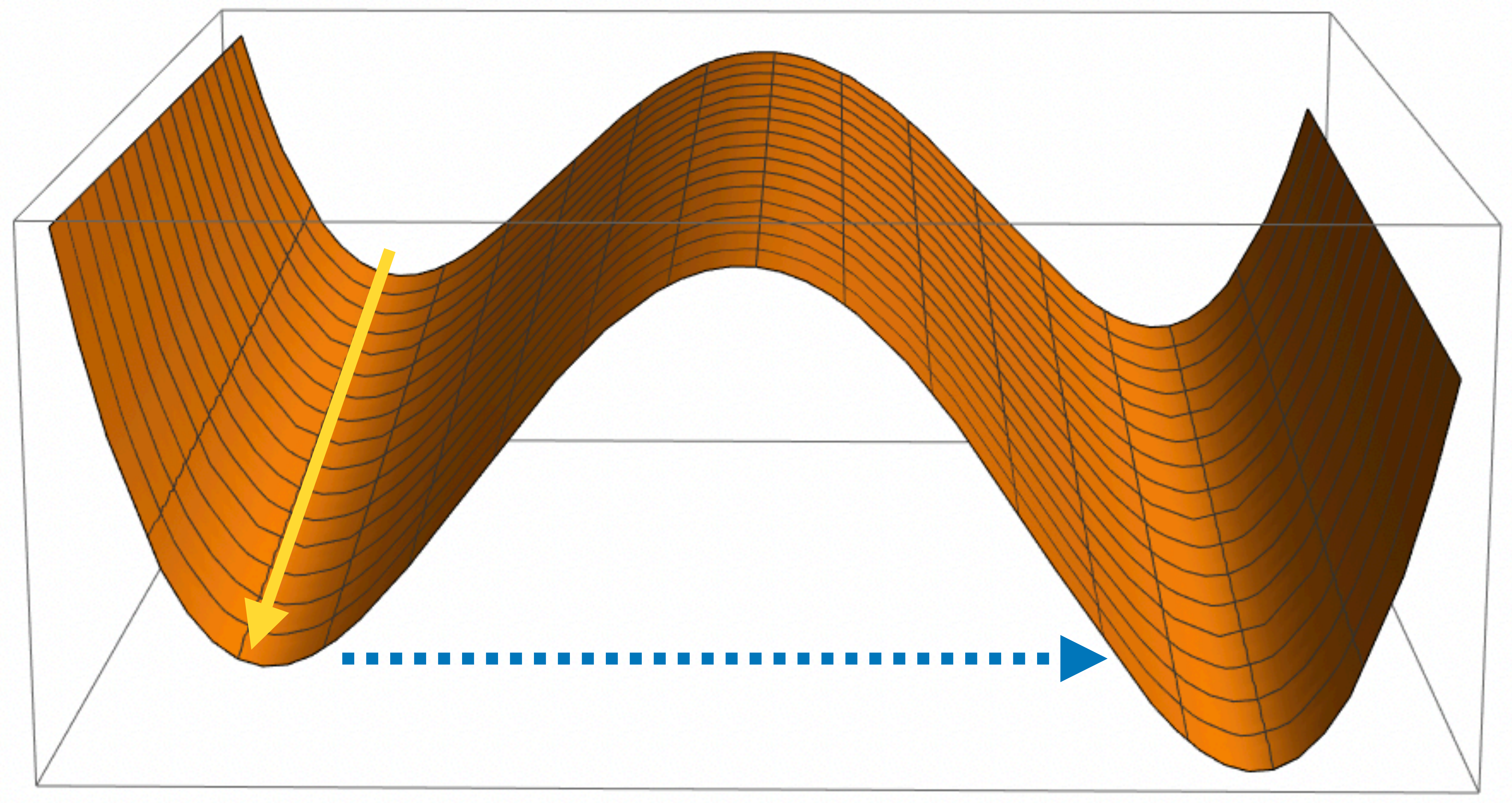
Tunneling with Time Dependence

Patrick Draper

Semiclassics provides useful tools/intuition for many problems in QFT. Vacuum decay is a prototypical example.



How do we approach semiclassics of bubble nucleation on time-dependent backgrounds?

