Windows on Quatum Gravity
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CONFORMAL FRAMES IN GRAVITY

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based on

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Chap I

Frame-Independence of Observables

1. Introducton

In cosmology, we encounter various frames of the metric which are conformally equivalent.

Einstein frame, Jordan frame, string frame, ...

They are mathematically equivalent, so one can work in any frame as long as mathematical manipulations are concerned.

But it is often said that there exists a unique physical frame on which we should consider actual ‘physics.’

Is it really so?

e.g. Dimensional reduction

D-dimensions → 4-dimensions

(D-tensor → 4-tensor + 4-vector + 4-scalar)

$$\begin{pmatrix} g^{(D)}_{\mu\nu}(x,y) & g^{(D)}_{\mu B}(x,y) \\ g^{(D)}_{A\nu}(x,y) & g^{(D)}_{AB}(x,y) \end{pmatrix} \rightarrow g_{\mu\nu}(x) = \begin{cases} \left\langle g^{(D)}_{\mu\nu} \right\rangle_{D-4} ? \\ f(x)\left\langle g^{(D)}_{\mu\nu} \right\rangle_{D-4} ? \\ g^{(D)}_{\mu\nu}(x,0) \ \ ? \end{cases}$$

$$x : 4 - \mathrm{dim}$$

$$y : (D-4) - \mathrm{dim}$$

or else?

Dilatonic scalars will almost always appear.

No natural conformal frame, *a priori*

no unique physical frame!

Two typical frames in scalar-tensor theory

$\left(\boldsymbol{\phi} + \boldsymbol{g} \right)$

- Jordan(-Brans-Dicke) frame

  "gravitational" part : $F(\phi)R+L(\phi)$

  matter part: $L(\psi, A, \ldots)$ ~ minimal coupling with g

  (matter assumed to be universally coupled with g
  ⋯ for baryons, experimentally consistent)

- Einstein frame

  "gravitational" part : $R+L(\phi)$ ~ minimal coupling between g and ϕ

  matter part: $G(\phi)L(\psi, A, \ldots)$ ψ : fermion, A : vector, ...

  (if non-universal coupling:

  $$\Rightarrow \sum_A G_A(\phi)L_A(Q_A); \quad Q_A = \psi, A, \cdots.$$)

2. Conformal transformations

metric and scalar curvature

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$R \to \tilde{R} = \Omega^{-2}\left[R - (D-1)\left(2\frac{\Box\Omega}{\Omega} - (D-4)g^{\mu\nu}\frac{\partial_\mu\Omega\partial_\nu\Omega}{\Omega^2} \right)\right]$$

matter fields

$$\phi \to \tilde{\phi} = \Omega^{-(D-2)/2}\phi$$ scalar

$$A_\mu \to \tilde{A}_\mu = \Omega^{-(D-4)/2}A_\mu$$ vector

$$\psi \to \tilde{\psi} = \Omega^{-(D-1)/2}\psi$$ fermion

Standard (baryonic) matter action in 4 dims

‘Jordan’ frame (= matter minimally coupled to gravity)

$$S=\int d^4x\sqrt{-g}\left[-i\bar{\psi}\,\gamma^{\mu}\left(\overleftrightarrow{D}_{\mu}-ieA_{\mu}\right)\psi-m\bar{\psi}\psi-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}+\cdots\right]$$

$$\bar{\psi}\gamma^{\mu}\overleftrightarrow{D}_{\mu}\psi=\frac{1}{2}\left[\bar{\psi}\gamma^{\mu}D_{\mu}\psi-(D_{\mu}\bar{\psi})\gamma^{\mu}\psi\right],$$

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}\,,\quad D_{\mu}=\partial_{\mu}-\frac{1}{4}\omega_{ab\mu}\Sigma^{ab}\,,$$

$$\Sigma^{ab}=\frac{1}{2}\left[\gamma^{a},\gamma^{b}\right],\quad \omega_{ab\mu}=e_{a\nu}\nabla_{\mu}e^{\nu}_{b}\,.$$

ψ : electron

A : electromagnetic 4-potential

For the moment, ignore/freeze dilatonic degrees of freedom.

