f(Lovelock) theories of gravity

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based on

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Introduction

Introduction

Higher order gravities:

- The EH action should be modified by quantum corrections. String Theory predicts an infinite series of higher order curvature terms.
- Cosmology. Inflation and accelerated expansion.
- Holography. The addition of higher curvature terms in the action allows us to extract information about general CFTs (*e.g.*, a free scalar). Bueno, Myers, Witczak-Krempa; Brigante; Kats; de Boer
- Well-known higher order theories:
  1. $f(R)$. Useful in cosmology models. Equivalent to a scalar-tensor theory. *e.g.*, Sotiriou, Faraoni
  2. Lovelock gravity. Most general theory with second order equations. Lanczos; Lovelock
- f(Lovelock) gravity is a natural generalization of $f(R)$ and Lovelock theories

Variational problem in f(Lovelock)

Variational problem in f(Lovelock)

The f(Lovelock) action is given by

$$S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g} f(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_{\lfloor D/2 \rfloor}), \tag{1}$$

where $\mathcal{L}_n$ are the Euler densities (ED)

$$\mathcal{L}_n = \frac{1}{2^n}\delta^{\mu_1 \dots \mu_{2p}}_{\nu_1 \dots \nu_{2n}} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \dots R^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}}. \tag{2}$$

$\mathcal{L}_n = 0$ if $D < 2n$ and $\sqrt{-g}\mathcal{L}_n$ is topological if $D = 2n$. The previous action (1) reduces to Lovelock-Lanczos and $f(R)$ theories when we choose f to be a linear combination of the ED or an arbitrary function of $R = \mathcal{L}_1$:

$$f_{LL} = \sum_{n=0}^{\lfloor D/2 \rfloor} \lambda_n L^{2n-2}\mathcal{L}_n, \quad f_R = f(R), \tag{3}$$

where L is a length scale and λ_n are dimensionless couplings.

Variational problem in f(Lovelock)

The variation of the action is

$$\delta S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{16\pi G}\int_{\partial M} d^{D-1}x\sqrt{|h|}\sum_{n=1}^{\lfloor D/2\rfloor}\partial_n f\,\delta v_n^{\mu} n_{\mu}, \quad (4)$$

where n_μ is the normal vector to the boundary and $h_{\mu\nu}$ is the induced metric on ∂M. Also, we have

$$\mathcal{E}_{\mu\nu} = \sum_{n=1}^{\lfloor D/2\rfloor}\left[\mathcal{E}^{(n)}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\mathcal{L}_n - 2P^{(n)}_{\lambda\mu\alpha\nu}\nabla^{\alpha}\nabla^{\lambda}\right]\partial_n f - \frac{1}{2}g_{\mu\nu}f, \quad (5)$$

$$\delta v_n^{\mu} = 2g^{\beta\sigma}P^{(n)\mu\nu}_{\alpha\beta}\nabla^{\alpha}\delta g_{\nu\sigma}. \quad (6)$$

The field equations are

$$\mathcal{E}_{\mu\nu} = 0. \quad (7)$$

- Fourth order equations
- Variational problem not well-defined: we need to set $\partial_\alpha \delta g_{\mu\nu}\big|_{\partial M} = 0$

Variational problem in f(Lovelock)

Boundary contribution

$$S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}\mathfrak{L} - \frac{1}{16\pi G}\int_{\partial M} d^{D-1}x\sqrt{|h|}\mathcal{B}. \tag{8}$$

	$\mathfrak{L}$	**Boundary term $\mathcal{B}$**	**B. conditions**
GR	R	$-2K$ Gibbons, Hawking; York	$\delta h_{\mu\nu}=0$
f(R)	$f(R)$	$-2f'(R)K$ Madsen, Barrow	$\delta h_{\mu\nu}=0,\ \delta R=0$
Lovelock	$\mathcal{L}_n$	Q_n Verwimp	$\delta h_{\mu\nu}=0$
f(Lovelock)	$f(\mathcal{L}_1,\ldots,\mathcal{L}_{\lfloor D/2\rfloor})$	???	???