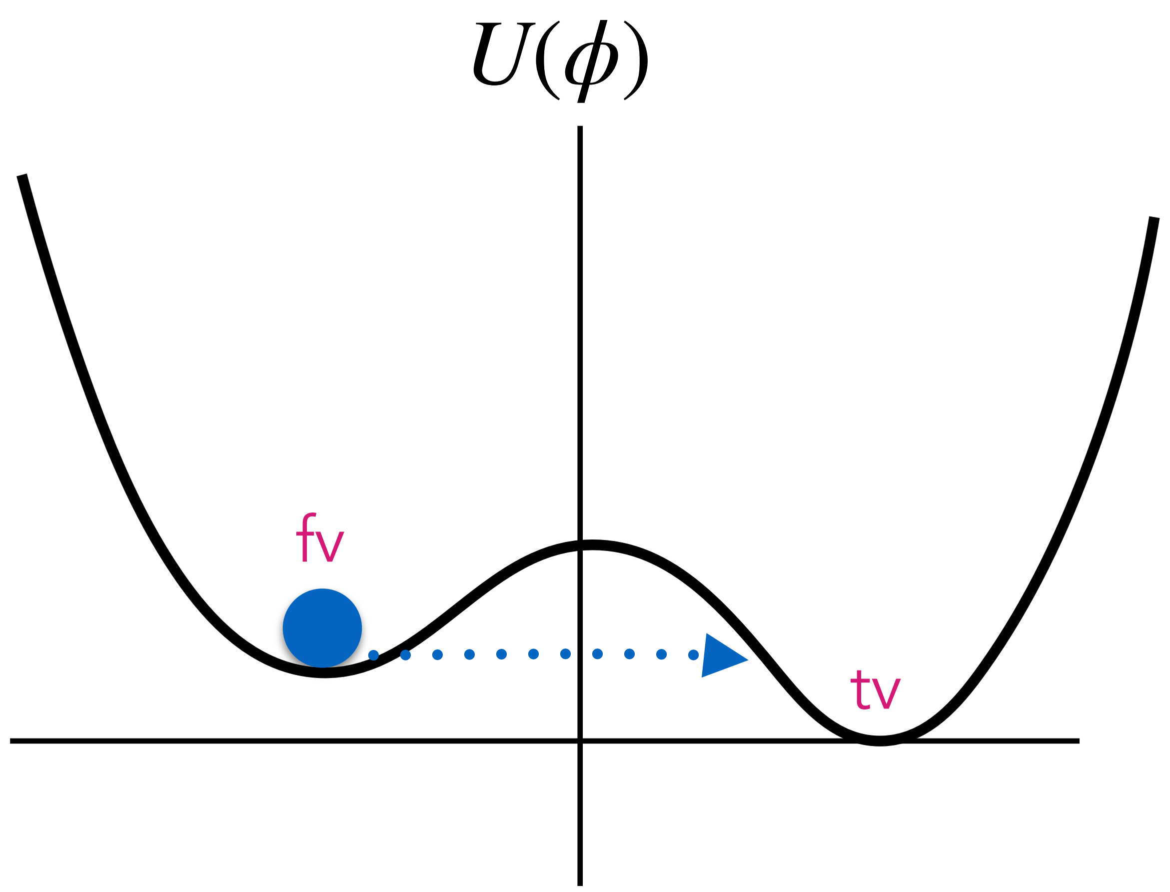
Euclidean continuation will result in a complex action

