Semi Classical Gravity = Statistical Physics

Jan de Boer, Amsterdam

Based on work in progress and:
- Alex Belin, JdB, arXiv:2006.05499
- Alex Belin, JdB, Diego Liska, arXiv:2110.14649
- Alex Belin, JdB, Pranyal Nayak, Julian Sonner, arXiv:2111.06373
- Tarek Anous, Alex Belin, JdB, Diego Liska, arXiv:2112.09143

Murcia
January 12, 2023

The celebrated AdS/CFT correspondence states that quantum gravity in a peculiar box (AdS) is equivalent to a suitable conformal field theory on the boundary. Maldacena '97

How much of the CFT is captured by considering only semi-classical gravity in AdS, i.e. the part of the gravitational theory accessible to low-energy observers?

One might naively think that semi-classical gravity simply corresponds to a suitable low-energy effective field theory for the CFT…..

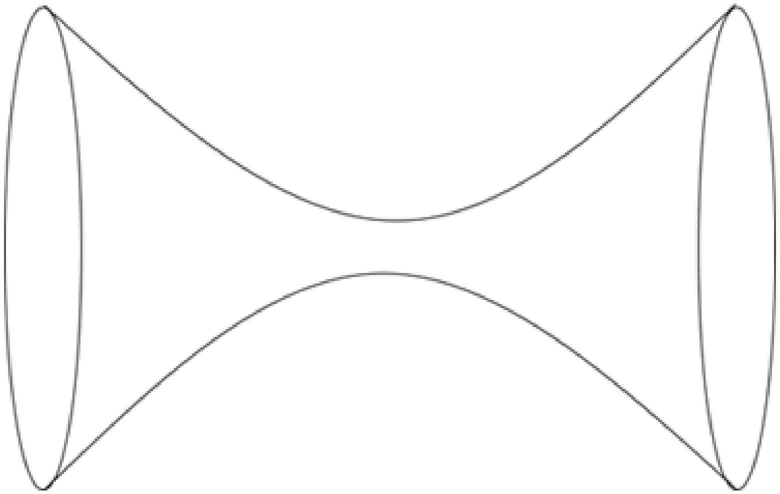
But semi-classical gravity also provides information beyond a standard low-energy effective field theory.

It knows for example about

- Black hole entropy which translates into the (approximate) high temperature partition function of the CFT.
- The partition function on various Euclidean manifolds in AdS/CFT (eg finite temperature correlators)
- The page curve (using island/replica wormholes): unitarity of a process where low energies → high energies → low energies

Penington ’19
Almheiri, Engelhardt, Marolf, Maxfield ‘19
Penington, Shenker, Stanford, Yang ‘19
Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini ‘19

Moreover, semi-classical gravity seems to give rise to correlations between copies of the same theory due to the existence of wormhole solutions



Such correlations (lack of factorization) could arise due to disorder averages but in standard AdS/CFT there was no need for (or a sign of) disorder.

"factorization puzzle"

A special case where we can answer this question is JT gravity, a 2d theory, where the semi-classical theory is UV complete. There, the semi-classical theory is described by a matrix model.

Saad, Stanford, Shenker ‘19

However, this is very different from the general case.

In higher dimensional semi-classical gravity, no computations resolve *exact* information about the UV physics of the theory.

Rather, they provide *coarse grained* information about the UV physics

Example: the spectral density

From the entropy of a black hole we obtain an *approximate* expression for

$$Z(\beta) = \int dE \rho(E) e^{-\beta E}$$

Typically, we do not have the required exp(-S) accuracy to resolve the exact density of states

$$\rho(E) = \sum_i \delta(E - E_i)$$

We would then be able to see all the individual microstates of the black hole in LEEFT. Exceptions could be integrable models, topological theories, or BPS black holes.

In JT gravity we found an average over theories (matrix model)

In higher dimensions we obtain coarse grained answers (of a single theory?)

Coarse graining and averaging are very similar, both erase detailed information.

Is it conceivable that coarse graining somehow imitates averaging and that this explains the appearance of wormholes in semi-classical gravity?

For example:

Consider a large set of N=e^S random phases $e^{i\phi_i}$

$$\left(\sum e^{i\phi_i}\right)\left(\sum e^{i\phi_i}\right)^* = \sum_{i=j} 1 + \sum_{i\neq j} e^{i(\phi_i - \phi_j)}$$