$$Q_n \equiv -2n\int_0^1 dt\,\delta^{\mu_1\ldots\mu_{2n-1}}_{\nu_1\ldots\nu_{2n-1}}K^{\nu_1}_{\mu_1}\left(\frac{1}{2}R^{\nu_2\nu_3}_{\mu_2\mu_3}-t^2K^{\nu_2}_{\mu_2}K^{\nu_3}_{\mu_3}\right)\ldots\left(\frac{1}{2}R^{\nu_{2n-2}\nu_{2n-1}}_{\mu_{2n-2}\mu_{2n-1}}-t^2K^{\nu_{2n-2}}_{\mu_{2n-2}}K^{\nu_{2n-1}}_{\mu_{2n-1}}\right),$$

$K_{\mu\nu}$ extrinsic curvature of ∂M. In f(Lovelock) we **propose** the following boundary term

$$\boxed{\mathcal{B}_{f(\mathrm{Lovelock})} = \sum_{n=1}^{\lfloor D/2\rfloor}\partial_n f(\mathcal{L})Q_n.} \tag{9}$$

Variational problem in f(Lovelock)

The full **f(Lovelock)** action is then

$$S_f = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}f(\mathcal{L}_1, ..., \mathcal{L}_{\lfloor D/2\rfloor}) - \frac{1}{16\pi G}\int_{\partial M} d^{D-1}x\sqrt{|h|}\sum_{n=1}^{\lfloor D/2\rfloor}\partial_n f(\mathcal{L})Q_n, \tag{10}$$

and the variation when $\delta h_{\mu\nu} = 0$ is

$$(16\pi G)\delta S_f\Big|_{\delta h_{\mu\nu}=0} = \int_M d^D x\sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} - \int_{\partial M} d^{D-1}x\sqrt{|h|}\sum_{n,m=1}^{\lfloor D/2\rfloor}\partial_m\partial_n f\delta\mathcal{L}_m Q_n. \tag{11}$$

In order to extremize the action we must fix also the partial derivatives of f on the boundary:

$$\boxed{\delta\big(\partial_n f(\mathcal{L})\big)\Big|_{\partial M} = 0, \quad n = 1, ..., \lfloor D/2\rfloor.} \tag{12}$$

The number of independent conditions is equal to $r = \text{rank}(\partial_n\partial_m f)$. Since we have to fix the induced metric $h_{\mu\nu}$ and the derivatives $\partial_n f$, we conclude that the number of physical degrees of freedom in f(Lovelock) theory is

$$n_{\text{dof}} = \frac{D(D-3)}{2} + r. \tag{13}$$

With respect to GR there are r extra degrees of freedom.

Equivalence with scalar-tensor theory

Equivalence with scalar-tensor theory

Non-degenerate case

Let us consider again the f(Lovelock) action

$$S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}f(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_k), \qquad (14)$$

where $k = \lfloor D/2 \rfloor$.We want to construct an equivalent scalar-tensor theory.

Equivalence with scalar-tensor theory

Non-degenerate case

Let us consider again the f(Lovelock) action

$$S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}f(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_k), \tag{14}$$

where $k = \lfloor D/2 \rfloor$.We want to construct an equivalent scalar-tensor theory. If $\det(\partial_n\partial_m f) \neq 0$, f(Lovelock) is equivalent to

$$S' = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}\left\{\sum_{p=1}^{k}\varphi_n\mathcal{L}_n - \tilde{V}(\varphi_1, ..., \varphi_k)\right\}. \tag{15}$$

- $\tilde{V}$ is the Legendre transform of f (which exists because $\det(\partial_n\partial_m f) \neq 0$)
- The equivalence can be checked by using the field equation for φ_n: $\mathcal{L}_n = \partial_n\tilde{V}(\varphi)$. It is the inverse Legendre transform.
- This generalizes the case of $f(R)$, which is equivalent to Barrow, Cotsakis

$$S'_{f_R} = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}\left\{\varphi R - \tilde{V}(\varphi)\right\}. \tag{16}$$

