Dynamical Domain Wall and Localization

Shin'ichi Nojiri

Department of Physics &

Kobayashi-Maskawa Institute

for the Origin of Particles and the Universe (KMI),

Nagoya Univ.

Based on

Y. Toyozato, K. Bamba and S. Nojiri, "Scalar Domain Wall as the Universe", Phys. Rev. D 87 (2013) 6, 063008 [arXiv:1202.5375 [hep-th]].

M. Higuchi and S. Nojiri, "Reconstruction of Domain Wall Universe and Localization of Gravity", Gen. Rel. Grav. 46 (2014) 11, 1822 [arXiv:1402.1346 [hep-th]].

Y. Toyozato, M. Higuchi and S. Nojiri, "Dynamical Domain Wall and Localization", arXiv:1510.01099.

Construction of the dynamical domain wall universe, where the four dimensional FRW universe is realized on the domain wall in the five dimensional space-time.

Localization of the fields (graviton, chiral spinor, vector, ...)

Introduction

Our universe could be a brane or domain wall embedded in a higher dimensional space-time.

K. Akama, "An Early Proposal of 'Brane World'," Lect. Notes Phys. 176 (1982) 267 [hep-th/0001113].

V. A. Rubakov and M. E. Shaposhnikov, "Do We Live Inside a Domain Wall?," Phys. Lett. B 125 (1983) 136.

D-brane in string theories.

J. Dai, R. G. Leigh and J. Polchinski, "New Connections Between String Theories," Mod. Phys. Lett. A 4 (1989) 2073.

J. Polchinski, "Tasi lectures on D-branes," hep-th/9611050.

Brane world scenario.

L. Randall and R. Sundrum, "A Large mass hierarchy from a small extra dimension," Phys. Rev. Lett. 83, 3370 (1999) [hep-ph/9905221].

L. Randall, R. Sundrum, "An Alternative to compactification," Phys. Rev. Lett. 83, 4690-4693 (1999). [hep-th/9906064].

Dynamics of brane?
Brane: a limit where the thickness of domain wall vanishes
$\Rightarrow$ Dynamics of domain wall

Domain wall model with two scalar fields

General FRW in the five dimensional space-time, the metric is given by

$$ds^2 = dw^2 + L^2 \mathrm{e}^{u(w,t)} ds^2_{\mathrm{FRW}} \, .$$

$$ds^2_{\mathrm{FRW}} = - dt^2 + a\left(t\right)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right\} \, .$$

Action with two scalar fields ϕ and χ:

$$S_{\phi\chi} = \int d^5x\sqrt{-g}\left\{ \frac{R}{2\kappa^2} - \frac{1}{2}A(\phi,\chi)\partial_\mu\phi\partial^\mu\phi - B(\phi,\chi)\partial_\mu\phi\partial^\mu\chi \right.$$
$$\left. -\frac{1}{2}C(\phi,\chi)\partial_\mu\chi\partial^\mu\chi - V(\phi,\chi) \right\} \, .$$

Energy-momentum tensor

$$
\begin{aligned}
T^{\phi\chi}_{\mu\nu} =& g_{\mu\nu}\left\{-\frac{1}{2}A(\phi,\chi)\partial_\rho\phi\partial^\rho\phi - B(\phi,\chi)\partial_\rho\phi\partial^\rho\chi - \frac{1}{2}C(\phi,\chi)\partial_\rho\chi\partial^\rho\chi - V(\phi,\chi)\right\} \\
&+ A(\phi,\chi)\partial_\mu\phi\partial_\nu\phi + B(\phi,\chi)\left(\partial_\mu\phi\partial_\nu\chi + \partial_\nu\phi\partial_\mu\chi\right) + C(\phi,\chi)\partial_\mu\chi\partial_\nu\chi\,.
\end{aligned}
$$