erratic invisible in gravity

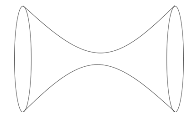
erratic invisible in gravity

N=e^S visible in gravity

erratic invisible in gravity

0 x 0 = N + 0

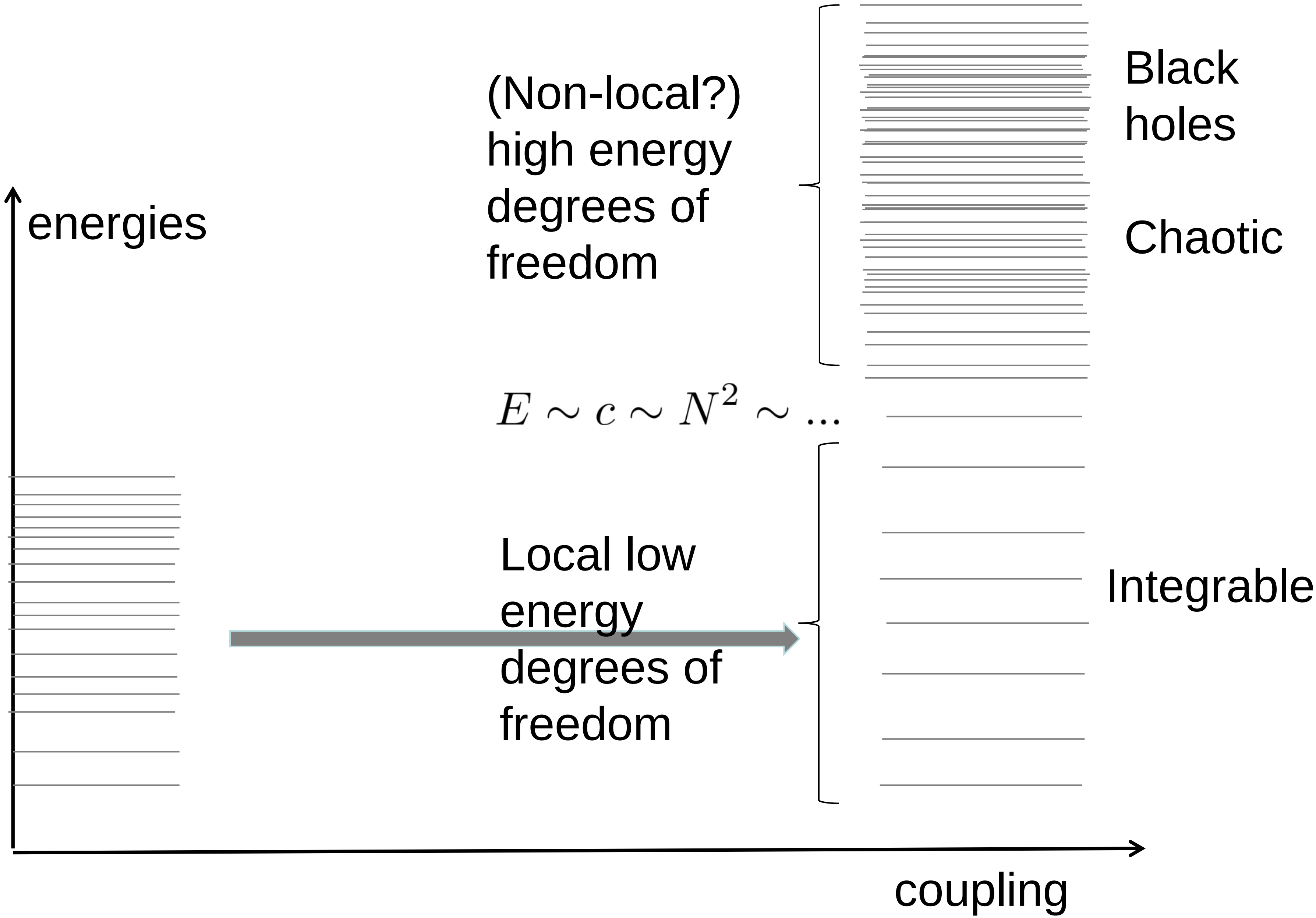
So semi-classical gravity is sensitive to the average size of fluctuations (stable under coarse graining) but not to the individual fluctuations themselves which disappear under coarse graining.



$$\left(\sum e^{i\phi_i}\right)\left(\sum e^{i\phi_i}\right)^* = \sum_{i=j} 1 + \sum_{i\neq j} e^{i(\phi_i - \phi_j)}$$

What happens in the full UV theory? Possibilities:

- The relevant gravitational solution (eg wormhole) is unstable and factorization is restored (but solution remains as off-shell configuration). UV is single theory.
- UV physics adds the fluctuating contributions $\sum_{i\neq j} e^{i(\phi_i - \phi_j)}$ and factorization is restored. UV is single theory
- The UV theory is an average of theories, averaging makes the fluctuating term exactly zero, and factorization is not restored



energies
(Non-local?) high energy degrees of freedom
Black holes
Chaotic
$E \sim c \sim N^2 \sim ...$
Local low energy degrees of freedom
Integrable
coupling

MAIN CLAIM:

- Semi-classical gravity is the theory of the statistics of the chaotic sector of the theory.

- It can probe (coarse-grained) higher moments of the relevant would-be statistical distributions but not individual values.

- It cannot distinguish averaged from non-averaged theories as long as the averages yield the same moments of the statistical distribution (up to the accuracy of semi-classical gravity).

Is this a fundamental limitation on how much information low-energy observers can obtain?

Suppose e.g. that the chaotic sector of the theory is like the digits of π.

There is no coarse grained measurement of say a block of N digits of π which will distinguish it from an average over a uniform distribution of N digits (up to a “non-perturbative” error of order ~1/N)

To make more precise what we mean by “a theory of the statistic of the chaotic sector” we are going to use standard principles of statistical physics.

One nice way to think about statistical physics is that gives us the best description of a system given limited information.

Suppose for example that we want to find a state ρ such that the expectation value of the energy E is fixed while maximizing our ignorance (=entropy). So extremize

$$-\mathrm{Tr}(\rho \log \rho) + \lambda(\mathrm{Tr}(\rho H) - E)$$

Result:

$$\rho = Z^{-1} \exp(\lambda H)$$

We fix the Langrange multiplier by computing E and find the canonical ensemble with

$$\lambda = -\beta(E)$$

As another example, suppose we have a system where we know

1. The approximate spectral density
2. The approximate finite temperature two-point function of some operator A.

We can then find the classical probability distribution for the matrix elements $A_{ij} = \langle E_i | A | E_j \rangle$ by maximizing the classical entropy with infinitely many constraints

$$\int \prod_{i,J} dA_{i,j} \; - \mu[\{A_{ij}\}] \log \mu[\{A_{ij}\}] + \mu[\{A_{ij}\}] \int dt d\beta \lambda(t,\beta) \left[\langle A(0) A(t) \rangle_\beta - f(t,\beta) \right]$$

This yields a quadratic matrix model

$$\mu[\{A_{ij}\}] \sim \exp(-\sum_{i,j} c_{i,j} |A_{i,j}|^2)$$

This reproduces the random matrix part of the Eigenstate Thermalization Hypothesis (ETH):

$$\langle E_i | O_a | E_j \rangle = \delta_{ij} f_a(\bar{E}) + e^{-S(\bar{E})/2} g_a(\bar{E}, \Delta E) R^a_{ij}$$

Deutsch ’91
Srednicki ’94
Foini, Kurchan ‘19

$f_a(\bar{E})$: one point functions of simple operators
$g_a(\bar{E}, \Delta E)$: two point functions of simple operators
R^a_{ij} : Gaussian random variables

$$\langle R^a_{ij} \rangle = 0, \qquad \langle R^a_{ij} R^b_{kl} \rangle = \delta^{ab} \delta_{il} \delta_{jk}$$

Cf Jafferis, Kolchmeyer, Mukhametzhanov, Sonner ‘22

ETH correctly reproduces the thermal one- and two-point functions and implies that typical states look thermal.

