Non-invertible symmetries and theories of class $\mathcal{S}$

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Symmetries in Quantum Field Theory I

- Symmetries first appear in the form of unitary operators $\mathcal{U}(\Sigma_t)$, acting on the Hilbert space and on operators:

$$\mathcal{U}(\Sigma_t)\,|\psi\rangle = |\psi'\rangle, \qquad \mathcal{U}(\Sigma_t)\,\mathcal{O}\,\mathcal{U}(\Sigma_t)^{-1} = \mathcal{O}'$$

  In a Lorentzian or Euclidean QFT there is no canonical choice of time direction: Σ_t can be tilted.

- Also, for continuous symmetries

$$\mathcal{U}(\Sigma_t) \;=\; \exp(2\pi i\,\alpha\,Q), \quad Q = \int_{\Sigma_t} *j, \quad d*j = 0,$$

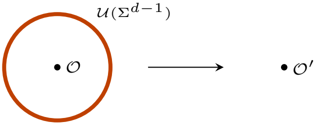
  so path integral is invariant under (small enough) deformations of Σ_t. This motivates to consider operators with supports on arbitrary submanifolds: $\mathcal{U}(\Sigma^{d-1})$.

Symmetries in Quantum Field Theory II

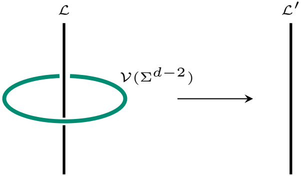
Symmetries correspond to topological defects.

[Gaiotto, Kapustin, Seiberg, Willett '14]

- Conventional symmetries a topological defects of co-dimension one.
- Charged operators are local operators, and charged objects are particles.



- An immediate generalization: consider topological defects of higher co-dimension $\longrightarrow$ higher-form symmetries.
- Charged operators are lines, surfaces etc., charged objects are strings and branes.

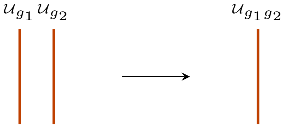


Symmetries in Quantum Field Theories III

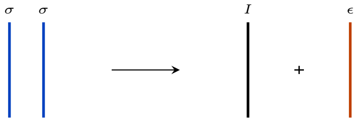
Symmetries correspond to topological defects.

[Gaiotto, Kapustin, Seiberg, Willett '14]

- Conventional symmetries are associated with groups, and fuse according to the group law.



- However, more general fusion rules, with several operators on the r.h.s., are possible. Examples are lines in Non-Abelian Chern-Simons theories and line operators in the 2d RCFTs.



Example from the 2d Ising model.

- Defects, fusing not according to the group law, are reffered to as <u>non-invertible</u> (or categorical) symmetries.

Overview

Non-invertible symmetries in four dimensions from a mixed anomaly I

[Kaidi, Ohmori, Zheng '21]

- Consider a theory $\mathcal{T}$ with a 0-form symmetry and a 1-form symmetry, e.g. $\mathbb{Z}_2^{(0)}$ and $\mathbb{Z}_2^{(1)}$. The corresponding background fields are $A^{(1)}$ and $B^{(2)}$. Let us also assume that there is the mixed 't Hooft anomaly, encoded in the 5d anomaly theory

$$S_{\text{Anomaly}} = \pi i \int_{X_5} A^{(1)} \cup \frac{\mathcal{P}(B^{(2)})}{2}, \qquad M_4 = \partial X_5$$

- Let also $\mathcal{D}(\mathcal{M}_3, B^{(2)})$ be the defect, corresponding to $\mathbb{Z}_2^{(0)}$. Due to the anomaly it is not invariant under the $B^{(2)}$ gauge transformations, the combination which is invariant is given by

$$\mathcal{D}(\mathcal{M}_3, B^{(2)}) e^{\pi i \int_{X_4} \mathcal{P}(B^{(2)})/2}, \qquad M_3 = \partial X_4$$

- We now turn to the theory $\mathcal{T}/\mathbb{Z}_2^{(1)}$ with gauged 1-form symmetry, promoting $B^{(2)}$ into a dynamical field $b^{(2)}$. $\mathcal{D}$ now happens to be a non-genuine defect, since it depends on the values of dynamicsl fields outside of its support.

