divergence [Kaplunovsky, 1987]

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau) \longrightarrow \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left(F(\tau) - c\tau_2\right), \qquad F(\tau) \sim c\tau_2, \quad \tau_2 \gg 1$$

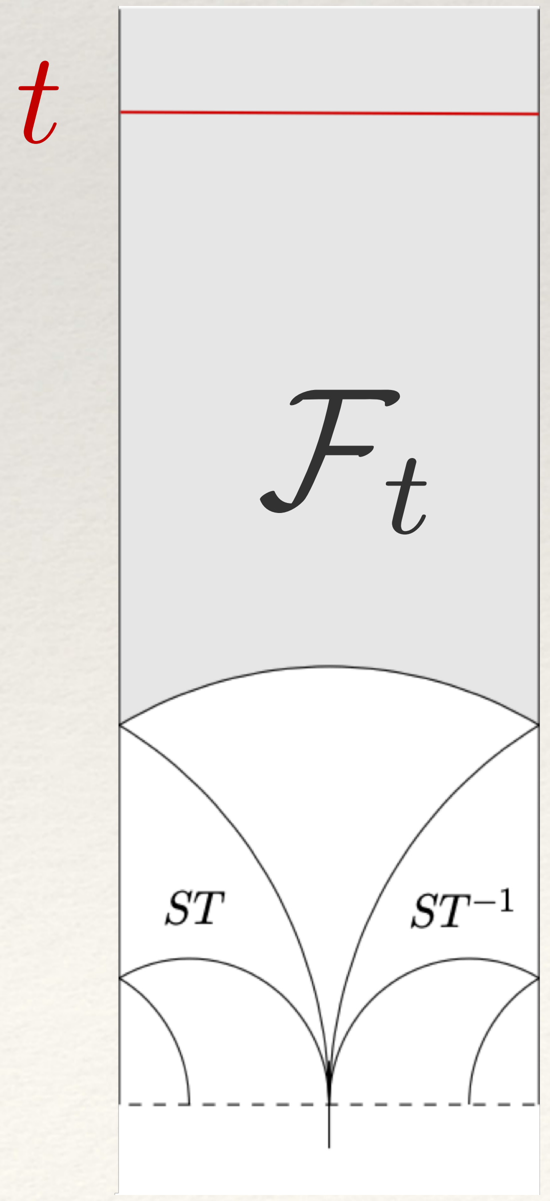
Breaks modular invariance!

*It corresponds to **remove massless states** => definition of **threshold correction.***

- Introducing a **cut-off** in the fundamental domain

$$\widehat{I}(t) \equiv \int_{\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} F + \int_{\mathcal{F}-\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} \left(F - c\tau_2\right)$$

Breaks modular invariance!



[Don Zagier, 1981; C. Angelantonj, I. Florakis, B. Pioline, 2011, ArXiv: 1110.5318.]

Many different regulators are possible

- **Deform** the integrand in a modular invariant way

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F \longrightarrow \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau)\mathcal{G}(\tau)$$

The **regulator**:

- Should be **modular invariant**;
- Must **suppress any power-law divergence** for large τ_2;
- Should **leave the rest of the theory intact as much as possible**.

A modular invariant regulator

A possible choice

$$\mathcal{G}_\rho(a,\tau) \equiv \frac{1}{1+\rho a^2}\frac{\rho}{\rho-1}a^2\frac{\partial}{\partial a}\left[Z_{\text{circ}}(\rho a,\tau) - Z_{\text{circ}}(a,\tau)\right]$$

