Universal feature of charged entanglement entropy

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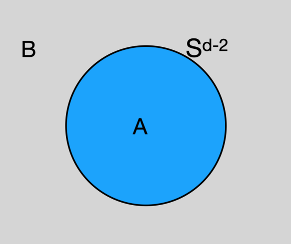
Based on Phys. Rev. Lett. 129 (2022) 2, 021601
w/ P. Bueno, Á. Murcia and A. Rivadulla

Iberian Strings - Murcia 11-13 January 2023

Entanglement entropy (EE) of CFTs across a spherical entangling region $\mathbb{S}^{d-2}$



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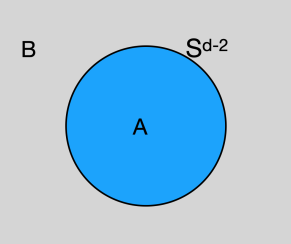

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The universal (regularized) part of the EE is given by Myers, Sinha '10, '11

$$S_{\rm EE} = v_{d-1} a^{\star}\,, \quad \text{where} \quad v_{d-1} \equiv \begin{cases} (-)^{\frac{d-2}{2}} 4 \log(\frac{R}{\delta}) & \text{even } d \\ (-)^{\frac{d-1}{2}} 2\pi & \text{odd } d \end{cases}$$

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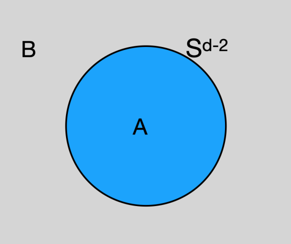

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Goal: generalize this result to the case of finite chemical potential

Introduction

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More generally: **charged Rényi entropies**

$$S_n(\mu) = \frac{1}{1-n} \log \operatorname{Tr} \left[\frac{\rho_A e^{\mu Q_A}}{n_A(\mu)} \right]^n$$

$$Q_A = \int_A d^{d-1}x J^t \,, \qquad n_A(\mu) = \operatorname{Tr} \rho_A e^{\mu Q_A}$$

$$S_{\rm EE}(\mu) = \lim_{n \to 1} S_n(\mu)$$

- $\mu =$ chemical potential
- For $\mu = 0$ we recover the usual RE and EE
- We wish to study the expansion of $S_{\rm EE}(\mu)$ around $\mu = 0$

Tools we use to attack this problem:

(1) Relation between entanglement and thermodynamic entropies

S_n across $\mathbb{S}^{d-2}$ $\sim$ thermal entropy on the $\mathbb{R}\times\mathbb{H}^{d-1}$ Belin, Hung, Maloney, Matsuura, Myers, Sierens '13

$$ds^2 = -dt^2 + dx^a dx^a \quad \Rightarrow \quad ds^2 = -d\tau^2 + R^2\left(d\chi^2 + \sinh^2\chi\, d\Omega^2_{d-2}\right)$$

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(2) AdS/CFT

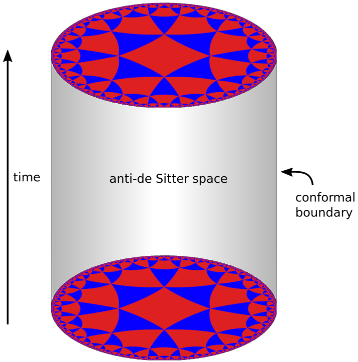
Thermal entropy of a CFT $\leftrightarrow$ thermodynamic entropy of a black hole

CFT on $\mathbb{R}\times\mathbb{H}^{d-1}$ $\leftrightarrow$ black holes with hyperbolic horizons

Chemical potential $\leftrightarrow$ black holes with electric charge

Especially useful: holography with higher derivatives

AdS/CFT correspondence



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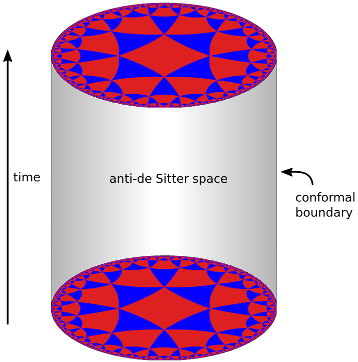