Equivalence with scalar-tensor theory

Degenerate case

In the case in which $\det(\partial_n \partial_m f) = 0$ we cannot construct the Legendre transform of f, but it can be shown that f(Lovelock) is equivalent to

$$S' = \int_M d^D x \sqrt{-g} \Big\{ \sum_{i \in I} \varphi_i \mathcal{L}_i + \sum_{j \in J} g_j(\varphi_i) \mathcal{L}_j - \tilde{V}(\varphi_{i_1}, ..., \varphi_{i_r}) \Big\}. \tag{17}$$

- I is a subset of r indices and J the complementary set. Therefore there are r scalars, where $r = \text{rank}(\partial_n \partial_m f)$
- $g_j(\varphi_i)$ are certain functions and $\tilde{V}(\varphi_{i_1}, ..., \varphi_{i_r})$ is the *semi*-Legendre transform of f
- In conclusion, **f(Lovelock) gravity is equivalent to a scalar-tensor theory with a number of scalars equal to the rank of the Hessian matrix of f.**
- Note that the number of scalars coincide with the number of extra degrees of freedom in f(Lovelock)

Equivalence with scalar-tensor theory

Degenerate case

As an example, let us consider the theory

$$S=\frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\Big\{-2\Lambda_0+R+\alpha L^2R^2+\beta L^4 R\mathcal{L}_2+\gamma L^6\mathcal{L}_2^2\Big\}, \tag{18}$$

If $4\alpha\gamma\neq\beta^2$ then $r=\text{rank}(\partial_n\partial_m f)=2$, and the equivalent scalar-tensor theory is

$$S'=\frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\Big\{-2\Lambda_0+\varphi_1 R+\varphi_2\mathcal{L}_2-2\frac{\gamma L^4(\varphi_1-1)^2-\beta L^2(\varphi_1-1)\varphi_2+\alpha\varphi_2^2}{L^6(4\alpha\gamma-\beta^2)}\Big\}. \tag{19}$$

On the contrary, if $4\alpha\gamma=\beta^2$, then $r=1$ and there is an equivalent scalar-tensor theory with only one scalar:

$$S'=\frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\Big\{-2\Lambda_0+\varphi R+\varphi\frac{\beta}{2\alpha}L^2\mathcal{L}_2-\frac{(\varphi-1)^2}{4\alpha L^2}\Big\}. \tag{20}$$

Equivalence with scalar-tensor theory

Conformal transformation

It is well-known that $f(R)$ theories are equivalent, through a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, to GR plus a minimally coupled scalar:

$$\tilde{S}'_{f_R} = \frac{1}{1\pi G} \int_M d^D x \sqrt{-\tilde{g}} \Big\{ \tilde{R} - \frac{1}{2} \big(\tilde{\nabla}\phi\big)^2 - U(\phi) \Big\}. \tag{21}$$

Conformal transformation in f(Lovelock)?

- Consider, for example, $f(R, \mathcal{L}_2)$ in $D = 4$.
- If f is non-degenerate, we have seen that the theory is equivalent to a scalar-tensor theory with two non-dynamical scalars
- If we perform a conformal transformation, the resulting theory is

$$\begin{aligned} \tilde{S}'_{f_{LL}} =& \frac{1}{1\pi G} \int_M d^4 x \sqrt{-\tilde{g}} \Big\{ \tilde{R} + \varphi \tilde{\mathcal{L}}_2 + 8 \tilde{\nabla}^\mu \varphi \tilde{\nabla}^\nu \phi \tilde{G}_{\mu\nu} - 6 \big(\tilde{\nabla}\phi\big)^2 \\ & - 8 \tilde{\nabla}^\mu \varphi \tilde{\nabla}_\mu \phi \tilde{\Box} \phi - 4 \tilde{\Box} \varphi \big(\tilde{\nabla}\phi\big)^2 + 8 \tilde{\nabla}^\mu \varphi \tilde{\nabla}_\mu \phi \big(\tilde{\nabla}\phi\big)^2 - U(\phi, \varphi) \Big\} \end{aligned} \tag{22}$$

- Second order equations
- Hordenski-like theory, with two scalars and a coupling to $\mathcal{L}_2$