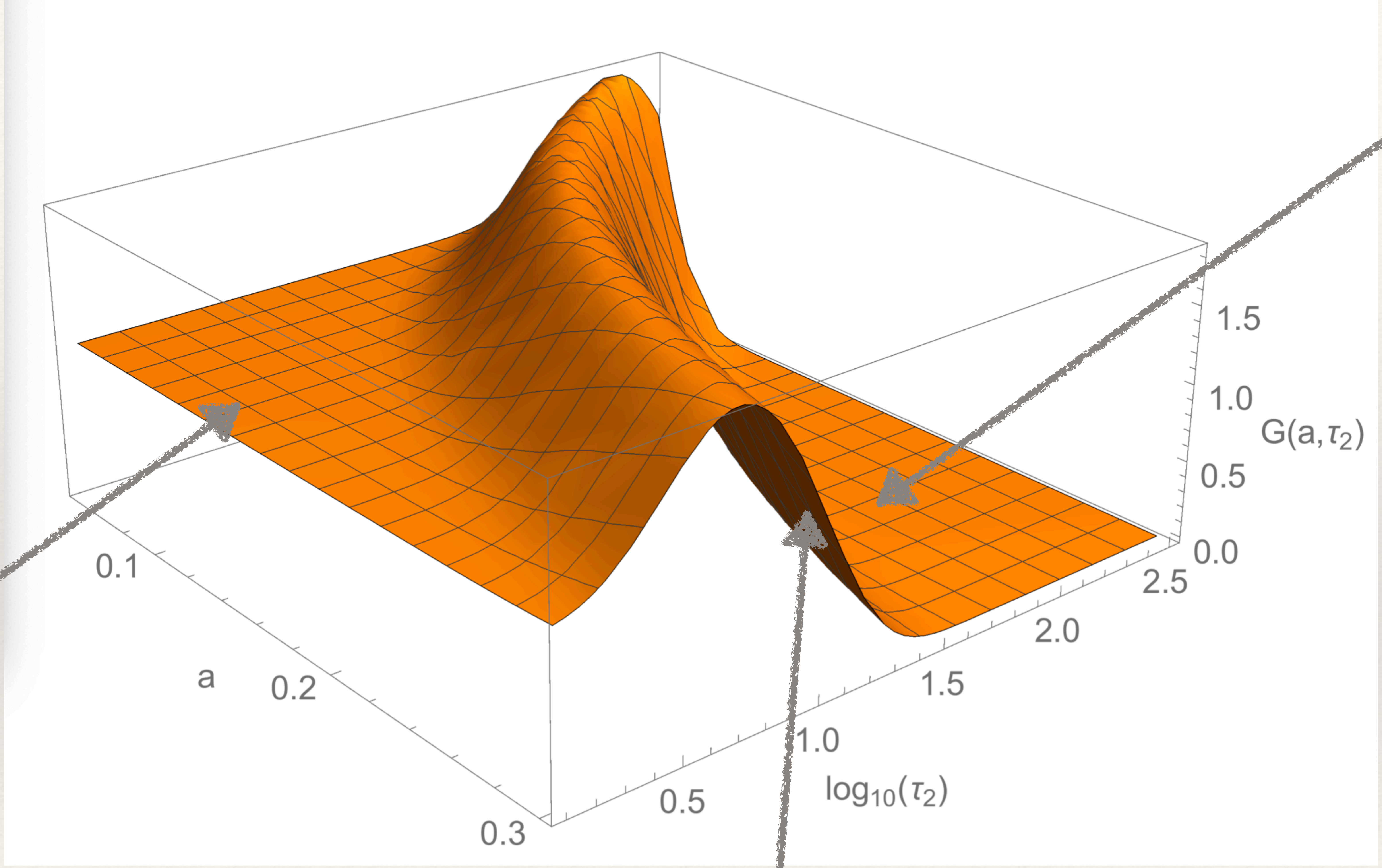
where $Z_{\text{circ}} \equiv \sqrt{\tau_2}\sum_{m,n\in\mathbb{Z}} e^{-\pi\tau_2(m^2a^2+n^2/a^2)}e^{2\pi i m n \tau_1}$

- Based upon Kiritsis and Kounnas regulator [E. Kiritsis, C. Kounnas, 1995] but **modified in a critical way.** Indeed:
    - We can define $\mu^2 = \rho a^2/\alpha'$ as an **energy scale;**
    - **Modular invariance** → **UV/IR mixing** → **Scale duality symmetry**

$$\mu/M_s \longrightarrow M_s/\mu$$

[S. A. Abel, K. R. Dienes, 2021, ArXiv: 2106:04622]