Non-invertible symmetries in four dimensions from a mixed anomaly II

[Kaidi, Ohmori, Zheng '21]

$$\mathcal{N}_S = \mathcal{D}\mathcal{A}^{(2,1)}$$

$$e^{\pi i \int \mathcal{P}(b^{(2)})/2 - \pi i \int \mathcal{P}(b^{(2)})/2} = 1$$

- Nevertheless, the defect $\mathcal{D}'(\mathcal{M}_3, b^{(2)})$ can be turned into a genuine defect by coupling it to the 3d TQFT $\mathcal{A}^{(2,1)}(\mathcal{M}_3, b^{(2)})$ [Hsin, Lam, Seiberg '18]

$$\mathcal{N}_S(\mathcal{M}_3, b^{(2)}) = \mathcal{D}(\mathcal{M}_3, b^{(2)})\, \mathcal{A}^{(2,1)}(\mathcal{M}_3, b^{(2)})$$

$$\mathcal{A}^{(2,1)}(\mathcal{M}_3, b^{(2)}) = e^{\frac{i}{2\pi}\int_{\mathcal{M}_3} -xdx + 2xdy + 2b^{(2)}y}$$

- While in the original theory $\mathcal{T}$ we had

$$\mathcal{D} \times \mathcal{D} = 1,$$

in the gauged theory $\mathcal{T}/\mathbb{Z}_2^{(1)}$ the corresponding fusion rule reads

$$\mathcal{N}_S \times \mathcal{N}_S = \frac{1}{|H^0(\mathcal{M}_3, \mathbb{Z}_2)|} \sum_{\Sigma \in H_2(\mathcal{M}_3, \mathbb{Z}_2)} (-1)^{Q(\Sigma)} L(\Sigma) \equiv \mathcal{C}^{(1)}(\mathcal{M}_3),$$

where $L(\Sigma) := e^{i\pi \int_\Sigma b^{(2)}}$ and $Q(\Sigma)$ is the triple intersection number of Σ in $\mathcal{M}_3$.

Non-invertible symmetries in four dimensions from a mixed anomaly III

- To summarize, while we managed to construct the genuine gauge-invariant defect $\mathcal{N}_S$, it turns out to be non-invertible.
- The right hand side of the fusion rule

$$\mathcal{C}^{(j)}(\mathcal{M}_3) \equiv \frac{1}{|H^0((\mathcal{M}_3,),\mathbb{Z}_2)|} \sum_{\Sigma \in H_2(\mathcal{M}_3,\mathbb{Z}_2)} (-1)^{jQ(\Sigma)} L(\Sigma)$$

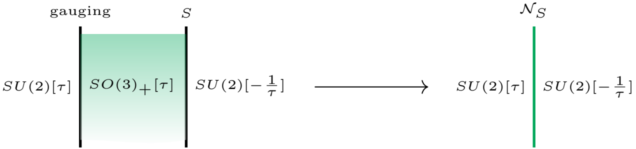
  is known as a condensate, and provides an example of *higher gauging*. [Roumpedakis, Seifnashri, Shao '22]
- Examples of this setup include: $SO(3)$ YM at $\theta = \pi$ (0-form symmetry is the time-reversal symmetry), $SO(3)$ $\mathcal{N} = 1$ SYM (0-form symmetry is the R-symmetry), $SO(3)_-$ $\mathcal{N} = 4$ SYM at $\tau = i$ (0-form symmetry is the S-self-duality).
- The setup can be extended to other types of ABJ anomalies. [Choi, Lam, Shao '22]
- The type of non-invertible defects considered above was dubbed *non-intrinsically non-invertible defects*, since one can choose the global form where all the symmetries are invertible. [Kaidi, Zafrir, Zheng '22].