Effect of conformal transformation

For $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$

$$S = \int d^4x \sqrt{-\tilde{g}} \left[i \bar{\tilde{\psi}} \tilde{\gamma}^{\mu} \left(\overleftrightarrow{\tilde{D}}_{\mu} - ieA_{\mu} \right) \tilde{\psi} - \tilde{m} \bar{\tilde{\psi}} \tilde{\psi} - \frac{1}{4} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \cdots \right]$$

where $\tilde{\gamma}^{\mu} = \Omega^{-1} \gamma^{\mu}, \quad \tilde{\psi} = \Omega^{-3/2} \psi, \quad \tilde{m} = \Omega^{-1} m$.

(A_μ is invariant in 4 dim)

Conformal transformation from `Jordan frame' to any other frame results in spacetime-dependent mass.

And this is the only effect, provided dynamics of dilatons (at short distances) can be neglected.

(dilatons may be dynamical on cosmological scales)

3. Standard Cosmology

Conventional wisdom

$$ds^2 = -dt^2 + a^2(t)d\sigma^2_{(K)} \;;$$

$d\sigma^2_{(K)}$: homogeneous and isotropic 3-space $\left(K = \pm 1, 0\right)$

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$ ··· expanding universe

⇨ cosmological redshift $E_{\text{obs}} = \dfrac{E_{\text{emit}}}{1+z}$

This is how we interpret observational data.

This is regarded as a `proof’ of cosmic expansion.

But ....

Conformal transformation:

$$ds^2 \rightarrow d\tilde{s}^2 = \Omega^2 ds^2; \quad \Omega = \frac{1}{a}$$

$$\Rightarrow \quad d\tilde{s}^2 = -d\eta^2 + d\sigma_{(3)}^2; \quad d\eta = \frac{dt}{a(t)}$$

In this conformal frame, the universe is static.

no Hubble flow.

photons do not redshift...

Is this frame unphysical?

Ex 1: Cosmological Redshift

In this static frame,

- electron mass varies in time: $\tilde{m}(\eta) = m\,\Omega^{-1} = \dfrac{m}{1+z}$

  where “z” is defined by

$$1+z \equiv \Omega = \frac{1}{a(\eta)} \qquad \left(a_0 = a(\eta_0) = 1\right)$$

- Bohr radius $\propto m^{-1}$ ⇔ atomic energy levels $\propto m$:

energy level in ‘static’ frame

$$\tilde{E}_n = \frac{E_n}{1+z}$$

energy level in ‘Jordan’ frame

Thus frequency of photons emitted at time $z = z(\eta)$ from a level transition $n \to n'$ is

$$\tilde{E}_{nn'} = \frac{E_{nn'}}{1+z}$$

this is exactly what we observe as Hubble’s law!

Ex 2: CMB

- CMB photons have never redshifted.
- The universe was in thermal equilibrium when the electron mass was small by a factor $>10^3$, ie, at time $z>10^3$, at fixed temperature T=2.725K.

Just to check physics…

- Thomson cross section: $\tilde{\sigma}_T \propto \tilde{m}^{-2} \rightarrow \tilde{\sigma}_T = \sigma_T (1+z)^2$

  electron density: $\tilde{n}_e = \text{const.} = n_e (1+z)^{-3}$

⇨ rate of scattering/interaction per unit proper time:

$$\tilde{n}_e \tilde{\sigma}_T d\eta = \frac{n_e \sigma_T}{1+z} d\eta = n_e \sigma_T dt$$

observational results are indistinguishable

Chap II

FRAME-DEPENDENCE OF INFLATION

(made by Guillem Domenech: modified by MS)

Scalar-Tensor Theory

What is the Scalar-tensor theory of gravity? It considers a scalar field non-minimally coupled to gravity:

$$S \sim \int d^4x \sqrt{-\tilde{g}} \left\{ F(\phi)\tilde{R} + \tilde{L}(\phi) \right\}$$

This form of the action is called Jordan frame.