Linearized equations

Linearized equations

Let us parametrize the f(Lovelock) theory in the following way

$$S_f = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}\Big\{-2\Lambda_0 + R + \lambda f(\mathcal{L}_1, ..., \mathcal{L}_k)\Big\}, \tag{23}$$

so we make explicit the Einstein-Hilbert term and the cosmological constant. Then, we assume that our background is maximally symmetric, with metric $\bar{g}_{\mu\nu}$. The Riemann tensor of such space is given by

$$\bar{R}^{\mu\nu}_{\alpha\beta} = \Lambda\delta^{\mu\nu}_{\alpha\beta}, \tag{24}$$

where Λ is a constant. If we plug this Riemann tensor in the field equations, we find the constraint equation for Λ:

$$\begin{aligned}2\Lambda_0 = {}&(D-1)(D-2)\Lambda\Big(1 - \frac{2}{D-2}\lambda\partial_1 f\left(\bar{\mathcal{L}}\right)\Big)\\ &-(D-1)(D-2)\lambda\sum_{n=2}^{k}\partial_n f\left(\bar{\mathcal{L}}\right)2n\frac{(D-3)!}{(D-2n)!}\Lambda^n + \lambda f\left(\bar{\mathcal{L}}\right).\end{aligned} \tag{25}$$

where the bar means that we evaluate at the background. This equation gives us the possible vacua of the theory.

Linearized equations

We perturbe the metric on this background: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The linearized equations for the metric perturbation read

$$\alpha\Big(\bar{\nabla}_{(\mu|}\bar{\nabla}_{\sigma}h^{\sigma}_{|\nu)} - \frac{1}{2}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h - \frac{1}{2}\Box h_{\mu\nu} + \Lambda h_{\mu\nu} - \Lambda h\bar{g}_{\mu\nu}\Big) +$$

$$+\left[\bar{g}_{\mu\nu}\big(\Lambda\beta - \frac{\alpha}{2}\big) + \frac{\beta}{D-1}\big(\bar{g}_{\mu\nu}\Box - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\big)\right]\Big(\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}h^{\alpha\beta} - \Box h - \Lambda(D-1)h\Big) = 0. \quad (26)$$

where α and β are the following constants

$$\alpha = 1 + \lambda\sum_{n=1}^{k} n\partial_p f\left(\bar{\mathcal{L}}\right)\frac{(D-3)!}{(D-n-1)!}\Lambda^{n-1}, \quad (27)$$

$$\beta = \lambda\sum_{n,m=1}^{k} nm\partial_n\partial_m f\left(\bar{\mathcal{L}}\right)\frac{(D-2)!(D-1)!}{(D-2n)!(D-2m)!}\Lambda^{n+m-2}. \quad (28)$$

As an important observation, there is no term $\Box^2 h_{\mu\nu}$, which is related to the presence of massive gravitons.

Linearized equations

Finally, we can choose the transverse gauge, $\bar{\nabla}_\mu h^{\mu\nu} = \bar{\nabla}^\nu h$, and we can identify the physical fields. We have a traceless, massless spin-2 field, $t_{\mu\nu}$, which satisfies the equation

$$\boxed{-\frac{\alpha}{2}\left(\Box t_{\mu\nu} - 2\Lambda t_{\mu\nu}\right) = 0,} \tag{29}$$

and a scalar mode, $h = h^\mu_\mu$, which satisfies

$$\boxed{-\Lambda(D-1)\Big[\big(D\Lambda\beta - \alpha(D/2-1)\big)h + \beta\Box h\Big] = 0.} \tag{30}$$

The metric perturbation $h_{\mu\nu}$ can be reconstructed by means of the relation

$$t_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{2}{D-2}\frac{\beta}{\alpha}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu h - \frac{\bar{g}_{\mu\nu}}{D}\Box h\right), \tag{31}$$

where $\hat{h}_{\mu\nu}$ is the traceless part of $h_{\mu\nu}$:

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{D}\bar{g}_{\mu\nu}h. \tag{32}$$

Linearized equations

We have found that in f(Lovelock) gravity there are no massive gravitons. This is a nice property, because massive gravitons usually behave as ghosts. Are there more theories free of massive gravitons?