This is all information which is available to semi-classical gravity throught the AdS/CFT correspondence.

Note: this does *not* prove the validity of ETH, nor does ETH require more input that the thermal one- and two-point functions.

One can thus argue that ETH is simply a consequence of applying statistical physic principles to simple finite temperature correlators.

Yet another example: suppose that we know the approximate spectral density or equivalently the partition function $Z(\beta)$

We can consider a probability distribution for the Hamiltonian and extermize the classical entropy while fixing the expectation value of the partion function to be $Z(\beta)$

$$\int dH \;\; -\mu[H]\log\mu[H] + \mu[H]\int d\beta\lambda(\beta)\left(\mathrm{Tr}(e^{-\beta H}) - Z(\beta)\right)$$

One finds

$$\mu[H] \sim \exp\left(\int d\beta\lambda(\beta)\mathrm{Tr}(e^{-\beta H})\right) \sim \exp(-\mathrm{Tr}V(H))$$

where V is arbitrary but needs to be fixed to yield the right partition function or spectral density.

In the absence of additional information, our best guess for the connected two-point function is then

$$\langle Z(\beta_1)Z(\beta_2)\rangle = Z(\beta_1)Z(\beta_2) + \frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2} + \dots$$

Ambjørn, Jurkiewicz, Makeenko ’90
Saad, Shenker, Sanford ‘19

One can play a similar game for other choices of data.

The general picture is one where if one e.g. inputs connected ≤k-point correlators, one gets a “matrix model” with up to k-th order interactions in the exponent.

Lagrange multipliers

$$\int dA d\lambda_i \left(-\mu[A] \log \mu[A] + \sum_i \lambda_i (f_i[A] - c_i) \right)$$

Shannon entropy

Input observations

$$\Rightarrow \mu[A] \sim e^{-\sum_i \lambda_i f_i[A]}$$

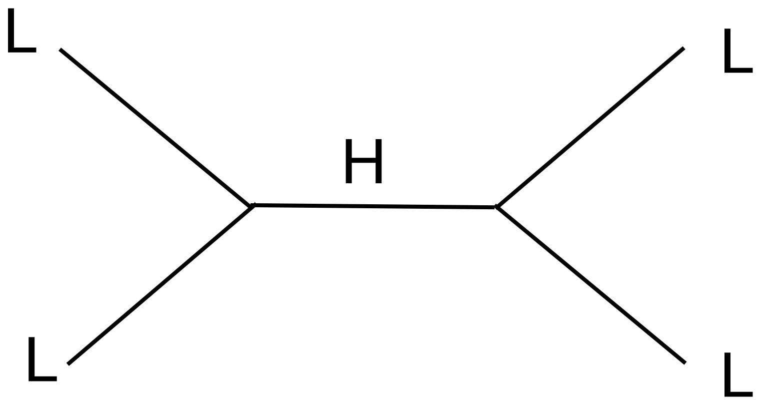
One can apply this philosophy for example to OPE coefficients in a chaotic CFT. We cannot directly compute these when one or more of the operators is in the high-energy, chaotic regime of the theory.

Application to operator statistics.

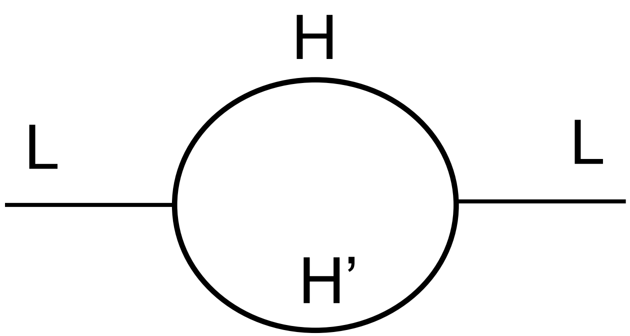
OPE coefficients $$C_{ijk} = \langle \mathcal{O}_i(\infty)\mathcal{O}_j(1)\mathcal{O}_k(0)\rangle$$

Distinguish light (L) and heavy (H) operators depending on whether operator is in integrable or in chaotic sector of the theory.

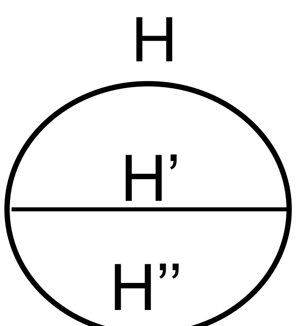
Then fact is then that semi-classical gravity has access to the statistics of C_{LLH}, C_{LHH} and C_{HHH} coarse grained over H indices but not to their individual values.



$$\sum_{H} C^2_{LLH}$$



$$\sum_{H,H'} C^2_{LHH'}$$



$$\sum_{H,H',H''} C^2_{HH'H''}$$

Input gives rise to quadratic matrix model for the C's

The same principles the imply the OPE randomness hypothesis (Belin, JdB ’20):