Field equations read

$$
\begin{aligned}
0 =& \frac{1}{2}A_\phi\partial_\mu\phi\partial^\mu\phi + A\nabla^\mu\partial_\mu\phi + A_\chi\partial_\mu\phi\partial^\mu\chi + \left(B_\chi - \frac{1}{2}C_\phi\right)\partial_\mu\chi\partial^\mu\chi \\
&+ B\nabla^\mu\partial_\mu\chi - V_\phi\,, \\
0 =& \left(-\frac{1}{2}A_\chi + B_\phi\right)\partial_\mu\phi\partial^\mu\phi + B\nabla^\mu\partial_\mu\phi + \frac{1}{2}C_\chi\partial_\mu\chi\partial^\mu\chi + C\nabla^\mu\partial_\mu\chi \\
&+ C_\phi\partial_\mu\phi\partial^\mu\chi - V_\chi\,.
\end{aligned}
$$

$A_\phi = \partial A(\phi,\chi)/\partial\phi$, **etc.**
We now choose $\phi = t$ and $\chi = w$.

We can construct a model to realize the arbitrary metric in the form,
$ds^2 = dw^2 + L^2 \mathrm{e}^{u(w,t)} ds^2_{\mathrm{FRW}}$**,**
$ds^2_{\mathrm{FRW}} = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right\}$**,**
by choosing $(H \equiv (1/a)(da/dt))$

$$A = \frac{1}{\kappa^2}\left(\frac{2k}{a^2} - \ddot{u} - 2\dot{H} + \frac{(\dot{u})^2}{2} + \dot{u}H\right) ,$$

$$B = -\frac{3u'}{2\kappa^2 L^2 \mathrm{e}^u}(\dot{u} + 2H) ,$$

$$C = \frac{1}{\kappa^2}\left(-\frac{3}{2}u'' + \frac{2k}{L^2\mathrm{e}^u a^2} - \frac{1}{2\mathrm{e}^u}\left(\ddot{u} + 2\dot{H} + (\dot{u})^2 + 5\dot{u}H + 6H^2\right)\right) ,$$

$$V = \frac{1}{\kappa^2}\left(-\frac{3}{4}\left(u'' + 2\left(u'\right)^2\right) + \frac{3k}{L^2\mathrm{e}^u a^2} + \frac{1}{4L^2\mathrm{e}^u}\left(3\ddot{u} + 6\dot{H} + 3(\dot{u})^2 + 15\dot{u} + 18H^2\right)\right) .$$

Einstein equation is satisfied.
Field equations are nothing but the Bianchi identities.

If any eigenvalue of the matrix $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ is negative, there appears a ghost.

Example without ghost

$$a(t) \propto t^{h_0}\,, \quad \mathrm{e}^{u(w,t)} = W\left(w\right) T\left(t\right)\ ,$$

$$T(t) = T_1 t^{1-3h_0} + T_2 t^{-2h_0}\,, \quad W(w) = \mathrm{e}^{-\sqrt{1+\frac{w^2}{w_0^2}}}\,,$$

$$\Rightarrow A = \frac{3}{2\kappa^2}\left(\frac{\dot{T}(t)}{T(t)} + \frac{2h_0}{t}\right)^2 > 0\,, \quad B = 0\,, \quad C = \frac{3}{2\kappa^2}\frac{\frac{1}{w_0^2}}{\left(1 + \frac{w^2}{w_0^2}\right)^{\frac{3}{2}}} > 0\,.$$

$\cdots$ **No ghost.**

General FRW universe can be realized by the Brans-Dicke type model.