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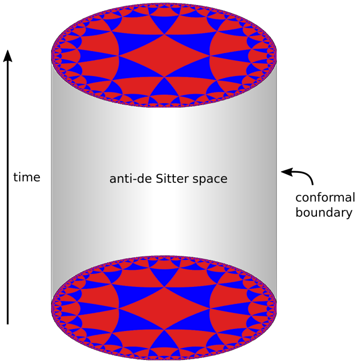

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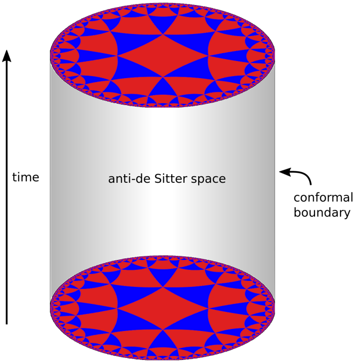

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Higher-derivative gravity in the bulk: two different views

1. Corrections to EG $\sim$ finite N and λ corrections in the CFT
2. Non-perturbative higher-derivative gravity $\sim$ more general CFTs

Higher-derivative gravity as holographic models

Many applications

- Holographic theories with $a \neq c$ in $d = 4$. General form of the correlator $\langle TTT \rangle$ e.g., Hofman, Maldacena '08; Boer, Kulaxizi, Parnachev '09; Myers, Paulos, Sinha '10; etc
- Find new phenomena, e.g., new types of phase transitions e.g., Camanho, Edelstein, Giribet, Gomberoff '14; Frassino, Kubiznak, Mann, Simovic '14; Hennigar, Brena, Mann '15
- Test universality of holographic EG results. For instance, $\eta/s \neq 1/4\pi$ e.g., Kats, Petrov '07; Brigante, Liu, Myers, Shenker, Yaida '07; Buchel, Myers, Sinha '08; Cai, Nie, Sun '08; Myers, Paulos, Sinha '10
- Provide evidence of true universality (e.g., holographic c-theorem Myers, Sinha '10, Renyi entropy Perlmutter '14; Chu, Miao '16, corner contribution to EE Bueno, Myers, Witczak-Krempa '15, free energy on squashed spheres Bueno, Cano, Hennigar, Mann '19...)

Introduction

Plan of the talk

- Holographic computation of charged EE
- Universal result from holography
- Proof of the universal relation from first principles

We consider bulk theories with a $(d-2)$-form B,

$$I = \int d^{d+1}x\sqrt{|g|}\,\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, H_{\mu_1\cdots\mu_{d-1}})\,, \quad H = dB$$

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Four-derivative "Electromagnetic Quasitopological Gravities" (see Javier's and Ángel's talks!)

$$\begin{aligned}I_{\rm EQG,4}=&\frac{1}{16\pi G}\int d^{d+1}x\sqrt{|g|}\Big[R+\frac{d(d-1)}{L^2}-\frac{2}{(d-1)!}H^2+\frac{\lambda}{(d-2)(d-3)}L^2\mathcal{X}_4\\&+\frac{2\alpha_1L^2}{(d-1)!}\left(H^2R-(d-1)(2d-1)R^{\mu\nu}{}_{\rho\sigma}(H^2)^{\rho\sigma}{}_{\mu\nu}\right)+\\&+\frac{2\alpha_2L^2}{(d-1)!}\left(R^{\mu}{}_{\nu}(H^2)^{\nu}{}_{\mu}-(d-1)R^{\mu\nu}{}_{\rho\sigma}(H^2)^{\rho\sigma}{}_{\mu\nu}\right)+\frac{\beta L^2}{(d-1)!^2}(H^2)^2\Big]\end{aligned}$$

$$(H^2)^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_n}\equiv H^{\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_{d-1}}H_{\nu_1\cdots\nu_n\mu_{n+1}\cdots\mu_{d-1}}$$