By means of a conformal transformation, i.e.

$$\tilde{g}_{\mu\nu} = F^{-1} g_{\mu\nu} \ldots$$

Frame independence

...one can bring the Jordan frame into the Einstein-Hilbert action, that is

$$S \sim \int d^4x\sqrt{-g}\,\{R + L(\phi)\}\ ,$$

the so-called Einstein frame.

What is the advantatge of such a transformation?

- Very well know how to deal with EH action (and much easier!).
- Physical observables are in fact frame independent.

Frame independence

However, what about the matter sector?

- Matter minimally couples to $\tilde{g}$.
- As long as we have successful inflation in the Einstein frame we can choose the matter metric $\tilde{g}$ by a conformal transformation.
- How different can $\tilde{g}$ and g be?
- Can this matter point of view leave observational imprints?

Inflation: definition

How do we define “Inflation" in a frame-independent way?

- Expansion law depends on the choice of conformal frame:

$$ds^2 = a^2(\eta)d\tilde{s}^2, \quad d\tilde{s}^2 = -d\eta^2 + d\vec{x}^2$$

$$\Rightarrow \text{ no expansion in the tilde frame}$$

- Here we assumes the existence of the Einstein frame, or at least a frame in which the linear perturbation behaves like that in the Einstein frame.

- This means an almost scale-invariant tensor spectrum.

- This definition fits all inflation models proposed so far.

Review: PL Inflation

We consider power-law inflation to illustrate these points. The Inflaton ϕ with potential $V(\phi) = V_0 e^{-\lambda\phi}$ gives rise to $(p = 2/\lambda^2)$: (Lucchin and Matarrese, 1985)

$$a = a_0\,(t/t_0)^p \qquad \phi = \frac{2}{\lambda}\ln(t/t_0) \qquad H = p/t \qquad \epsilon = 1/p\,.$$

The curvature and tensor power spectrum under the slow-roll approximation are given by

$$\mathcal{P}_{\mathcal{R}_c}(k) = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 = \frac{p}{8\pi^2}\frac{H_0^2}{M_{pl}^2}\left(\frac{k}{k_0}\right)^{\frac{-2}{p-1}} \;,$$

$$\mathcal{P}_{\mathcal{T}}(k) = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2} = \frac{16}{p}\mathcal{P}_{\mathcal{R}_c}(k)\,.$$

We need $p \gg 1$ for a successful inflation. $(r = 16/p)$

Curvaton model

For simplicity, let us take a curvaton as a representative of matter. The curvaton is a scalar field χ that:

- Initially is subdominant
- Has a non-vanishing initial energy density
- Dominates after inflaton decays
- and contributes to the scalar power spectrum

Curvaton model

Our curvaton is a matter field and therefore lives in the Jordan frame, i.e.

$$S_m \sim \int d^4x \sqrt{-\tilde{g}} \left(-\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \tilde{m}^2\chi^2\right) .$$

The power-spectrum for the curvaton under the sudden decay approximation is given by (Lyth and Wands, 2002)

$$\mathcal{P}_\chi(k) = r_\star \frac{\delta\chi^2}{\chi_\star^2} = r_\star \frac{\widetilde{H}^2}{(2\pi M_{pl}\chi_\star)^2} ,$$

where $r_\star$ is the energy density fraction of the curvaton at decay.

Matter point of view

Matter is coupled to the Jordan $\tilde{g}$ so our conformal transformation yields

$$\tilde{a} = F^{-1/2} a \qquad \text{and} \qquad d\tilde{t} = F^{-1/2} dt\,.$$

Let us take a concrete example inspired in a dilationic coupling, that is

$$F(\phi) = \mathrm{e}^{\gamma\lambda\phi/M_{pl}} = (t/t_0)^{2\gamma}\,.$$

Matter point of view

After integrating time, the Jordan scale factor is given by another power-law

$$\tilde{a} = \tilde{a}_0 \left(\tilde{t}/\tilde{t}_0\right)^{\tilde{p}} \qquad \tilde{H} = \tilde{p}/\tilde{t}\,,$$

where $\tilde{p} - 1 = \frac{p-1}{1-\gamma}$. ($\tilde{p}$ can be negative!)