Based *Phys.Rev.D* 108 (2023) 9, 096038 with Manthos Karydas and Hao Zhang

Semiclassical theory of vacuum decay by bubble nucleation

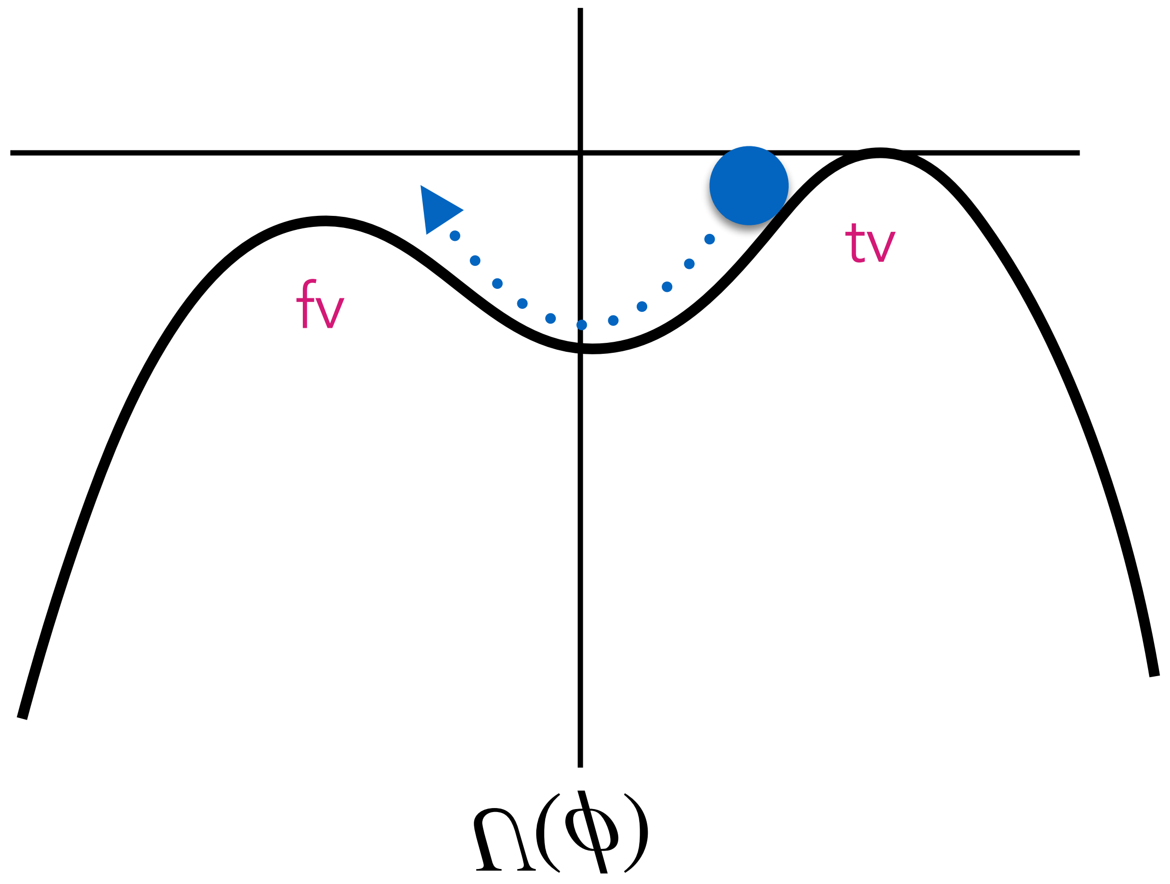
Typical tunneling potential

$$V(\phi) = \frac{1}{2}\lambda(\phi^2 - a^2)^2 - (\epsilon/2a)\phi$$



We take $t \to -i\tau$ and look for O(4) invariant saddle points. Solve a shooting problem:

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = U'(\phi)$$



$$\Gamma \approx v^4 \exp(-S_E/\hbar)$$

- Simple because O(4) invariance => ODEs
- nucleation point is a classical bubble with zero momentum
- real-valued fields remain real

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Two generalizations:

- $t \to e^{i\gamma}\tau$
- $V(\phi) \to V(\phi, \chi_0(t))$

O(4) invariance broken, DOFs complexified, final states may carry momentum

Loss of O(4) invariance is a technical complication because ODEs => PDEs.

To avoid this complication we will work with effective QM models
for collective coordinate DOFs.   QM => back to ODEs.

Start by reviewing how this works for the standard, time-indep, Euclidean case,
then extend to $t \to e^{i\gamma}\tau$,
then add time dependence

standard, time-independent analysis:

$$\int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \right)$$

$\phi_{\epsilon=0} \sim \tanh(z - z_0)$ Born-Oppenheimer "fast mode" is a domain wall of tension ~ σ

$\phi \sim \tanh(R(t) - R_0)$ R(t) is the "slow mode"

$\Big\downarrow$ Integrate over space

$$L = -\sigma R^2 \sqrt{1 - (\partial_t R)^2} + \epsilon R^3$$

thin-wall effective Lagrangian captures leading semiclassics.