Electromagnetic dual theory

$$\mathcal{L}_{\text{dual}} = \mathcal{L} + \frac{1}{4\pi G(d-1)!}(\star F)_{\alpha_1 \ldots \alpha_{D-2}} H^{\alpha_1 \ldots \alpha_{D-2}} \,, \qquad F = dA$$

$$F = 4\pi G(d-1)! \star \frac{\partial \mathcal{L}}{\partial H}$$

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$$\begin{aligned}\tilde{I}_{\text{EQT},4} = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{|g|}\Bigg[& R + \frac{d(d-1)}{L^2} - F^2 + \frac{\lambda}{(d-2)(d-3)}L^2\mathcal{X}_4 \\ & + \frac{L^2}{d-2}RF^2\left(3d\alpha_1 + \frac{d\alpha_2}{(d-1)}\right) \\ & - \frac{2L^2}{d-2}F_{\mu\alpha}F_{\nu}{}^{\alpha}R^{\mu\nu}\left(4(2d-1)\alpha_1 + \frac{(3d-2)\alpha_2}{(d-1)}\right) \\ & + \frac{2L^2}{d-2}F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma}((2d-1)\alpha_1 + \alpha_2) + \frac{\beta}{4}L^2(F^2)^2 + \mathcal{O}(L^4)\Bigg]\end{aligned}$$

Holographic computation

Setup: we consider the theory $I_{\mathrm{EQT},4}$ as a model for a holographic CFT

To make contact with holography we will use its electromagnetic dual theory

$$h_{ab} \to T^{ab}\,, \quad A_a \to \ell_* J^a$$

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AdS vacuum:

$$ds^2 = \tilde{L}^2 \frac{dr^2}{r^2} + \frac{r^2}{L^2}\eta_{ab}dx^a dx^b$$

The AdS radius is denoted by $\tilde{L} = L/\sqrt{f_\infty}$, and we have

$$f_\infty = \frac{1}{2\lambda}\left[1 - \sqrt{1-4\lambda}\right]$$

Observe that the value of λ is bounded to $\lambda \leq 1/4$

Hyperbolic black holes

$$ds^2 = -\frac{\tilde{L}^2}{R^2} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Xi^2 \,, \quad H = Q\omega_{d-1}$$

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$$f(r) = -1 + \frac{r^2}{2\lambda L^2}\left[\Gamma(r) \pm \sqrt{\Gamma^2(r) - 4\lambda(1+\Upsilon(r))}\right]$$

$$\begin{aligned}\Gamma(r) =& 1 - \frac{2\alpha_1 L^2 Q^2}{r^{2(d-1)}} \,, \quad \Upsilon(r) = -\frac{mL^2}{(d-1)r^d} - \frac{\beta L^4 Q^4}{(3d-4)(d-1)r^{4(d-1)}} \\ &+ \frac{2L^2Q^2}{(d-1)(d-2)r^{2(d-1)}}\left(1 - (d-2)\frac{L^2}{r^2}(4(d-1)\alpha_1 + \alpha_2)\right)\end{aligned}$$

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In the dual frame we have a electrostatic potential $A = \Phi(r)dt$,

$$\Phi(r) = -\frac{\tilde{L}Q}{R}\left[\frac{1}{(d-2)r^{d-2}} - \frac{L^2\alpha_1}{r^d}\left(3(d-1) + 3(d-1)f(r) + rf'(r)\right) - \frac{L^2\alpha_2}{r^d}(1+f(r)) - \frac{L^2Q^2\beta}{(3d-4)r^{3d-4}}\right] + \Phi_\infty$$

Holographic computation

The black hole horizon is the largest root of $f(r_+) = 0$. From this, we get

- Hawking temperature $T = \frac{f'(r_+)\tilde{L}}{4\pi R}$
- Chemical potential $\mu = \Phi_\infty/\ell_\star$ (from $\Phi(r_+) = 0$)
- Entropy (Wald's formula) $S = -2\pi \int_{\mathcal{H}} d^{d-1}x\sqrt{h}\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\alpha\beta}}$

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We have

$$S = \frac{r_+^{d-1} V_{\mathbb{H}^{d-1}}}{4G}\left[1 + 2\alpha_1 \frac{Q^2 L^2}{r_+^{2d-2}} - \frac{2(d-1)L^2\lambda}{(d-3)r_+^2}\right]$$

and we should understand $S = S(T, \mu)$.