A modular invariant regulator



Exponential decay

Preserves the theory here $\mathcal{G} \approx 1$

It starts eating at $\tau_2 \approx 1/a^2$

The case of 6D $\mathcal{N}=1$ => 4D $\mathcal{N}=2$ toroidal compactifications

Famous result [L. Dixon, V. Kaplunovsky, J. Louis, 1990]

$$\Delta_a(T,U) \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\left(\tau_2 \hat{U}_{2,2}(\tau) - \tau_2\right) \;=\; -\log\left(\frac{8\pi e^{1-\gamma}}{3\sqrt{3}} T_2 U_2 |\eta(T)|^4 |\eta(U)|^4\right)$$

where:

- $\hat{U}_{2,2}(\tau) \;=\; \sum_{\vec{k},\vec{\ell}\in\mathbb{Z}^2} e^{-\pi\tau_2\alpha' M^2} e^{2\pi i\tau(k_2\ell_1 - k_1\ell_2)}$
- $\alpha' M^2 \;=\; \frac{|k_1 + U k_2 + T\ell_1 + TU\ell_2|^2}{U_2 T_2}; \; (k_1,k_2,\ell_1,\ell_2)\in\mathbb{Z}^4$
- T and U are the so called **moduli of the (space-time) torus**.

The case of 6D $\mathcal{N}=1$ => 4D $\mathcal{N}=2$ toroidal compactifications

Famous result [L. Dixon, V. Kaplunovsky, J. Louis, 1990]

$$\Delta_a(T,U) \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\left(\tau_2 \hat{U}_{2,2}(\tau) - \tau_2\right) \;=\; -\log\left(\frac{8\pi e^{1-\gamma}}{3\sqrt{3}} T_2 U_2 |\eta(T)|^4 |\eta(U)|^4\right)$$

- ✓ **Does not spoil the moduli dependence of the threshold;**
- ✗ **Worldsheet modular invariance is broken!**

The case of 6D $\mathcal{N}=1$ => 4D $\mathcal{N}=2$ toroidal compactifications

Famous result [L. Dixon, V. Kaplunovsky, J. Louis, 1990]

$$\Delta_a(T,U) \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\left(\tau_2 \hat{U}_{2,2}(\tau) - \tau_2\right) \;=\; -\log\left(\frac{8\pi e^{1-\gamma}}{3\sqrt{3}} T_2 U_2 |\eta(T)|^4 |\eta(U)|^4\right)$$

$$\left.\frac{1}{g^2}\right|_{\text{1-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2 \hat{U}_{2,2}(\tau) \mathcal{G}_\rho(a,\tau)$$

Everything is **regulated** and **modular invariant!**

Our result

DKL result

Restore the unregulated theory when $a \to 0$

$$\frac{1}{g(a,T,U)}\Big|_{\text{1-loop}} = -\frac{1}{1+2a^2}\Big\{\log(c\,T_2U_2\,|\eta(T)\eta(U)|^4) + 2\log(\sqrt{2}a)$$

$$+4\sum_{\gamma,\gamma'\in\Gamma_\infty\backslash\Gamma}\left[2\tilde{\mathcal{K}}_0^{(0,1)}\left(\frac{2\pi}{a\sqrt{\gamma\cdot T_2\,\gamma'\cdot U_2}}\right)-\tilde{\mathcal{K}}_1^{(1,2)}\left(\frac{2\pi}{a\sqrt{\gamma\cdot T_2\,\gamma'\cdot U_2}}\right)\right]\Big\}$$