- **GR** (second order equations)
- **Lovelock** (second order equations)
- $f(R)$
- f(**Lovelock**)
- **Quasitopological gravity** (cubic curvature theory) Myers, Robinson
- ...

However, most of higher order gravities contain massive gravitons. For example, $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$.
Which conditions must a theory satisfy so it is free of massive gravitons?

We want to determine the presence of massive gravitons in a theory of the form

$$S = \int_M d^D x \sqrt{-g}\mathfrak{L}, \tag{33}$$

where $\mathfrak{L}$ is a scalar function of the Riemann tensor $R_{\mu\nu\sigma\rho}$ and the (inverse) metric $g^{\mu\nu}$. The presence of massive gravitons is related to the term $\Box^2 h_{\mu\nu}$ in the linearized equations.

The result of our analysis is the following: We define

$$\boxed{C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = \frac{\partial}{\partial R_{\sigma\rho\lambda\eta}} \frac{\partial \mathfrak{L}}{\partial R^{\mu\alpha\beta\nu}}\bigg|_{\bar{g}_{ab}}.} \tag{34}$$

On a MSB, the most general form of this tensor is

$$C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = a B^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} + b\left(\bar{g}_{\mu\beta}\bar{g}_{\alpha\nu} - \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\right)\left(\bar{g}^{\sigma\lambda}\bar{g}^{\rho\eta} - \bar{g}^{\sigma\eta}\bar{g}^{\rho\lambda}\right) + c g_{ab} g^{cd} B^{\sigma\rho\lambda\eta}_{cidj} B^{aibj}_{\mu\alpha\beta\nu}, \tag{35}$$

where $B^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} \equiv \delta^{[\sigma}_{\mu}\delta^{\rho]}_{\alpha}\delta^{[\lambda}_{\beta}\delta^{\eta]}_{\nu} + \delta^{[\lambda}_{\mu}\delta^{\eta]}_{\alpha}\delta^{[\sigma}_{\beta}\delta^{\rho]}_{\nu}$.

The parameters a, b and c depend on the Lagrangian $\mathfrak{L}$. We found that the condition for not having massive gravitons is

$$\boxed{a + 2c = 0.} \tag{36}$$

At the end, this is a constraint equation on the parameters of the theory. For example, there are a lot of cubic gravities, most of them not studied yet, which satisfy this condition.

Corollary

If the Lagrangians L_1,...,L_n are free of massive gravitons $\Rightarrow$ any theory with Lagrangian $f(L_1, ..., L_n)$ is also free of them.

Black holes

1 Introduction

2 Variational problem in f(Lovelock)

3 Equivalence with scalar-tensor theory
- Non-degenerate case
- Degenerate case
- Conformal transformation

4 Linearized equations
- Massive gravitons in general $\mathcal{L}(R_{\mu\nu\alpha\beta})$ theories

5 Black holes
- BPS solution
- Homogeneous function
- One ED

6 Conclusions

We consider the theory

$$S=\frac{1}{16\pi G}\int_M d^Dx\sqrt{-g}\left\{\frac{(D-1)(D-2)}{L^2}+R+\alpha L^2\mathcal{L}_2+\beta L^2R^2+\gamma L^4R\mathcal{L}_2+\delta L^6\mathcal{L}_2^2\right\}. \quad (37)$$

When the parameters are given by

$$\begin{aligned}\alpha &= \frac{\lambda}{(D-2)(D-3)},\quad \beta=\frac{1}{4(D-1)(D-2)},\\ \gamma &= \frac{\lambda}{2(D-1)(D-2)^2(D-3)},\quad \delta=\frac{\lambda^2}{4(D-1)(D-2)^3(D-3)^3},\end{aligned}$$

where λ is arbitrary, we find the following solution

$$\boxed{ds^2=-\left(1+\frac{r^2}{L^2}h(r)\right)dt^2+\frac{1}{\left(1+\frac{r^2}{L^2}h(r)\right)}dr^2+r^2d\Omega^2_{(D-2)}.} \quad (38)$$

where $h(r)$ is the function

$$h(r)=\frac{1}{2\lambda}\left[1-\sqrt{1-4\lambda\left(\frac{2D-4}{D}+c_1\frac{L^D}{r^D}+c_2\frac{L^{D-1}}{r^{D-1}}\right)}\right]. \quad (39)$$