Here $V_{\mathbb{H}^{d-1}}$ is the volume of the hyperbolic space, whose regularized part reads

$$V_{\mathbb{H}^{d-1}} \equiv v_{d-1} \frac{\Omega_{d-1}}{4\pi}$$

Holographic entanglement entropy on a disk

- For $T = 1/(2\pi R)$, the hyperbolic black hole entropy computes the EE, $S = S_{\rm EE}(\mu)$
- Dependence on μ is complicated. Expansion around $\mu = 0$ yields

Holographic entanglement entropy on a disk

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- Dependence on μ is complicated. Expansion around $\mu = 0$ yields

$$\boxed{\frac{S_{\rm EE}^{\rm EQG}(\mu)}{v_{d-1}} = a^{\star}_{\rm GB} + \frac{\pi^{(d-2)/2}(d-2)^2[1-3d(d-1)\alpha_1 f_\infty - d\alpha_2 f_\infty]}{(d-1)8\Gamma(d/2)\alpha_{\rm eff}^2}\frac{\tilde{L}^{d-3}\ell_*^2}{G}(\mu R)^2}$$

where $\alpha_{\rm eff} = 1 - f_\infty\alpha_1(3d^2-7d+2) - f_\infty\alpha_2(d-2)$ and

$$a^{\star}_{\rm GB} = \frac{\tilde{L}^{d-1}}{8G}\frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\left[1 - \frac{2(d-1)}{d-3}\lambda f_\infty\right].$$

- $a^{\star}_{\rm GB}$ is the correct quantity for holographic GB gravity (conformal anomaly or free energy on the sphere)
- Can we interpret the μ^2 term?

Current two-point functions

$$\langle J_a(x) J_b(0) \rangle = \frac{C_J}{|x|^{2(d-1)}} I_{ab}(x)$$

where

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Result:

$$C_J = \frac{1}{\alpha_{\mathrm{eff}}} \frac{\Gamma(d)}{\Gamma(d/2-1)} \frac{\ell_*^2 \tilde{L}^{d-3}}{4\pi^{d/2+1} G}\,, \qquad \alpha_{\mathrm{eff}} = 1 - f_\infty \alpha_1 (3d^2 - 7d + 2) - f_\infty \alpha_2 (d-2)$$

Energy fluxes

Consider a local insertion of T_{ab} or J_a and we measure the energy flux at null infinity Hofman, Maldacena '08

$$\langle \mathcal{E}(\vec{n}) \rangle_J = \frac{E}{\Omega_{(d-2)}} \left[1 + a_2 \left(\frac{|\epsilon \cdot n|^2}{|\epsilon|^2} - \frac{1}{d-1} \right) \right]$$

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The holographic computation of $\langle \mathcal{E}(\vec{n}) \rangle_J$ requires studying A_μ fluctuations on a shock-wave background

$$ds^2 = \frac{\tilde{L}^2}{u^2} \left[\delta(y^+) W\left(y^i, u\right) \left(dy^+\right)^2 - dy^+ dy^- + \sum_{i=1}^{d-2} \left(dy^i\right)^2 + du^2 \right]$$

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For holographic Einstein-Maxwell $a_2 = 0$. For our theory

$$a_2 = -\frac{2d(d-1)\left((2d-1)\alpha_1 + \alpha_2\right) f_\infty}{(d-2)\alpha_{\text{eff}}}$$

Three-point function $\langle TJJ \rangle$

$$\langle T_{ab}(x_1) J_c(x_2) J_d(x_3) \rangle = \frac{t_{abef}(X_{23}) I_c^e(x_{21}) I_d^f(x_{31})}{|x_{12}|^d |x_{13}|^d |x_{23}|^{d-2}}$$

where $t_{abcd}(X_{23})$ is given by

$$t_{abcd}(X^a) \equiv \hat{a} h^{(1)}_{ab}(\hat{X}^a)\delta_{cd} + \hat{b} h^{(1)}_{ab}(\hat{X}^a) h^{(1)}_{cd}(\hat{X}^a) + \hat{c} h^{(2)}_{abcd}(\hat{X}^a) + \hat{e} h^{(3)}_{abcd}(\hat{X}^a)$$