$$C_{LLH} \sim f(E_H) R_H$$

$$C_{LHH'} \sim f(\bar{E_H}, \Delta E_H) R_{HH'}$$

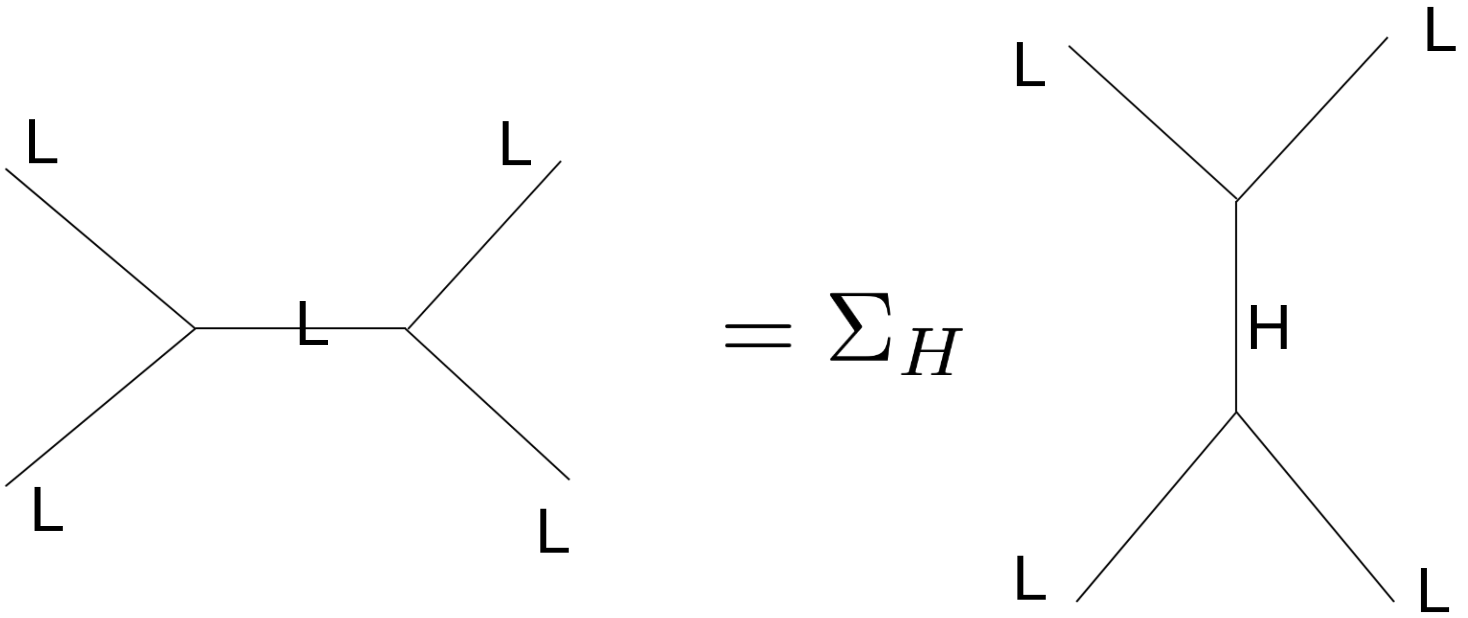
$$C_{HH'H''} \sim f(\bar{E_H}, \Delta E_H) R_{HH'H''}$$

Example of ETH

Slowly varying function of arguments

- Pseudorandom
- Mean=0
- Variance=1
- Can have higher moments which are exponentially suppressed.

How does compute all of this? Use field theory techniques which originate in large diffeomorphisms in the gravitational theory.

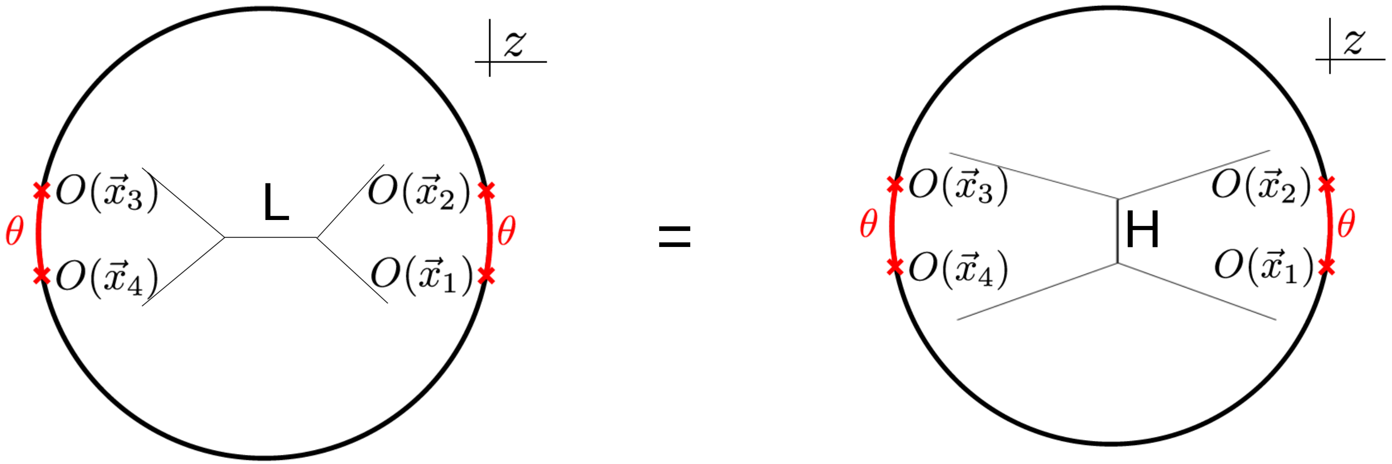


$$= \sum_H$$

Crossing symmetry

Pappadopulo, Rychkov, Espin, Ratazzi ‘12

More precisely:



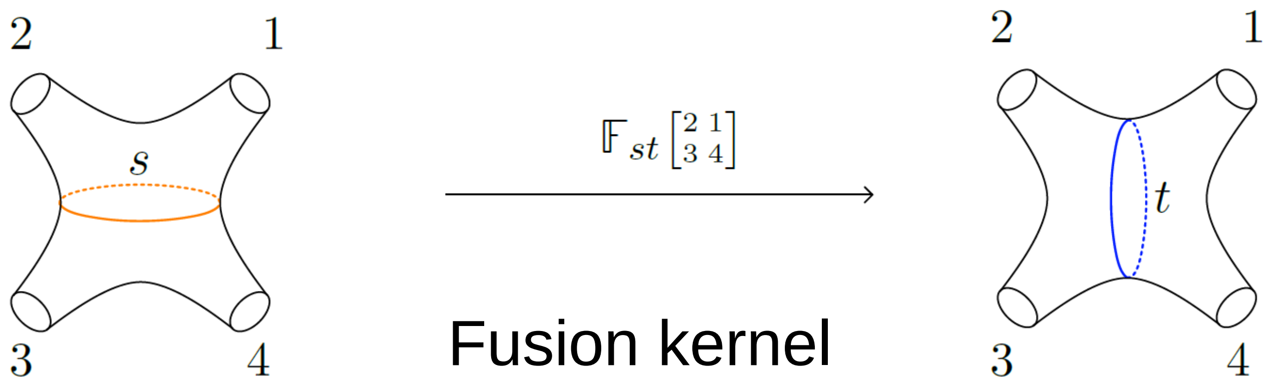
$$\frac{1}{\theta^{4\Delta_L}} \simeq \sum_H C_{LLH}^2 \left(\cos\frac{\theta}{2}\right)^{2\Delta_H}$$