Black holes

BPS solution

When the constants satisfy

$$0 < \lambda \leq \frac{D}{8D-16}, \quad c_1 \leq 0, \quad c_2 \leq -c_1\left[\frac{1-\lambda(8D-16)/D}{-4\lambda c_1}\right]^{1/D} D(D-1)^{1/D-1} \tag{40}$$

the solution exists $\forall r > 0$, it is asymptotically AdS with radius $\tilde{L}^2 = \frac{L^2 D}{4D-8}\left(1+\sqrt{1-\lambda\frac{8D-16}{D}}\right)$, there is a curvature singularity at $r=0$ and a horizon. Therefore, the solution is an asymptotically AdS black hole. In the limit $\lambda \to 0$ we get

$$ds^2 = -g(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2 d\Omega_{(D-2)}, \tag{41}$$

where

$$g(r) = 1 + \frac{r^2}{L^2}\frac{2D-4}{D} + c_1\frac{L^{D-2}}{r^{D-2}} + c_2\frac{L^{D-3}}{r^{D-3}}. \tag{42}$$

This is a well-known solution of R^2 gravity Ayon-Beato, Garbarz, Giribet, Hassaine . This solution reduces to Reissner-Nordstrom-AdS in $D=4$, and to Schwarzcshild-AdS if $c_1 = 0$.

In $D = 4$ the most general f(Lovelock) gravity is

$$S = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}f(R, \mathcal{L}_2). \quad (43)$$

When f is homogeneous of degree 1, this is $f(\alpha R, \alpha \mathcal{L}_2) = \alpha f(R, \mathcal{L}_2)$, and if the derivatives of f are not singular at $R = 0$, then this theory allows Ricci flat solutions

$$\boxed{R_{\mu\nu} = 0.} \quad (44)$$

We get solutions as Schwarzschild's or Kerr's.
In the case in which f is homogeneous of degree 1, another family of solutions can be found by imposing $\partial_R f(R, \mathcal{L}_2) = 0$. This gives us a equation of the form

$$\boxed{\alpha R + \beta L^2 \mathcal{L}_2 = 0.} \quad (45)$$

Black holes

One ED

Let us consider the theory $S = \int d^D x \sqrt{-g} f(\mathcal{L}_n)$. In some cases, a solution is given by imposing $\mathcal{L}_n = \Lambda^n D!/(D-2n)! = const$. A solution to this equation is given by

$$ds^2 = -\left(1 - \Lambda r^2 F(r)\right) dt^2 + \frac{dr^2}{1 - \Lambda r^2 F(r)} + r^2 d\Omega^2_{(D-2)}, \tag{46}$$

where

$$\boxed{F(r) = \left[1 + \frac{1}{\Lambda^n}\left(\frac{c_1}{r^{D-1}} + \frac{c_2}{r^D}\right)\right]^{1/n}.} \tag{47}$$

- In $D = 4$, $n = 1$, this is dS/AdS-RN black hole, solution of some R^2 gravities.
- If $c_2 = 0$, the previous is solution of pure Lovelock gravity $\mathcal{L}_n + const$
- If $c_2 = 0$ and the constant value of $\mathcal{L}_n$ is a solution of the equation $2n\mathcal{L}_n f'(\mathcal{L}_n) - Df(\mathcal{L}_n) = 0$, then the previous is a solution of $f(\mathcal{L}_n)$ theory.
- Funny situation: $\Lambda = c_2 = 0$, $D = 3n + 1$: Schwarzschild-like solution!

$$ds^2 = -\left(1 - r_0/r\right) dt^2 + \frac{dr^2}{1 - r_0/r} + r^2 d\Omega^2_{(D-2)}, \tag{48}$$

Conclusions

Conclusions

- We have computed the variation of f(Lovelock) action and we have proposed a generalized boundary contribution which sets the variational problem well-posed.
- By counting the quantities that must be fixed on the boundary we found that, with respect to GR, there are $r = \text{rank}(\partial_n \partial_m f)$ extra degrees of freedom.
- We have shown that f(Lovelock) gravity is equivalent to a scalar-tensor theory with r scalars.
- We have computed the linearized equations and we have found that in f(Lovelock) there is a massless, traceless spin-2 graviton and a scalar mode, but there is no massive graviton.
- We have developed a general procedure in order to determine the presence of massive gravitons in any $\mathcal{L}(R_{\mu\nu\alpha\beta})$ theory.
- We have found several exact solutions of certain f(Lovelock) theories, some of them represent static and regular black holes.