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Conservation laws imply

$$d\hat{a} - 2\hat{b} + 2(d-2)\hat{c} = 0\,, \quad \hat{b} - d(d-2)\hat{e} = 0$$

Ward identity yields:

$$C_J = \frac{2\pi^{d/2}}{\Gamma(d/2+1)}(\hat{c}+\hat{e})$$

Energy flux $\langle \mathcal{E}(\vec{n})\rangle_J \sim \int \langle TJJ\rangle / \int \langle JJ\rangle$ Chowdhury, Raju, Sachdev, Singh, Strack '12

$$a_2 = \frac{(d-1)(d(d-2)\hat{e}-\hat{c})}{(d-2)(\hat{c}+\hat{e})}$$

Universality of charged entanglement entropy

$$\frac{S_{\rm EE}^{\rm EQG}(\mu)}{v_{d-1}} = a^{\star}_{\rm EQG} + \frac{\pi^{(d-2)/2}(d-2)^2[1-3d(d-1)\alpha_1 f_\infty - d\alpha_2 f_\infty]}{(d-1)8\Gamma(d/2)\alpha_{\rm eff}^2}\frac{\tilde{L}^{d-3}\ell_*^2}{G}(\mu R)^2 + \mathcal{O}(\mu^4)$$

$$C_J = \frac{1}{\alpha_{\rm eff}}\frac{\Gamma(d)}{\Gamma(d/2-1)}\frac{\ell_*^2\tilde{L}^{d-3}}{4\pi^{d/2+1}G}$$

$$a_2 = -\frac{2d(d-1)\left((2d-1)\alpha_1+\alpha_2\right)f_\infty}{(d-2)\alpha_{\rm eff}}$$

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We are able to write the EE as follows

$$\boxed{\frac{S_{\rm EE}^{\rm EQG}(\mu)}{v_{d-1}} = a^\star_{\rm EQG} + \frac{\pi^d C_J^{\rm EQG}}{(d-1)^2\Gamma(d-2)}\left[1+\frac{(d-2)a_2^{\rm EQG}}{d(d-1)}\right](\mu R)^2 + \ldots}$$

Conjecture: universal relation for all CFTs in $d \geq 3$

Proof from first principles

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Twist operators: used as an alternative to the replica trick $\mathrm{Tr}\,\rho_A^n = \langle \sigma_n \rangle_n$

Calabrese, Cardy '04; Hung, Myers, Smolkin, Yale '11; Belin, Hung, Maloney, Matsuura, Myers, Sierens '13

Scaling dimension h_n and magnetic response k_n

$$\langle T_{ab}\sigma_n(\mu)\rangle_n = -\frac{h_n(\mu)}{2\pi}\frac{b_{ab}}{y^d}\,, \quad \langle J_a \sigma_n(\mu)\rangle_n = \frac{ik_n(\mu)}{2\pi}\frac{\tau_a}{y^{d-1}}$$

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For $\mathbb{S}^{d-2}$ entangling surface, these are related to thermo in hyperbolic space

$$h_n(\mu) = \frac{2\pi n}{d-1}R^d\left(\mathcal{E}(T_0,\mu=0) - \mathcal{E}(T_0/n,\mu)\right)\,, \quad k_n(\mu) = 2\pi n R^{d-1}\rho(n,\mu)$$

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Expansions around $n=1$, $\mu=0$ are related to integrated T and J correlators.

$$k_n\Big|_{n=1,\mu=0} = \partial_n k_n\Big|_{n=1,\mu=0} = 0\,,$$

$$\partial_\mu k_n\Big|_{n=1,\mu=0} = \frac{16R\pi^{d+1}}{\Gamma(d+1)}[\hat{c}+\hat{e}] \quad \leftarrow \langle JJ\rangle$$

$$\partial_n\partial_\mu k_n\Big|_{n=1,\mu=0} = \frac{16R\pi^{d+1}}{d\Gamma(d+1)}[2\hat{c} - d(d-3)\hat{e}] \quad \leftarrow \langle TJJ\rangle$$

Proof from first principles

Let us consider a CFT on $\mathbb{R} \times \mathbb{H}^{d-1}$ at temperature $T = T_0/n$.

First law for the grand potential

$$d\Omega = -SdT - Nd\mu$$

where $N = V_{\mathbb{H}^{d-1}} R^{d-1} \rho$ is the total charge.