Localization of graviton

Perturbation $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$**.**
Assume $h_{\mu\nu} = 0$ **if** μ **or** $\nu \neq 1, 2, 3$**.**
Einstein eq. $\Rightarrow$ **equation for graviton in five dimensions:**

$$0 = \left[\partial_w^2 - u'' - u'^2 + L^{-2}\mathrm{e}^{-u}\left(\ddot{u} + \frac{\dot{a}\dot{u}}{a} + \dot{u}\partial_t + 2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\partial_t - \partial_t^2 + \frac{\triangle}{a^2}\right)\right] h_{ij} \, .$$

By assuming $h_{ij}(w, x) = \mathrm{e}^{u(w,t)}\hat{h}_{ij}(x)$**,**

$$0 = \left(2\frac{\dot{a}\dot{u}}{a} - \dot{u}\partial_t + 2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\partial_t - \partial_t^2 + \frac{\triangle}{a^2}\right)\hat{h}_{ij} \, .$$

If u **goes to minus infinity sufficiently rapidly for large** $|w|$**,** $h_{ij}(w, x)$ **is normalized in the direction of** w **and therefore there occurs the localization of graviton.**

Graviton localized on the domain wall

$$0 = \left(2\frac{\dot{a}\dot{u}}{a} - \dot{u}\partial_t + 2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) \hat{h}_{ij} \, .$$

In the four dimensional FRW space-time,

$$0 = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) h_{ij} \, .$$

If $\dot{u}\left(2\frac{\dot{a}}{a} - \partial_t\right)\hat{h}_{ij} = 0$**, two expressions coincide with each other.**
If $\dot{u} \neq 0$**, there could appear some corrections proportional to** $\dot{u}$**.**

Localization of chiral spinor

There was an attempt to localize the fermion on the domain wall has been failed.
D. P. George, M. Trodden and R. R. Volkas, "Extra-dimensional cosmology with domain-wall branes," JHEP 0902 (2009) 035 [arXiv:0810.3746 [hep-ph]].

Dirac equation in five dimensions

$$\Gamma^M \nabla_M \Psi + \tilde{f}\chi\,(w)\,\Psi = 0\,.$$

$\tilde{f}\chi$**: a function of the scalar field** $\chi = w$**, general yukawa interaction.**
∇_M**: covariant derivative:**

$$\nabla_M \stackrel{\rm def}{=} \partial_M + \frac{1}{4}\omega_{ABM}\Gamma^{AB}\,, \quad \Gamma^{AB} \stackrel{\rm def}{=} \frac{1}{2}\left[\Gamma^A, \Gamma^B\right]\,.$$

$\Psi = \eta\,(t,w)\,\psi\,(x)$ **(ψ: four dimensional Dirac spinor)**

$$\begin{aligned}
&\Gamma^{M}\nabla_{M}\Psi + \tilde{f}\chi\,(w)\,\Psi \\
&\quad = L^{-1}\mathrm{e}^{-u/2}\left\{\Gamma^{\hat{0}}\partial_{0}\psi + \frac{1}{a\,(t)}\Gamma^{\hat{i}}\partial_{i}\psi\right\}\eta + \left\{\partial_{5}\eta + u'\,(t,w)\,\eta\right\}\Gamma^{\hat{5}}\psi \\
&\qquad + \tilde{f}\chi\,(w)\,\eta\psi + L^{-1}\mathrm{e}^{-u/2}\left\{\partial_{0}\eta + \frac{3}{2}\left(\frac{1}{2}\dot{u} + \frac{\dot{a}}{a}\right)\eta\right\}\Gamma^{\hat{0}}\psi\,.
\end{aligned}$$

Let $\tilde{C}_1\,(w)$ and $D_1\,(t)$ be arbitrary functions.

$$\begin{aligned}
\eta\,(t,w) &\stackrel{\mathrm{def}}{=} \zeta\,(t)\,\lambda\,(w)\,g\,(t,w) \\
g\,(t,w) &\stackrel{\mathrm{def}}{=} \exp\left[\tilde{C}_1\,(w) + D_1\,(t) - u\,(t,w)\right]\,,
\end{aligned}$$