$$\overline{|C_{LLH}|^2} \sim \frac{\Delta_H^{2\Delta_L - 1}}{\Gamma(2\Delta_L)\rho(\Delta_H)}$$

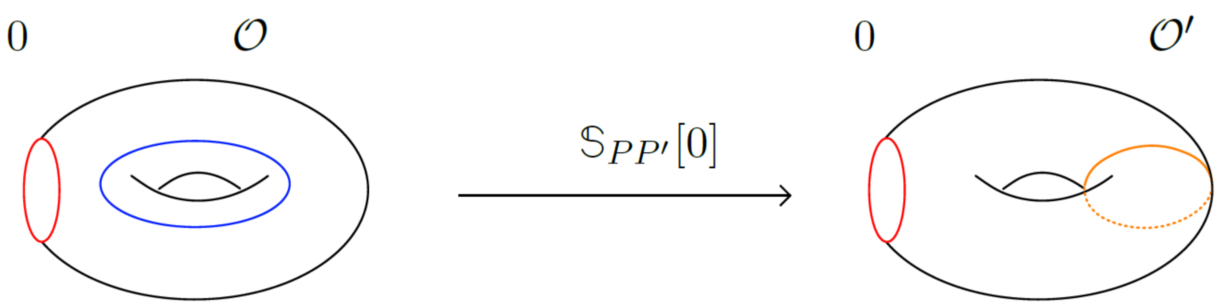
By looking at more complicated objects (like the 6-point function of light operators or a genus 3 partition functions) we can also obtain information about (coarse grained) expectation values of more than two C's.

These give rise to higher order interactions in the matrix model for the C's.

In d=2:



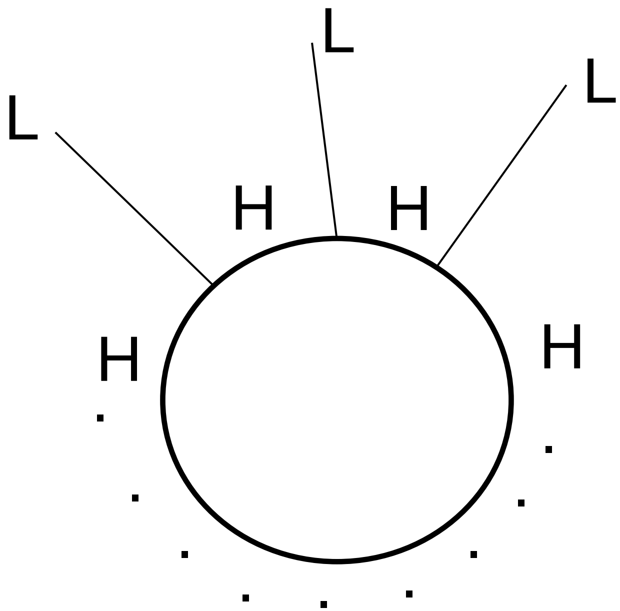
Fusion kernel



Modular kernel

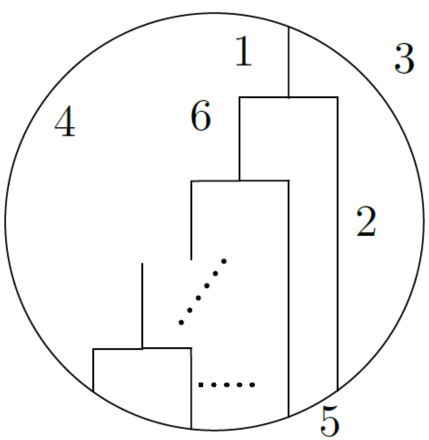
Ponsot, Teschner '99, '00
Teschner '03
Collier, Maloney, Maxfield, Tsiaris '19

Results:



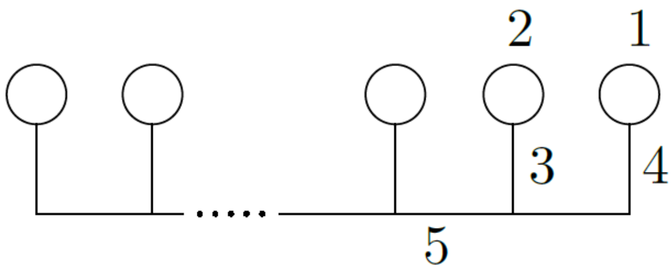
$$\langle C_{LHH}^k \rangle \sim e^{-(k-1)S}$$

Foini, Kurchan '19



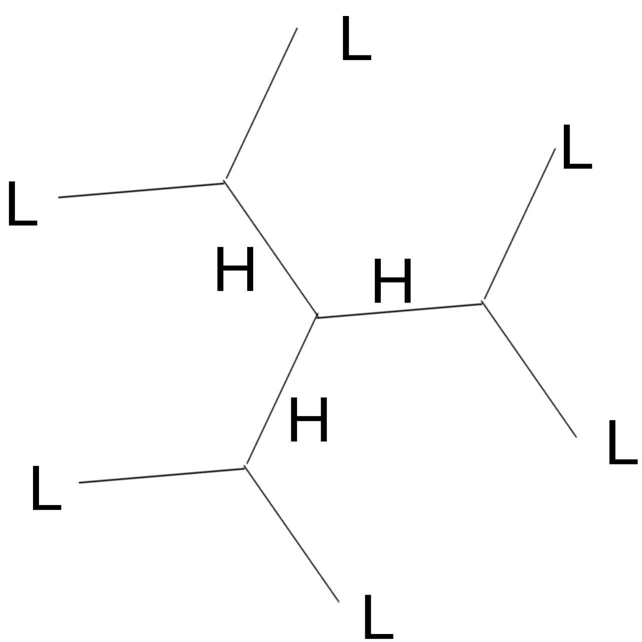
$$\langle C_{HHH}^k \rangle \sim e^{-\frac{5k-4}{4}S}$$