Non-invertible symmetries in four dimensions from a duality

[Choi, Cordova, Hsin, Lam, Shao '21]

- Non-invertible symmetry from invariance under the gauging of a 1-form symmetry.

$$\mathcal{T} \simeq \mathcal{T}/G^{(1)}$$



- The fusion rule is given by

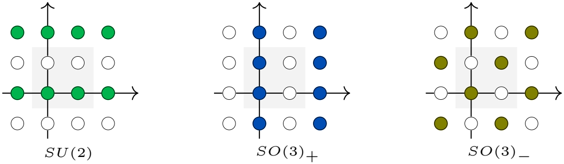
$$\mathcal{N}_S \times \mathcal{N}_S = \mathcal{C}^{(0)}(\mathcal{M}_3)$$

- Examples include 4d Maxwell theory at rational τ and $\mathcal{N} = 4$ SYM at $\tau = i$ with certain gauge groups and global forms. More examples follow.

Global forms of su(p) $\mathcal{N} = 4$ Super-Yang-Mills I

- We are going to consider $SU(N)$ theories with $N = p$ prime.
- Having specified the gauge algebra $su(p)$, we still have to make a choice of a global form, which is equivalent to choosing a point in the charge lattice: $(e, m) \in \mathbb{Z}_p \times \mathbb{Z}_p$. [Aharony, Seiberg, Tachikawa '13]
- Other charges are then required to satisfy the Dirac quantization condition:

$$em' - e'm = 0 \mod p \tag{1}$$



- There are $p + 1$ global forms ($SU(p)$ and $PSU(p)_{(k)}$, $k = 0, ..., p - 1$).
- In fact, slightly more refined spectroscopy turns out to be useful. For each chosen lattice we may stack with q copies of the invertible phase

$$e^{\frac{2\pi i q}{p} \int \frac{p+1}{2} B \cup B}, \qquad q = 0, ..., p - 1$$

This leads to a total of $p(p + 1)$ global variant of a theory.

Global forms of SU(p) $\mathcal{N} = 4$ Super-Yang-Mills II

- There are two important groups, acting on the space of global form.
- The first of them is the $SL(2,\mathbb{Z})$ Montonen-Olive duality (which also acts on the coupling τ):

$$S: \qquad (e,m) \longrightarrow (m,-e)$$
$$T: \qquad (e,m) \longrightarrow (e+m,m)$$

- Given a 4d theory with a $\mathbb{Z}_p^{(1)}$ 1-form symmetry, one can introduce two operations:

$$\sigma: \qquad \mathcal{Z}[B^{(2)}] \longrightarrow \sum_{b^{(2)} \in H^2(\mathcal{M}_4,\mathbb{Z}_p)} e^{2\pi i \int b^2 \cup B^{(2)}} \mathcal{Z}[b^{(2)}]$$
$$\tau: \qquad \mathcal{Z}[B^{(2)}] \longrightarrow e^{\frac{2\pi i}{p} \int \frac{p+1}{2} B \cup B} \mathcal{Z}[B^{(2)}]$$

Together they generate the $SL(2,\mathbb{Z}_p)$ group.

Global forms of SU(p) $\mathcal{N} = 4$ Super-Yang-Mills III

- A useful observation: global forms can be encoded in a ray matrix

$$M \in SL(2, \mathbb{Z}_p)/\mathbb{Z}_p^\times$$

  Indeed, $|SL(2, \mathbb{Z}_p)|/|\mathbb{Z}_p^\times| = p(p+1)$.