By assuming $\Gamma^{\hat{5}}\psi = \pm\psi$**,**

$$\begin{aligned}
&\Rightarrow \Gamma^M \nabla_M \Psi + \tilde{f}\chi(w)\Psi \\
&= L^{-1} \mathrm{e}^{-u/2} \left\{ \Gamma^{\hat{0}} \partial_0 \psi + \frac{1}{a(t)} \Gamma^{\hat{i}} \partial_i \psi \right\} \eta \\
&\quad \pm \zeta g \left\{ \partial_5 \lambda(w) + \tilde{C}_1'(w)\lambda(w) \pm \tilde{f}\chi(w)\lambda(w) \right\} \psi \\
&\quad + L^{-1} \mathrm{e}^{-u/2} \lambda g \left\{ \partial_0 \zeta(t) + \left(\frac{3}{2}\frac{\dot{a}}{a} - \frac{1}{4}\dot{u}(t,w) + \dot{D}_1(t) \right) \zeta(t) \right\} \Gamma^{\hat{0}} \psi \,.
\end{aligned}$$

$\mathrm{e}^{u(t,w)} = T(t)\,W(w)$ **and using the Dirac equation,**

$$\lambda(w) = C_3 \exp\left[-\tilde{C}_1(w) \mp \tilde{f} \int \mathbf{d}w\, \chi(w) \right]$$

$$\zeta(t) = C_4 a^{-3/2} T^{1/4} \exp\left[-D_1(t) \right] \,.$$

$\Rightarrow$

$$\eta\left(t,w\right)=C_3C_4a^{-3/2}T^{1/4}\exp\left[-u\left(t,w\right)\mp\tilde{f}\int\mathbf{d}w\,\chi\left(w\right)\right]$$
$$=C_5a^{-3/2}T^{-3/4}W^{-1}\exp\left[\mp\tilde{f}\int\mathbf{d}w\,\chi\left(w\right)\right]\,,$$

$\chi\left(w\right)=w\;\Rightarrow$

$$\eta\left(t,w\right)=C_5a^{-3/2}T^{-3/4}W^{-1}\exp\left[\mp\frac{\tilde{f}}{2}w^2\right]\,.$$

Condition of localization

$$I=\int_{-\infty}^{+\infty}\mathbf{d}w\,\mathrm{e}^{3u/2}\left|\eta\right|^2=C_5^2a^{-3}\int_{-\infty}^{+\infty}\mathbf{d}w\,W^{-1/2}\exp\left[\mp\tilde{f}w^2\right]<\infty\,.$$

Fermion can be localized on the domain wall and the localized fermion can be chiral or anti-chiral.

In

D. P. George, M. Trodden and R. R. Volkas, "Extra-dimensional cosmology with domain-wall branes," JHEP 0902 (2009) 035 [arXiv:0810.3746 [hep-ph]]

It was assumed that the warp factor does not depend on the time but we have used the time-dependent warp factor.

Localization of vector field

Action of vector field

$$S_V = \int d^5x\sqrt{-g}\left\{-\frac{1}{4}F_{MN}F^{MN} - \frac{1}{2}m(\chi)^2 A_M A^M\right\} ,$$
$$F_{MN} = \partial_M A_N - \partial_N A_M \, .$$

Background

$$ds^2 = dw^2 + L^2 W(w) T(t) ds^2_{\mathrm{FRW}} ,$$
$$ds^2_{\mathrm{FRW}} = - dt^2 + a(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 \, .$$

$\Rightarrow$

$$
\begin{aligned}
S_V = \int d^5x \Big\{ & \frac{1}{2} L^2 W(w) T(t) a(t)^3 F_{50}^2 - \frac{1}{2} L^2 W(w) T(t) a(t) F_{5i}^2 \\
& + \frac{1}{2} a(t) F_{0i}^2 - \frac{1}{4} a(t)^{-1} F_{ij}^2 \\
& - \frac{1}{2} m(\chi)^2 \big(L^4 W(w)^2 T(t)^2 a(t)^3 A_5^2 \\
& - L^2 W(w) T(t) a(t)^3 A_0^2 + L^2 W(w) T(t) a(t) A_i^2 \big) \Big\} \ .
\end{aligned}
$$