Belin, JdB, Liska '21



$$\langle C^k_{HHH} \rangle \sim e^{-\frac{9k-6}{8} S}$$

Belin, JdB, Liska '21



$$\overline{C^3_{LLH} C_{HHH}}\Big|_{d\geq 2,\ \text{all op}} \sim \frac{\Delta_H^{6\Delta_L - 3}}{\rho(\Delta_H)^3}$$

$$\overline{C^3_{LLH} C_{HHH}}\Big|_{d=2,\ \text{quasi-prim}} \sim \frac{\Delta_H^{6\Delta_L - 6}}{\rho(\Delta_H)^3}$$

$$\overline{C^3_{LLH} C_{HHH}}\Big|_{d=2,\ \text{Vir-prim}} \sim \left(\frac{3\sqrt{3}}{16}\right)^{3\Delta_H} \frac{\Delta_H^{6\Delta_L - \frac{19+11c}{36}}}{\rho_{\text{vip}}(\Delta_H)^{\frac{9}{4}}} .$$

$$\langle C^3_{LLH} C_{HHH} \rangle_{\text{all}} \sim e^{-3S}$$

$$\langle C^3_{LLH} C_{HHH} \rangle_{d=2\,\text{quasi}-\text{prim}} \sim e^{-3S}$$

$$\langle C^3_{LLH} C_{HHH} \rangle_{d=2\,\text{prim}} \sim e^{-\frac{9}{4} S}$$

Anous, Belin, JdB, Liska '21

More complicated tree-level diagram in d=2

$$\langle C^{m+2}_{LLH} C^{m}_{HHH} C^{\eta}_{HHL} \rangle_{d=2\,\mathrm{prim}} \sim e^{-\frac{2+4\eta+7m}{4}S}$$

Anous, Belin, JdB, Liska ‘21

General lesson:

Connected higher-point functions are exponentially suppressed

Results apply in any CFT, not necessarily chaotic – size of window one needs to coarse grain over depends on theory.

Qualitative behavior can also be obtained from

- Microcanonical unitary averages
- Altland-Sonner sigma model for chaos

Altland, Sonner '21
Belin, JdB, Nayak, Sonner '21

Results suggests statistics of OPE coefficients can be represented by a “generating functional” which captures the higher moments of the probability distributions

$$\mathcal{Z}(J_{abc}) = \exp\Big[f_1(\Delta) J_{abc} J^{abc} + f_2(\Delta) J^{a}{}_{ab} J^{bc}{}_{c} + \sum_{i=1}^{5} g_i(\Delta) JJJJ|_{\text{i-type contraction}} + \dots\Big]$$

so that

$$\langle C ... C \rangle = \frac{\delta}{\delta J} ... \frac{\delta}{\delta J} \mathcal{Z}(J)\Big|_{J=0}$$

Open questions:

- What index contractions appear? Approximate U(N) invariance?
- Is single-sided information sufficient to construct Z?

Note that everything was based on things that can be computed in AdS with one CFT boundary.

The “matrix model” structure then predicts connected correlators when there is more than one boundary (i.e. wormholes) – matrix model provides vertices and propagators.

There are then two possibilities:

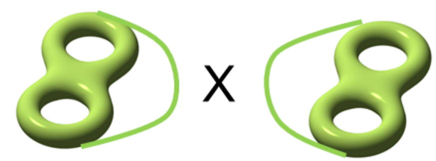
-the predictions agree with gravity computations and one finds no further refinement of the marix model, or

-the predictions do not agree with gravity computations and one needs to add the wormhole computation as additional input to refine the matrix model.

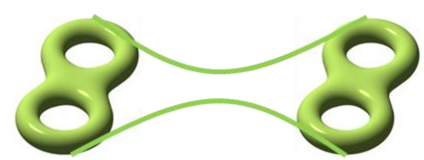
Example: the square of the high-temperature genus two partition function (dominated by high-energy states)

$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{ijk} C^*_{ijk} e^{-3\beta\Delta} \right) \left(\sum_{l,m,n} C_{lmn} C^*_{lmn} e^{-3\beta\Delta} \right) \right\rangle$$

$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{ijk} C^*_{ijk} e^{-3\beta\Delta} \right) \left(\sum_{l,m,n} C_{lmn} C^*_{lmn} e^{-3\beta\Delta} \right) \right\rangle$$



$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{ijk} C^*_{ijk} e^{-3\beta\Delta} \right) \left(\sum_{l,m,n} C_{lmn} C^*_{lmn} e^{-3\beta\Delta} \right) \right\rangle$$



(there is also a quartic vertex but it is subleading..)

In this example, the wormhole exists (Maldacena, Maoz ‘04) and agrees with the above prediction from “perturbation theory”.

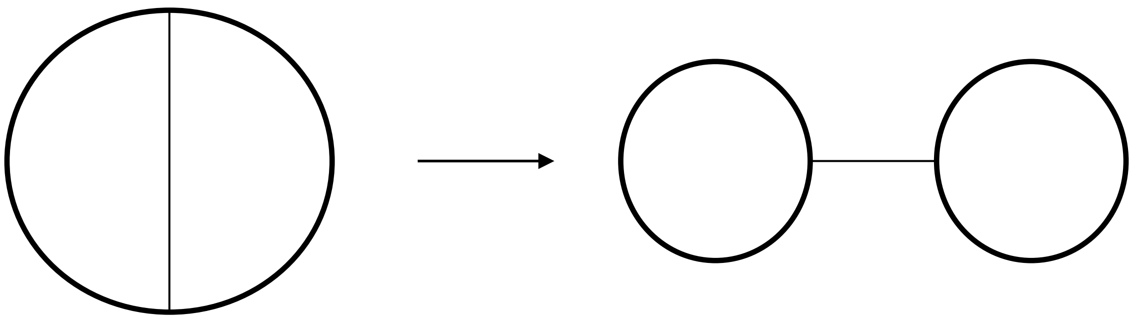
Belin, JdB ‘20

Any disagreement could in principle be fixed by adding additional quartic vertices in which the same index appears more than twice – still an open question whether such vertices appear or not.