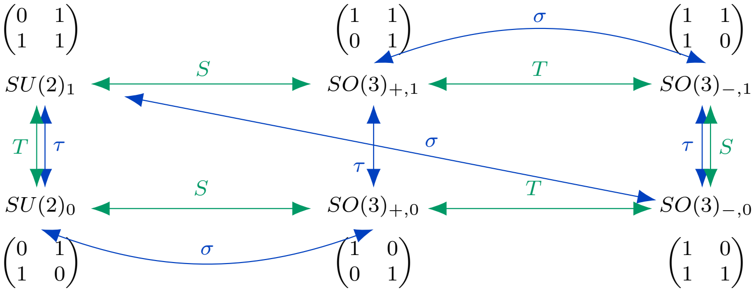
- Elements of $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}_p)$ act on M as

$$M \longrightarrow F(S,T)^T \, M \, G(\sigma, \tau), \qquad F(S,T) \in SL(2,\mathbb{Z}),\ G(\sigma,\tau) \in SL(2,\mathbb{Z}_p)$$

  Here S, T, σ, τ are represented by

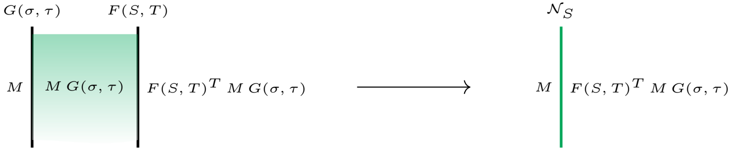
$$S, \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T, \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Global forms of SU(p) $\mathcal{N} = 4$ Super-Yang-Mills IV



- Every global variant can be mapped to every other by the action of both S, $T \in SL(2,\mathbb{Z})$ and σ, $\tau \in SL(2,\mathbb{Z}_p)$.
- Every action of the modular group $F(S,T)$ can be undone by an appropriate topological manipulation $G(\sigma,\tau)$. This is true independent of the global variant M under consideration.

Non-invertible symmetries of $\mathcal{N} = 4$ $su(p)$ Super-Yang-Mills I



- In fact, it is a generic situation that a duality gives rise to a non-invertible defect, so it is more reasonable to ask when the symmetry turns out to be invertible. The corresponding global form must then satisfy the equation

$$F(S,T)^T M = \lambda M \tau^n, \qquad \lambda \in \mathbb{Z}_p^\times,\ n \in \mathbb{Z}_p$$

- Let us retain from looking at a possible anomaly, then we can concentrate on the first column of M:

$$F(S,T)^T M_1 = \lambda M_1 \tau^n, \qquad \lambda \in \mathbb{Z}_p^\times$$

- The duality transformation should also leave the coupling invariant: $F(S,T)\tau_{\text{YM}} = \tau_{\text{YM}}$. These are S with $\tau_{\text{YM}} = i$ and ST with $\tau_{\text{YM}} = e^{2\pi i/3}$.

Non-invertible symmetries of $\mathcal{N} = 4$ Super-Yang-Mills II

- For $F(S,T) = S$ we are led to consider the characteristic equation

$$\det(S - \lambda\mathbb{I}) = \lambda^2 + 1 = 0 \mod p$$

with the result

$$\#\,\text{solutions} = \begin{cases} 1, & p = 2 \\ 0, & p = 3 \\ 1 + (-1)^{\frac{p-1}{2}} & p > 3 \end{cases}$$

- For $F(S,T) = ST$ the characteristic polynomial is

$$\det(ST - \lambda\mathbb{I}) = \lambda^2 - \lambda + 1 = 0 \mod p$$

with the number of solutions

$$\#\,\text{solutions} = \begin{cases} 0, & p = 2 \\ 1, & p = 3 \\ 1 + (-3|p) & p > 3 \end{cases}$$

p	2	3	5	7	11	13	17	19	23	29
S intrinsic?	✗	✓	✗	✓	✓	✗	✗	✓	✓	✗
ST intrinsic?	✓	✗	✓	✗	✓	✗	✓	✗	✓	✓