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Writing N in terms of the magnetic response $k_n(\mu)$, and using that $\partial_T = -\frac{T_0}{T^2}\partial_n$, we have

$$\partial_\mu S = -\frac{T_0 V_{\mathbb{H}^{d-1}}}{2\pi T^2}\partial_n \left(\frac{k_n(\mu)}{n}\right)$$

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Evaluating for $n = 1$ ($T = T_0$) and $\mu = 0$ it follows that the derivative vanishes

$$\partial_\mu S_{\text{EE}}\Big|_{\mu=0} = 0$$

Taking a second derivative with respect to μ

$$\partial_\mu^2 S = -\frac{T_0 V_{\mathbb{H}^{d-1}}}{2\pi T^2}\partial_\mu \partial_n \left(\frac{k_n(\mu)}{n}\right)$$

Evaluating again for $n = 1$ and $\mu = 0$, we have

$$\partial_\mu^2 S_{\text{EE}}\Big|_{\mu=0} = R V_{\mathbb{H}^{d-1}} \left[\partial_\mu k_n - \partial_\mu \partial_n k_n\right]\Big|_{n=1,\mu=0}$$

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Using the universal relations for the magnetic response we get

$$\begin{aligned}\partial_\mu^2 S_{\text{EE}}\Big|_{\mu=0} &= V_{\mathbb{H}^{d-1}} \frac{16(d-2)R^2\pi^{d+1}}{d\Gamma(d+1)}\left[\hat{c} + d\hat{e}\right] \\ &= \frac{2v_{d-1}R^2\pi^d C_J}{(d-1)^2\Gamma(d-2)}\left[1 + \frac{(d-2)a_2}{d(d-1)}\right]\end{aligned}$$

Proof from first principles

Conclusion: universal relation for the charged EE valid for any $d \geq 3$ CFT

$$\boxed{\frac{S_{\rm EE}(\mu)}{v_{d-1}} = a^\star + \frac{\pi^d C_J}{(d-1)^2\Gamma(d-2)}\left[1+\frac{(d-2)a_2}{d(d-1)}\right](\mu R)^2 + \ldots}$$

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Observation: $-\frac{d-1}{d-2} \leq a_2 \leq d-1$ and $C_J > 0$ implies

$$\partial_\mu^2 S_{\rm EE}\big|_{\mu=0} > 0$$

Example 1: Dirac fermion and conformally coupled scalar in $d = 4$ Belin, Hung, Maloney, Matsuura, Myers, Sierens '13

$$\frac{S_{\rm EE}^{\rm f}(\mu)}{v_3} = a_{\rm f}^{\star} + \frac{(\mu R)^2}{12}\,, \qquad \frac{S_{\rm EE}^{\rm s}(\mu)}{v_3} = a_{\rm s}^{\star} + \frac{(\mu R)^2}{24} + \frac{|\mu R|^3}{24}$$

where $a_{\rm f}^{\star} = 11/360$, $a_{\rm s}^{\star} = 1/360$. We have Osborn '93, Petkou '96; Hofman, Maldacena '08; Chowdhury, Raju, Sachdev, Singh, Strack '13

$$C_J^{\rm f} = \frac{1}{\pi^4}\,, \quad C_J^{\rm s} = \frac{1}{4\pi^4}\,, \quad a_2^{\rm f} = -\frac{3}{2}\,, \quad a_2^{\rm s} = 3$$

finding perfect agreement with our formula.

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Example 2: $d=4$, $\mathcal{N}=1$ SCFTs. $J=\mathrm{U}(1)$ R-current Osborn '99; Hofman, Maldacena '08

$$C_J^{\mathcal{N}=1,\,\mathrm{U}(1)_R}=\frac{4c}{\pi^4}\,,\qquad a_2^{\mathcal{N}=1,\,\mathrm{U}(1)_R}=3\left(1-\frac{a}{c}\right)$$

$$S_{\mathrm{EE}}^{\mathcal{N}=1,\,\mathrm{U}(1)_R}=v_3\left[a+\frac{2}{3}\left(c-\frac{a}{3}\right)(\mu R)^2+\ldots\right]$$

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- Presented a novel universal relation for the charged EE across spherical entangling surfaces
- Holography with higher derivatives very interesting tool to guess this kind of relations (previous examples)
- Proof from first principles was accessible in this case (in other cases this is much harder)
- Case $d = 2$ is special. $a_2 = 0$, C_J divergent, $S_{\text{EE}}(\mu)$ is non-analytical and has linear terms in μ.

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Thank you for your attention