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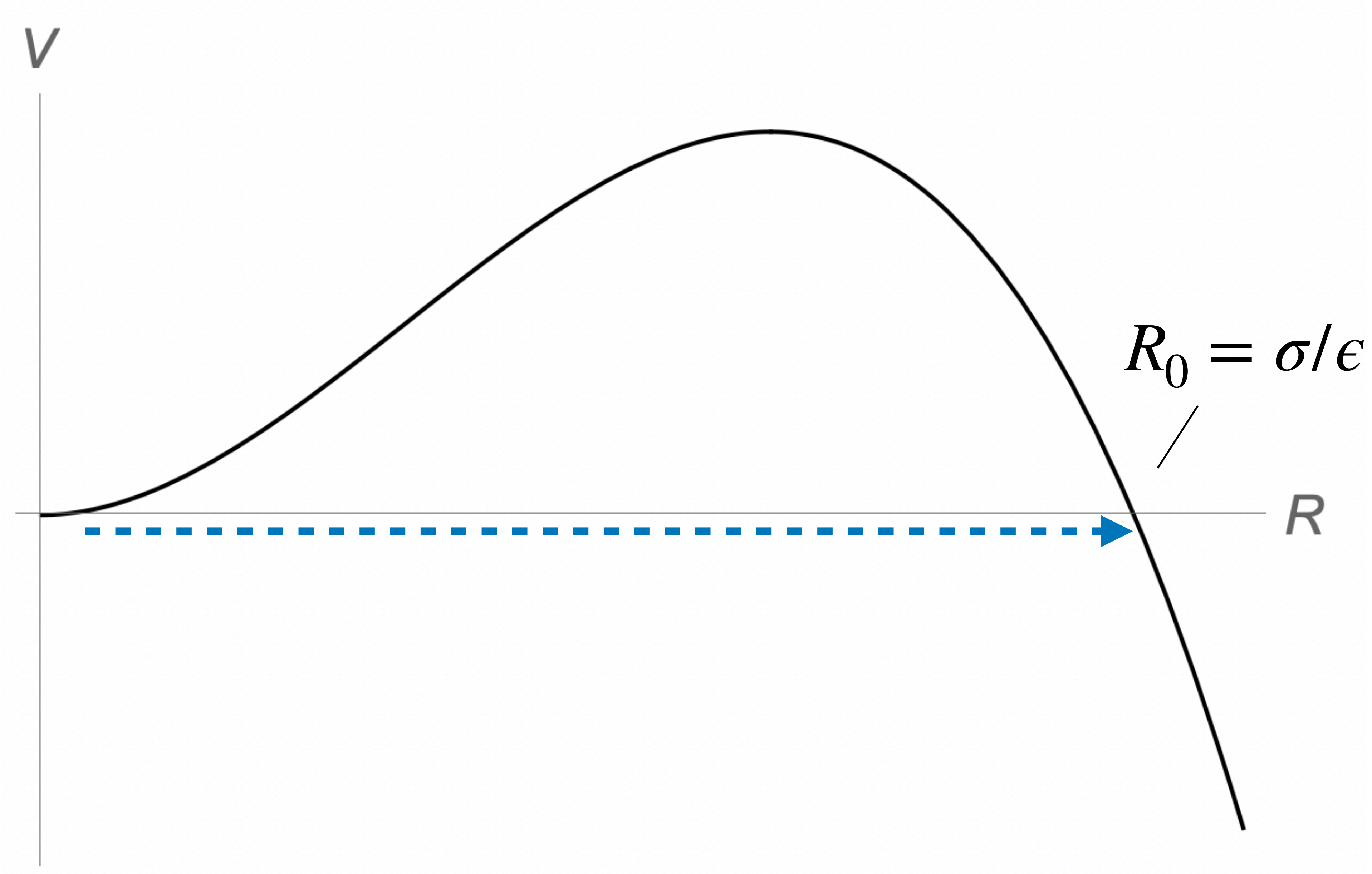