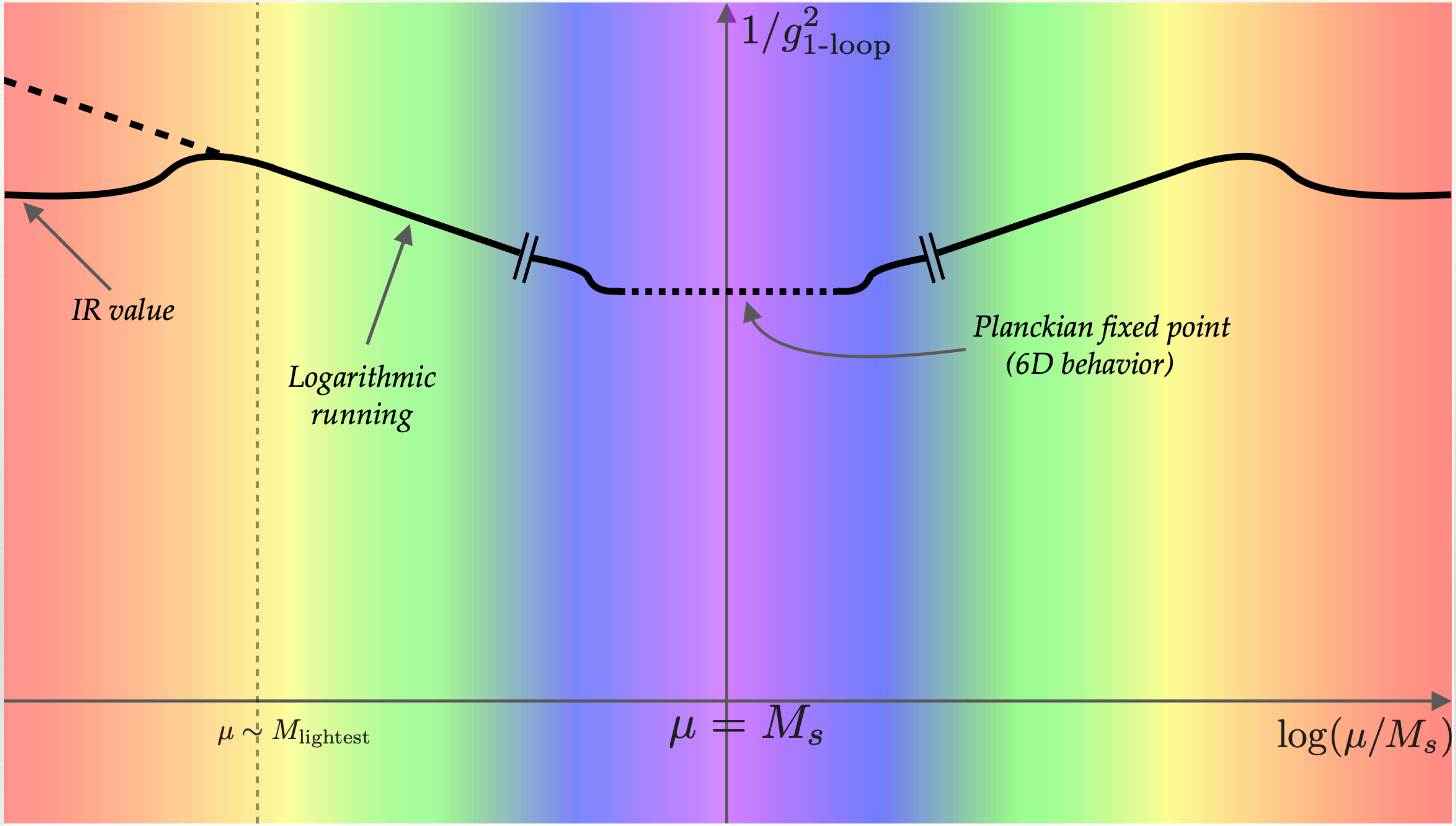
[Abel, Dienes, LAN, to appear]

Correction which restore full worldsheet modular invariance

- $\tilde{\mathcal{K}}_\nu^{(n,p)}(z,\rho) = \sum_{k,r=1}^{\infty}(krz)^n\Big(K_\nu(krz/\rho)-\rho^pK_\nu(krz)\Big)$
- $\Gamma \equiv \mathrm{SL}(2,\mathbb{Z}),\ \Gamma_\infty \equiv \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, \qquad n\in\mathbb{Z}$
- $c \equiv 2\pi^2\,e^{-2(\gamma+1)}$

…contains all the required behaviour in limits…

Running of the gauge coupling



Deep IR

The most UV possible (turnaround region)

Dual deep IR

Conclusions and Outlook

- We **regulated** the theory fully respecting **modular invariance** and **UV/IR mixing properties**;

- We have seen how gauge couplings **run** in a particular class of compactifications;

- This formalism can be applied in **full generality to any model**. Indeed, using the methods developed in arXiv:2106.04622 for calculating the Higgs mass in string theory, we worked out the **general case** (to appear soon, arXiv: 2301:nnnnn)

Thanks for your attention!