Two wormhole “predictions”

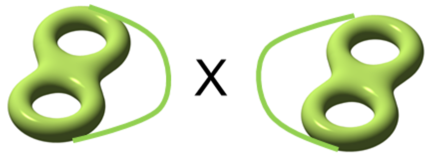
1. Take the high-temperature genus two partition function in a different corner of moduli space

$$\sum_{i,j,k} C_{ijk} C^*_{ijk} e^{-3\beta\Delta} \Longrightarrow \sum_{i,j,k} C_{iij} C^*_{jkk} e^{-3\beta\Delta}$$



We can repeat the previous computation for the square of this partition function

$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{iij} C^*_{jkk} e^{-3\beta\Delta} \right) \left(\sum_{lmn} C_{llm} C^*_{mmn} e^{-3\beta\Delta} \right) \right\rangle$$



$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{iij} C^*_{jkk} e^{-3\beta\Delta} \right) \left(\sum_{l,m,n} C_{llm} C^*_{mnn} e^{-3\beta\Delta} \right) \right\rangle$$

$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{iij} C^*_{jkk} e^{-3\beta\Delta} \right) \left(\sum_{l,m,n} C_{llm} C^*_{mnn} e^{-3\beta\Delta} \right) \right\rangle$$

????

Vertex computed independently from the genus 3 partition function

The quartic vertex dominates over the second wormhole contribution.

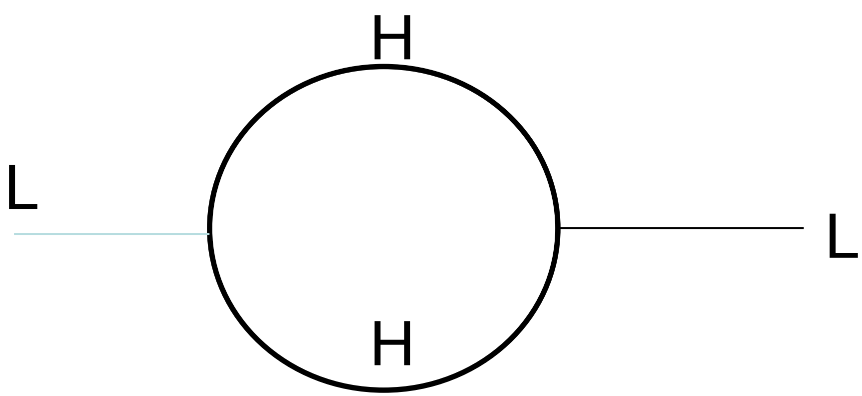
Suggests that there exists a new wormhole connecting two genus two Riemann surfaces with action

$$Z = e^{\frac{25c - 360\Delta_\chi}{288}\frac{\pi^2}{\beta}}$$

lightest scalar

Result suggest that this is a wormhole supported by matter fields. Would be interesting to construct it explicitly.

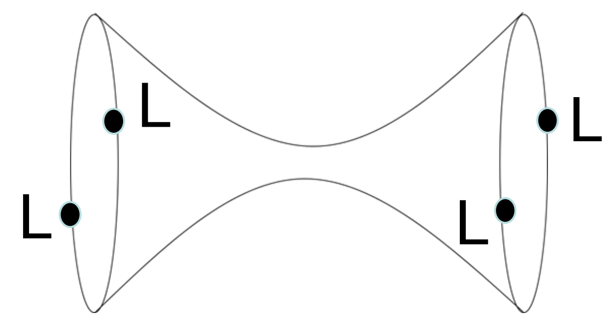
2. Consider the product of two finite temperature two-point functions $\langle (C_{LHH}C_{LHH})_1(C_{LHH}C_{LHH})_2\rangle$



$$\sim C_{LHH}^2$$

Connected Wick contraction "predicts" a new wormhole

$$\langle (C_{LHH}C_{LHH})_1(C_{LHH}C_{LHH})_2\rangle$$



Such wormholes seem to exist as partially on-shell solutions.. (Cotler, Jensen '21; Sasieta '22; Chandra, JdB, to appear)

In Chandra, Collier, Harman, Maloney ’22 various computations of this type are done more accurately.

They postulate a Cardy density of states and OPE coefficients with purely Gaussian statistics (and a spectrum of light but not too light operators) and show that it successfully reproduces many gravity computations.

The perspective is different: they consider the Gaussian statistics are resulting from a suitable averaging procedure and take this average as a possible dual description of pure 3d gravity.

Here, we interpret the same equations as capturing some of the statistics of the chaotic high-energy spectrum without requiring any type of averaging.