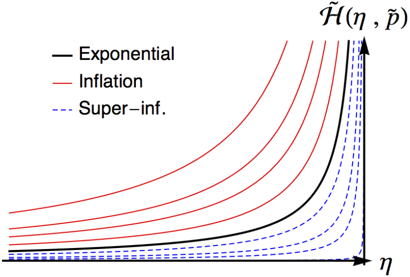


Figure: Jordan conformal hubble parameter $\tilde{\mathcal{H}}$ as a function of the conformal time η and $\tilde{p}$. For $\tilde{p} < 0$ we have super-inflation.

Jordan Power Law

The curvaton follows the Jordan power law. This time the power spectrum takes the same form but with $\tilde{p}$ instead of p. For $\tilde{p} < 0$ the spectrum is blue!

$$\tilde{n}_\chi - 1 = \frac{-2}{\tilde{p} - 1} \, .$$

Such a blue tilt might induce primordial black hole formation.

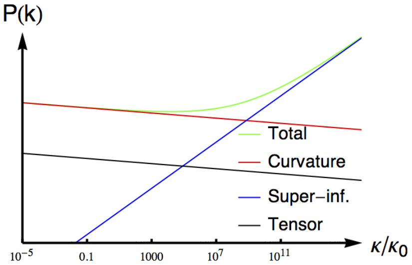


Figure: Power-spectrum for the Jordan power-law case.

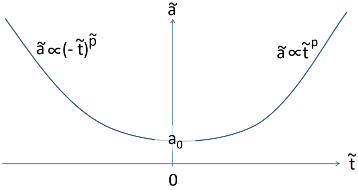
Jordan Bounce

We consider a slightly more complicated transformation, e.g.

$$F(\phi) = \left(1 + \mathrm{e}^{\frac{-\gamma\lambda}{2M_{pl}}\phi}\right)^{-2} = \left(1 + (t/t_0)^{-\gamma}\right)^{-2} .$$

It gives a bouncing universe in the Jordan frame!

$$\tilde{a} \approx \left\{ \begin{array}{ccc} a_0(-\tilde{t}/\tilde{t}_0)^{\tilde{p}} & |\tilde{t}| \gg \tilde{t}_0 & (\tilde{t} < 0) \\ a_0(\tilde{t}/\tilde{t}_0)^{p} & \tilde{t} \gg \tilde{t}_0 & \end{array} \right. .$$



The singularity has been sent to $\tilde{t} \to -\infty$.

Jordan Bounce

We find a **blue tilt** at short scales that gives an **apparent suppresion**.

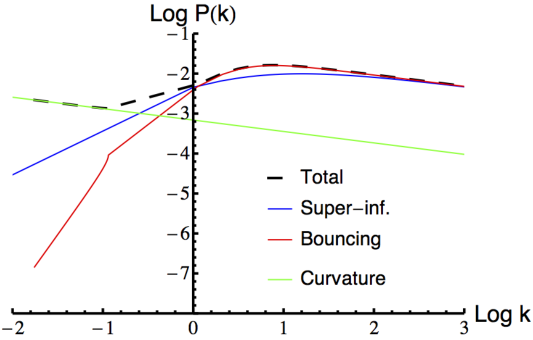


Figure: Power-spectrum for the Jordan bouncing frame.

Summary

With a simple analytic model we have shown that:

- In the scalar-tensor theory the matter point of view can be very different although we have inflation in the Einstein frame!
- Depending on which frame matter is minimally coupled, it can leave important features, e.g. to the power spectrum.
- We easily obtain a blue tilt on small scales (for the super-inf. case) and a blue tilt on large scales (for the bouncing case).