Review of class $\mathcal{S}$ I

- The starting point is a 6d $\mathcal{N} = (2,0)$ SCFT (of A, D or E type), which is then compactified on a Riemann surface of genus g with n punctures. This gives rise to a 4d $\mathcal{N} = 2$ SCFT. [Gaiotto, '09].
- In this talk we will restrict ourselves to theories of the type A_{p-1}, and to Riemann surfaces with no punctures.
- 6d $\mathcal{N} = (2,0)$ SCFT are *relative theories*, meaning that they have *partition vectors*, rather than partition functions. [Witten '09, Freed, Teleman '12] This leads to an extra choice to be made, a maximal isotropic sublattice $\mathcal{L} \in H_3(X_6, \mathbb{Z}_p)$:

$$\langle M_3, M_3' \rangle = 0, \qquad M_3,\, M_3' \in \mathcal{L}$$

- For the geometry of the form $X_6 = X_4 \times \Sigma_g$ (and assuming $H_{1,3}(X_4, \mathbb{Z}) = 0$) we have

$$H_3(X_6, \mathbb{Z}) \;=\; H_1(\Sigma_g, \mathbb{Z}) \,\otimes\, H_2(X_4, \mathbb{Z})$$

Review of class $\mathcal{S}$ I

- Correspondingly, we have the splitting of the maximal isotropic lattices:

$$\mathcal{L} = L \otimes H(X_2, \mathbb{Z}), \qquad L \in H(\Sigma_g, \mathbb{Z})$$

So, in order to completely specify a class $\mathcal{S}$ theory, we need to pick up a maximal isotropic sublattice $\gamma \in H(\Sigma_g, \mathbb{Z})$ [Tachikawa, '13].

- Example: $su(2)$ $\mathcal{N} = 4$ SYM, with $H(\mathbb{T}^2, \mathbb{Z}_p) = \{1,\, A,\, B,\, A+B\}$:

$$L_A = \{1,\, A\} \longleftrightarrow SU(2)$$

$$L_B = \{1,\, B\} \quad \longleftrightarrow \quad SO(3)_+$$

$$L_{A+B} = \{1,\, A+B\} \quad \longleftrightarrow \quad SO(3)_-$$

- Besides the sublattice L, also a representative of a non-trivial class in $L_\perp \otimes H(X_4, \mathbb{Z}_p)$ should be chosen, where $L_\perp = H_1(\Sigma_g, \mathbb{Z}_p)/L$.
- This is summarized in the following notation for the theories under consideration:

$$\mathcal{T}_L^{p,g}[\Omega,\, \mathcal{B}], \qquad \mathcal{B} \in L_\perp \otimes H_2(X_4, \mathbb{Z}_p)$$

Above, Ω is the period matrix of the Riemann surface, encoding the coupling constants of the theory.

Non-invertible symmetries in class $\mathcal{S}$ II

- Conceptually, the story is very similar to what we have seen for the $\mathcal{N} = 4$ SYM.
- The global forms are encoded in symplectic "ray" matrices

$$M \in Sp(2g, \mathbb{Z})/GL(g, \mathbb{Z}_p,)$$

- There can be defined topological manipulations, gaugings of 1−form symmetries and stacking with invertible phases. Together they generate the group $Sp(2g, \mathbb{Z}_p)$. The elements of this group acts on the global structure of a theory:

$$M \longrightarrow M\,G, \qquad G \in Sp(2g, \mathbb{Z}_p$$

- There is the duality group, the Mapping Class Group of the Riemann surface (a.k.a. Gaiotto duality), which acts both on the period matrix (couplings) and the global forms:

$$\begin{aligned} M &\longrightarrow F^T\,M, \qquad F \in Sp(2g, \mathbb{Z}) \\ \Omega &\longrightarrow F[\Omega] \end{aligned}$$

- One can look for the fixed points (loci) in the space of couplings, $F(\Omega) = \Omega$. This is done in the math literature, up to $g = 5$.
- Finally, the elements of the duality groups, having fixed points, can be studied. The question is whether there are invariant isotropic sublattices in $H_1(\Sigma_g, \mathbb{Z}_p$. If so, we say that the non-invertible symmetry is non-intrinsic.