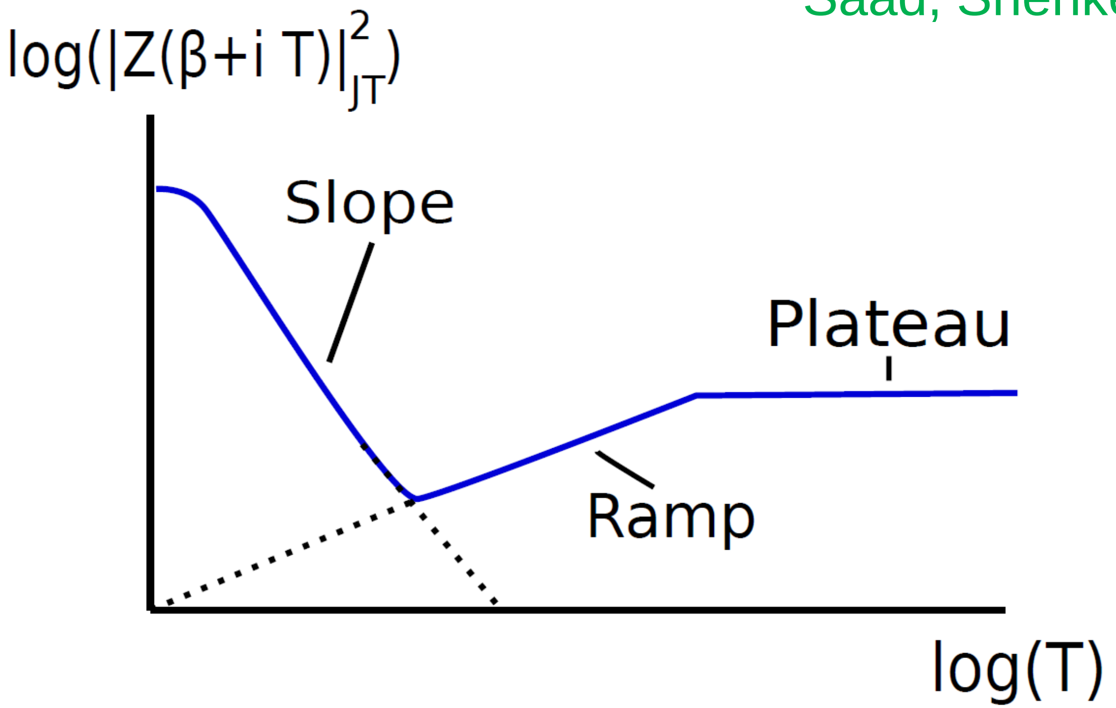
Besides operator statistics there is also spectral statistics related to e.g. $\langle \rho(E)\rho(E')\rangle$

This can be studied explicitly e.g. in

- JT gravity (Saad Shenker Stanford '19)
- Pure 3d gravity (Cotler, Jensen '20)
- Using a sigma model based on symmetry breaking (Altland, Sonner, '21)

More precisely, using wormholes in JT gravity, one can find the following picture for the spectral form factor

Saad, Shenker, Stanford, ‘19



Knows about discrete features of the spectrum.. But spoils factorization

The ramp is related to a particular wormhole configuration known as the "double-cone". It has been studied in d>3 e.g. by Mahajan, Marolf, Santos '21; Cotler, Jensen '21

The plateau requires a non-perturbative resummation of wormholes.

It is not clear to what extent the ramp and especially the plateau are part of "semi-classical gravity" as they involve very late time physics of order $t \sim e^S$.

To "prove" the chaotic, random matrix nature of the high-energy sector of a strongly coupled CFT would presumably require finding a precise reduction of late time gravitational physics to a 2d JT sector?

Coarse graining versus averaging.

It is a subtle question to distinguish the two notions. Explicit averaging comes with an intrinsic scale, which could possibly be detected by sufficiently accurate low-energy observations.

The amount of coarse graining required, on the other hand, depends on the amount of input/number of measurements provided.

Averaging over approximate CFT's?

One can also simply declare some low-energy information as complete information, obtain an ensemble of corresponding theories, and consider the average over this ensemble.

This gives rise to the notion of an "approximate CFT" (Belin, JdB, Jafferis, Nayak, Sonner, to appear) as low energy observers cannot prove the theory is an exact CFT.

For 3d gravity with a few light operators, the corresponding ensemble turns out to be given by a quadratic/quartic matrix/tensor model. It is tempting to speculate that this matrix/tensor model somehow generates a sum over 3d geometries but this remains to be shown.

A more general statistical framework?

It is an interesting question whether there is a single general formalism which captures both operator and spectral statistics and also ignorance of the precise state of the theory.

The similarity with averaged theories, and the relation of wormholes to superselection sectors and "alpha-vacua" (Marolf, Maxfield '20 '21) suggest to also include a probability distribution on the space of density matrices in the statistical description

$$\rho_{\text{micro}} \Longrightarrow \rho_{\text{LEEFT}} = \int \mu[\rho] d\rho$$

As shown earlier, in statistical physics, the thermal state arises by requiring (i) maximal entropy and (ii) the right expectation value of the energy.

What replaces those notions for ensembles of density matrices? If one replaces (i) by a combination of classical and quantum entropy and keeps (ii) one arises at the following measure on the space of density matrices

$$\mu[\rho] = \mathcal{N} e^{-S(\rho|\rho_\beta)}$$

which has several nice properties (but also several problems).

Work in progress:
Arav, Chapman, JdB
JdB, Liska, Post, Sasieta

More generally, what one finds is that computations behave well if the matrix elements of $\log \rho$ obey ETH themselves…. Not entirely clear what the meaning of this is.

Uncovering the UV

It is an interesting question what the minimum number of ingredients are that we need to add to semiclassical gravity in order to uncover more detailed features of the UV and restore factorization.

Several suggestions exist in the literature, like half-wormholes, various branes, non-local interactions, ….

See e.g.
Gao, Jafferis, Kolckmeyer ‘21
Saad, Shenker, Stanford, Yao ’21
Blommaert, Kruthoff ‘21
Mukhametzhanov ‘21
Blommaert, Iliesiu, Kruthoff ‘21

CONCLUSIONS

Semi-classical gravity is the theory of the statistics of the high-energy, chaotic sector of the theory.

Picture is consistent with known wormhole solutions and predicts new wormhole solutions.

Is single-sided information enough or do wormholes yield genuine new information (so far not!).

What is the right overarching statistical framework?

Can we get more insight from e.g. bootstrap approaches?

Can one prove the chaotic random matrix theory nature of the high-energy sector of a strongly coupled CFT?

Or there any deep lessons for other chaotic systems in nature?

It seems very difficult to probe interesting aspects of quantum gravity using semi-classical gravity alone.

Is there a fundamental (gravitational) principle which restores factorization?

Semi-classical gravity is “averaging agnostic”.