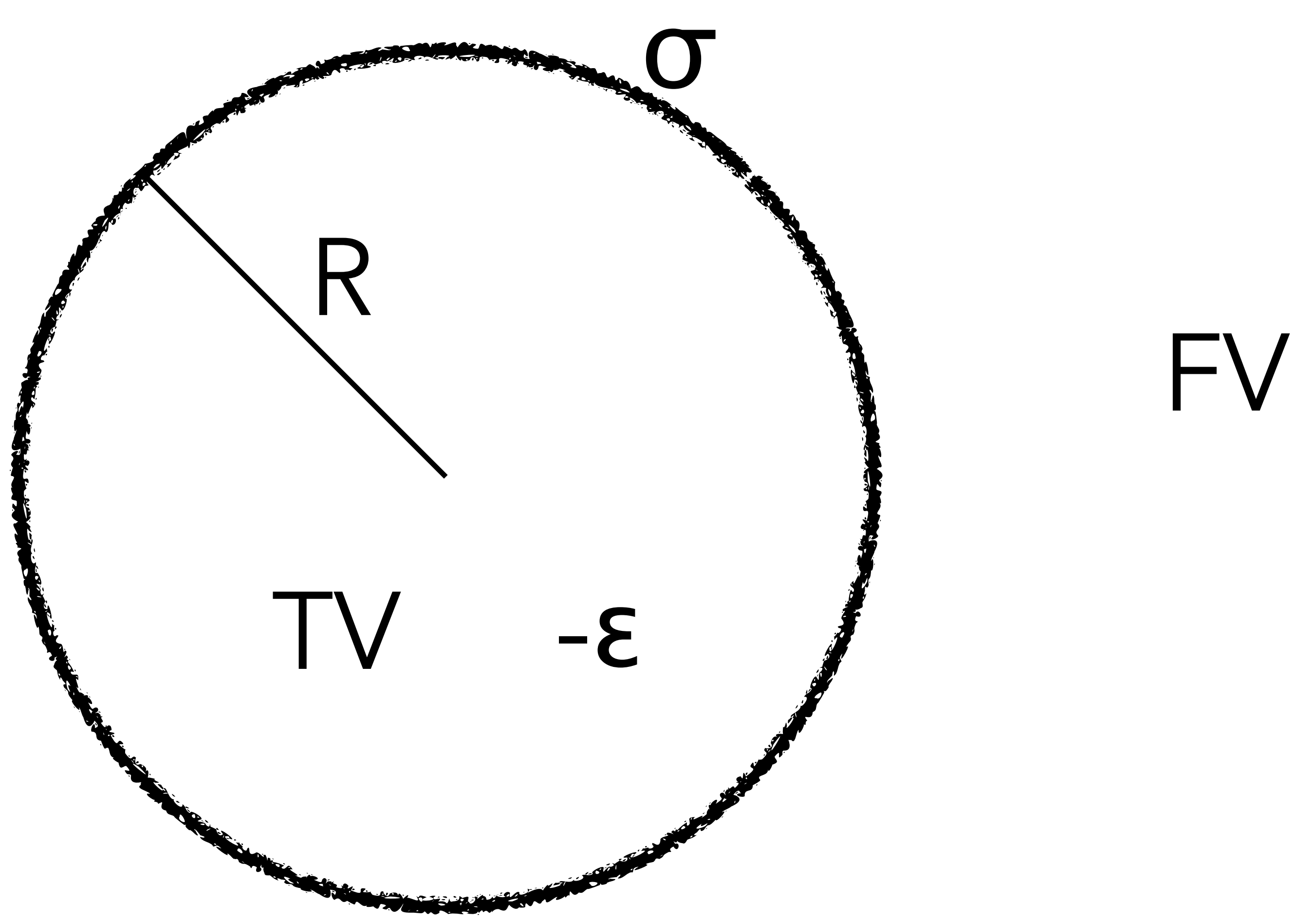
tension+lorentz contraction