Non-invertible symmetries in class $\mathcal{S}$ II

- As an example, consider the case of $g = 2$.
- Singular moduli of $Sp(4,\mathbb{Z})$ occurring at isolated values of Ω, as well as loci of complex dimensions one and twp. Here we have defined $\rho := e^{\frac{2\pi i}{3}}$ and $\varepsilon := e^{\frac{2\pi i}{5}}$.

Order	Subgroup	Generators	Ω
10	$\mathbb{Z}_{10}$	ϕ	$\begin{pmatrix} \varepsilon & \varepsilon+\varepsilon^{-2} \\ \varepsilon+\varepsilon^{-2} & -\varepsilon^{-1} \end{pmatrix}$
24	$(\mathbb{Z}_2\times\mathbb{Z}_6)\rtimes\mathbb{Z}_2$	M_1, M_2, M_3	$\frac{i}{\sqrt{3}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	C, M_4	$\begin{pmatrix} \rho & 0 \\ 0 & i \end{pmatrix}$
32	$(\mathbb{Z}_4\times\mathbb{Z}_4)\rtimes_2$	M_5, M_6, M_7	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$
48	$GL(2,3)$	M_7, M_8	$\frac{1}{3}\begin{pmatrix} 1+2i\sqrt{2} & -1+i\sqrt{2} \\ -1+i\sqrt{2} & 1+2i\sqrt{2} \end{pmatrix}$
72	$\mathbb{Z}_3\times(\mathbb{Z}_6\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2$	M_7, M_9, M_{10}	$\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$
8	$\mathbb{Z}_2\times\mathbb{Z}_4$	C, N_1	$\begin{pmatrix} i & 0 \\ 0 & \tau_3 \end{pmatrix}$
	D_8	M_7, N_2	$\begin{pmatrix} \tau_1 & \\ & \tau_1 \end{pmatrix}$
	D_8	M_7, N_3	$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}$
12	$\mathbb{Z}_2\times\mathbb{Z}_6$	C, N_4	$\begin{pmatrix} \rho & 0 \\ 0 & \tau_3 \end{pmatrix}$
	D_{12}	N_5, N_6	$\begin{pmatrix} \tau_1 & \tau_1 \\ \tau_1 & \tau_1 \end{pmatrix}$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	C, P_1	$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}$
	$\mathbb{Z}_2\times\mathbb{Z}_2$	C, M_7	$\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_1 \end{pmatrix}$

Non-invertible symmetries in class $\mathcal{S}$ III

p	2	3	5	7	11	13	17	19	23	29
ϕ intrinsic?	✓	✓	✗	✓	✗	✓	✓	✓	✓	✓
M_1 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
M_2 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
M_3 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
M_7 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
M_8 intrinsic?	✗	✗	✗	✓	✗	✗	✗	✗	✓	✗
N_2 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
N_5 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
N_6 intrinsic?	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗

- We see that two generators indeed give rise to intrinsically non-invertible defects.
- There are cases where the group, leaving a particular period matrix invariant, is generated by two or more generators, and while each of them separately have an invariant sublattice, all of them together do not have one. In certain sence, this non-invertibility can be considered as intrinsic.

Conclusion

- We have discussed non-invertible symmetries in $\mathcal{N}=4$ SYM and class $\mathcal{S}$ at $g=2$. The cases of $g=3,4,5$ are also analyzed, while the further progress is hindered at the math side.
- The story of non-invertible symmetries in class S have an interesting perspective from the *Symmetry TFT* point of view.
- Geometrical methods, applied here, can be used in other contecst, e.g. to study non-invertible symmetries in theories, obtained by compactifications from 6d to 3d or 2d.
- Another intriguing direction, left uncovered, and which is currently a subject of active investigations, is the dynamical consequences that follow from the presence of non-invertible symmetries.

Thank you for your attention!