Variation of A_5, A_0, and A_i $\Rightarrow$

$$
\begin{aligned}
0 =& L^2 W(w)\partial_0 \left(T(t)a(t)^3 \left(\partial_5 A_0 - \partial_0 A_5\right)\right) \\
& - L^2 W(w)T(t)a(t) \left(\partial_5\partial_i A_i - \partial_i^2 A_5\right) \\
& - m(\chi)^2 L^4 W(w)^2 T(t)^2 a(t)^3 A_5 \,, \\
0 =& - L^2 T(t)a(t)^3 \partial_5 \left(W(w)\left(\partial_5 A_0 - \partial_0 A_5\right)\right) \\
& + a(t)\left(\partial_0\partial_i A_i - \partial_i^2 A_0\right) + m(\chi)^2 L^2 W(w)T(t)a(t)^3 A_0 \,, \\
0 =& L^2 T(t)a(t)\partial_w \left(W(w)\left(\partial_5 A_i - \partial_i A_5\right)\right) - \partial_0\left(a\left(\partial_0 A_i - \partial_i A_0\right)\right) \\
& - a(t)^{-1}\left(\partial_i\partial_j A_j - \partial_j^2 A_i\right) - m(\chi)^2 L^2 W(w)T(t)a(t)A_i \,.
\end{aligned}
$$

Assuming

$$A_5 = 0\,, \quad A_\mu = X(w)C_\mu\left(x^\nu\right)\,, \quad \mu,\nu = 0,1,2,3\,,$$

and choosing $\qquad m\left(\chi = w\right)^2 = \frac{\left(W(w)X'(w)\right)'}{W(w)X(w)}\,,$

$$\Rightarrow\ 0 = \partial_5 X(w)\left\{\partial_0\left(T(t)a(t)^3 C_0\right) - T(t)a(t)\partial_i C_i\right\}\,,$$

$$0 = \partial_0\partial_i C_i - \partial_i^2 C_0\,,$$

$$0 = \partial_0\left(a(t)\left(\partial_0 C_i - \partial_i C_0\right)\right) + a(t)^{-1}\left(\partial_i\partial_j C_j - \partial_j^2 C_i\right)\,.$$

Last two eqs. : nothing but the field equations of the vector field in four dimensions.
First eq. : a gauge condition, which is a generalization of the Landau gauge, $\partial^\mu A_\mu = 0$**.**

If $X(w) \to 0$ sufficiently when $|w| \to \infty$, A_μ: normalizable.
$\Rightarrow$ Localization of vector field on the domain wall.

In case

$$(W(w)X'(w))' = 0 \;\Rightarrow\; m(\chi) = 0\,.$$

$\Rightarrow$ non-abelian gauge theory?
(In general, non-normalizable)

Summary

- We have constructed the dynamical domain wall, where arbitrary four dimensional FRW universe is realized on the domain wall in the five dimensional space-time.
- Graviton, Chiral Spinor, and Vector can localize on the brane.

Graviton in FRW Universe in Four Dimensions

Perturbation $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \Rightarrow$

$$\delta R_{\mu\nu} = \frac{1}{2}\left[\nabla_\mu \nabla^\rho h_{\nu\rho} + \nabla_\nu \nabla^\rho h_{\mu\rho} - \nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu \left(g^{\rho\lambda} h_{\rho\lambda}\right) \right.$$
$$\left. -2R^\lambda{}_\nu{}^\rho{}_\mu h_{\lambda\rho} + R^\rho{}_\mu h_{\mu\nu} + R^\rho{}_\nu h_{\rho\mu}\right] ,$$
$$\delta R = - h_{\mu\nu} R^{\mu\nu} + \nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 \left(g^{\mu\nu} h_{\mu\nu}\right) .$$

Imposing the gauge condition $\nabla^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0$
Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \Rightarrow$

$$\frac{1}{2}\left[-\nabla^2 h_{\mu\nu} - 2R^\lambda{}_\nu{}^\rho{}_\mu h_{\lambda\rho} + R^\rho{}_\mu h_{\mu\nu} + R^\rho{}_\nu h_{\rho\mu} \right.$$
$$\left. -h_{\mu\nu} R + g_{\mu\nu} R^{\rho\lambda} h_{\rho\lambda}\right] = \kappa^2 \delta T_{\mu\nu} .$$