Conclusions

Thank you for your attention

Bonus

Let us consider the f(Lovelock) action

$$S = \frac{1}{16\pi G}\int_M d^Dx\sqrt{-g}f(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_{\lfloor D/2 \rfloor}). \tag{49}$$

Is this theory equivalent to this other one, with $k = \lfloor D/2 \rfloor$ scalar fields?

$$S' = \frac{1}{16\pi G}\int_M d^Dx\sqrt{-g}\Big\{\sum_{n=1}^{k}\partial_n f(\phi_1, ..., \phi_k)\mathcal{L}_n - V(\phi_1, ..., \phi_k)\Big\}, \tag{50}$$

where $V(\phi_1, ..., \phi_k) = \sum_{n=1}^{k}\partial_n f(\phi_1, ..., \phi_k)\phi_n - f(\phi_1, ..., \phi_k)$.
The variation of the action with respect to the scalar fields yields

$$\sum_{n=1}^{k}\partial_n\partial_m f(\phi)\big(\mathcal{L}_n - \phi_n\big) = 0, \quad m = 1, ..., k \tag{51}$$

If the only solution to these equations is $\phi_n = \mathcal{L}_n$, we recover the f(Lovelock) action and the theories are equivalent.

Bonus

If $\det(\partial_n\partial_m f) \neq 0$, then the only solution is $\phi_n = \mathcal{L}_n$. Moreover, we can perform the Legendre transform of f:

$$\begin{aligned} \varphi_n &= \partial_n f(\phi_1, ..., \phi_k), \quad n = 1, ..., k, & (52) \\ \tilde{V}(\varphi_1, ..., \varphi_k) &= \sum_{n=1}^{k} \varphi_n \phi_n - f(\phi_1, ..., \phi_k) = V(\phi(\varphi)). & (53) \end{aligned}$$

Then, in terms of the fields φ_n, it is clear that the action (50) takes the form

$$S' = \frac{1}{16\pi G} \int_M d^D x \sqrt{-g} \Big\{ \sum_{p=1}^{k} \varphi_n \mathcal{L}_n - \tilde{V}(\varphi_1, ..., \varphi_k) \Big\}. \quad (54)$$

This theory is equivalent to f(Lovelock).

Bonus

If $\det(\partial_n \partial_m f) = 0$, the solution is not unique and there are $k - r$ non-physical degrees of freedom, where $r = \text{rank}(\partial_n \partial_m f)$. Therefore, we should keep only r scalars. If we define

$$\varphi_n = \partial_n f(\phi_1, ..., \phi_k), \quad n = 1, ..., k. \tag{55}$$

Then, there is a subset $I \subset \{1, ..., k\}$ of r indices such that $\{\varphi_i\}_{i \in I}$ are independent variables. The rest of fields depend on the formers: $\varphi_j = g_j(\varphi_i)$, $j \in J = \{1, ..., k\} - I$. We take as independent variables (φ_i, ϕ_j), and we can define the semi-Legendre transform of f:

$$\tilde{V}(\varphi_i) = \sum_{i \in I} \varphi_i \phi_i + \sum_{j \in J} g_j(\varphi_i) \phi_j - f(\phi_1, ..., \phi_k). \tag{56}$$

It can be shown that it only depends on φ_i. Then, f(Lovelock) is equivalent to a scalar-tensor theory with r scalars:

$$S' = \int_M d^D x \sqrt{-g} \Big\{ \sum_{i \in I} \varphi_i \mathcal{L}_i + \sum_{j \in J} g_j(\varphi_i) \mathcal{L}_j - \tilde{V}(\varphi_{i_1}, ..., \varphi_{i_r}) \Big\}. \tag{57}$$