By using the formulation in

S. Nojiri and S. D. Odintsov, “Unifying phantom inflation with late-time acceleration: Scalar phantom-non-phantom transition model and generalized holographic dark energy”, Gen. Rel. Grav. 38 (2006) 1285 [hep-th/0506212].

consider scalar field theory

$$S_\phi = \int d^4x \sqrt{-g}\mathcal{L}_\phi \,, \quad \mathcal{L}_\phi = -\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\,,$$

$$\Rightarrow \qquad T_{\mu\nu} = -\omega(\phi)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi \,,$$

$$\Rightarrow \qquad \delta T_{\mu\nu} = h_{\mu\nu}\mathcal{L}_\phi + \frac{1}{2}g_{\mu\nu}\omega(\phi)\partial^\rho\phi\partial^\lambda\phi h_{\rho\lambda} \,.$$

Because we are now interested in the graviton, we may assume $h_{\mu\nu} = 0$ except the components with $\mu, \nu = 1, 2, 3$.
In the FRW universe ($k = 0$), assume $\phi = t$, FRW equations

$$\frac{3}{\kappa^2}\frac{\dot{a}^2}{a^2} = \frac{\omega}{2} + V\,, \quad \frac{1}{\kappa^2}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \frac{\omega}{2} - V\,,$$

$$\Rightarrow \qquad \omega = \frac{1}{\kappa^2}\left(2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2}\right)\,, \quad V = -\frac{1}{\kappa^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\,.$$

$$\Rightarrow \qquad 0 = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\partial_t - \partial_t^2 + \frac{\triangle}{a^2}\right) h_{ij}\,.$$

Explicit expressions of connections and curvatures in five dimensions

Metric

$$g_{AB} = \begin{pmatrix} -L^2 \mathrm{e}^{u(w,t)} & & & & \\ & L^2 \mathrm{e}^{u(w,t)} \frac{a(t)^2}{1-kr^2} & & & \\ & & L^2 \mathrm{e}^{u(w,t)} a(t)^2 r^2 & & \\ & & & L^2 \mathrm{e}^{u(w,t)} a(t)^2 r^2 \sin^2\theta & \\ & & & & 1 \end{pmatrix} .$$

Connections

$$\Gamma^t_{tt} = \frac{1}{2}\dot{u} , \quad \Gamma^w_{tt} = \frac{1}{2} L^2 e^u u' , \quad \Gamma^r_{rt} = \Gamma^\theta_{\theta t} = \Gamma^\phi_{\phi t} = \frac{\dot{a}}{a} + \frac{1}{2}\dot{u} , \quad \Gamma^t_{tw} = \Gamma^r_{rw} = \Gamma^\theta_{\theta w} = \Gamma^\phi_{\phi w} = \frac{1}{2} u' ,$$

$$\Gamma^t_{ij} = L^{-2} \mathrm{e}^{-u} \left(\frac{\dot{a}}{a} + \frac{1}{2}\dot{u} \right) g_{ij} , \quad \Gamma^r_{rr} = \frac{kr}{1-kr^2} , \quad \Gamma^w_{ij} = -\frac{1}{2} u' g_{ij} , \quad \Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = \frac{1}{r} ,$$

$$\Gamma^r_{\theta\theta} = -r(1-kr^2) , \quad \Gamma^\phi_{\phi\theta} = \cot\theta , \quad \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta , \quad \Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta .$$

Ricci curvatures

$$R_{tt} = \left[-\frac{1}{2}u'' - u'^2 + \frac{3}{2}L^{-2}\mathrm{e}^{-u}\left(\ddot{u} + \frac{\dot{a}\dot{u}}{a} + 2\frac{\ddot{a}}{a}\right)\right] g_{tt}\,,$$

$$R_{ij} = \left[-\frac{1}{2}u'' - u'^2 + \frac{1}{2}L^{-2}\mathrm{e}^{-u}\left(\ddot{u} + 5\frac{\dot{a}\dot{u}}{a} + 2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + \dot{u}^2 + 4\frac{k}{a^2}\right)\right] g_{ij}\,,$$

$$R_{ww} = -2u'' - u'^2$$

$$R_{tw} = -\frac{3}{2}\dot{u}'\,.$$

Scalar curvature

$$R = -4u'' - 5u'^2 + 3L^{-2}\mathrm{e}^{-u}\left(\ddot{u} + \frac{1}{2}\dot{u}^2 + 3\frac{\dot{a}\dot{u}}{a} + 2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right)\,.$$

Explicit form of spin connection

Vierbein field $e^{\hat{B}}$, $g_{MN} \stackrel{\text{def}}{=} e^{\hat{A}}{}_M \eta_{\hat{A}\hat{B}} e^{\hat{B}}{}_N$

$$\Rightarrow \quad e^{\hat{A}}{}_M = \begin{bmatrix} Le^{u/2} & 0 & 0 & 0 & 0 \\ 0 & Le^{u/2}a & 0 & 0 & 0 \\ 0 & 0 & Le^{u/2}a & 0 & 0 \\ 0 & 0 & 0 & Le^{u/2}a & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$e_{\hat{A}M} \stackrel{\text{def}}{=} \eta_{\hat{A}\hat{B}}\, e^{\hat{B}}{}_M = \begin{bmatrix} -Le^{u/2} & 0 & 0 & 0 & 0 \\ 0 & Le^{u/2}a & 0 & 0 & 0 \\ 0 & 0 & Le^{u/2}a & 0 & 0 \\ 0 & 0 & 0 & Le^{u/2}a & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e_{\hat{A}}{}^{M} \stackrel{\text{def}}{=} e_{\hat{A}N} g^{MN}$$

$$= \begin{bmatrix} L^{-1}\mathrm{e}^{-u/2} & 0 & 0 & 0 & 0 \\ 0 & L^{-1}\mathrm{e}^{-u/2}a^{-1} & 0 & 0 & 0 \\ 0 & 0 & L^{-1}\mathrm{e}^{-u/2}a^{-1} & 0 & 0 \\ 0 & 0 & 0 & L^{-1}\mathrm{e}^{-u/2}a^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Ricci rotation coefficients and spin connections

$$\Omega_{ABM} \stackrel{\text{def}}{=} \left(\partial_A e^{\hat{C}}{}_B - \partial_B e^{\hat{C}}{}_A\right) e_{\hat{C}M} = -\Omega_{BAM} ,$$

$$\omega_{ABM} \stackrel{\text{def}}{=} -\frac{1}{2}\left(\Omega_{ABM} - \Omega_{BMA} - \Omega_{MAB}\right) = -\omega_{BAM} .$$

Non-vanishing components

$$\begin{cases} \Omega_{050} & = -\Omega_{500} = \frac{1}{2}u' L^2 \mathrm{e}^u \\ \Omega_{0ij} & = -\Omega_{i0j} = \left(\frac{1}{2}\dot{u}a^2 + \dot{a}a\right) L^2 \mathrm{e}^u \delta_{ij} \,, \\ \Omega_{5ij} & = -\Omega_{i5j} = \frac{1}{2}u' L^2 \mathrm{e}^u a^2 \delta_{ij} \,, \end{cases}$$

$$\begin{cases} \omega_{050} & = -\omega_{500} = -\frac{1}{2}u' L^2 \mathrm{e}^u \,, \\ \omega_{0ij} & = -\omega_{i0j} = -\left(\frac{1}{2}\dot{u}a^2 + \dot{a}a\right) L^2 \mathrm{e}^u \delta_{ij} \,, \\ \omega_{5ij} & = -\omega_{i5j} = -\frac{1}{2}u' L^2 \mathrm{e}^u a^2 \delta_{ij} \,. \end{cases}$$