pressure





similar L can be used for Schwinger model, BON,...

$$A = \langle \mathrm{TV\ bubble}, \mathrm{T_f} = 0 | \mathrm{FV}, \mathrm{T_i} \rangle = ?$$

$$e^{iS} = \exp\left(i \int_{T_i}^{T_f} dt \left[-\sigma R^2 \sqrt{1 - (\partial_t R)^2} + \epsilon R^3 \right] \right)$$

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$$t \to -i\tau$$

continuation of T_i

$$e^{-S_c} = \exp\left(- \int_{T_i}^{T_f} d\tau \left[\sigma R^2 \sqrt{1 + (\partial_\tau R)^2} - \epsilon R^3 \right] \right)$$

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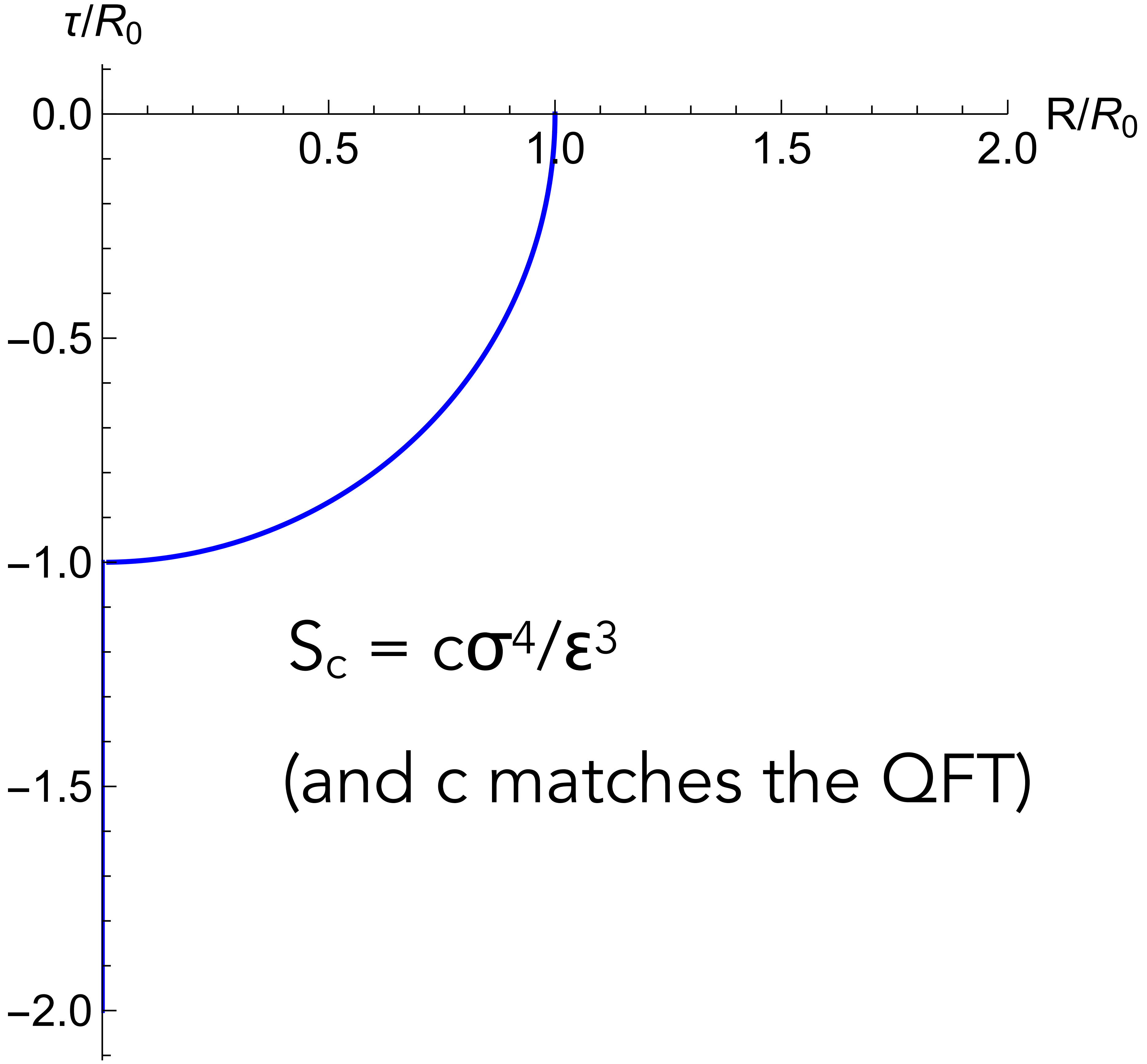
Zero-energy saddle point for which R=0 at some τ_0:

$$R = \Theta(\tau + R_0)\sqrt{R_0^2 - \tau^2}.$$

$$R_0 = \sigma/\epsilon \ , \qquad \tau_0 = -R_0$$

"Amplitude to nucleate a bubble at rest at t=0"



$S_c = c\sigma^4/\epsilon^3$

(and c matches the QFT)

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Technically this joins two zero-energy solutions piecewise.
Still a stationary point of the action: $\sigma R^2 \sqrt{1 + (\partial_\tau R)^2}$

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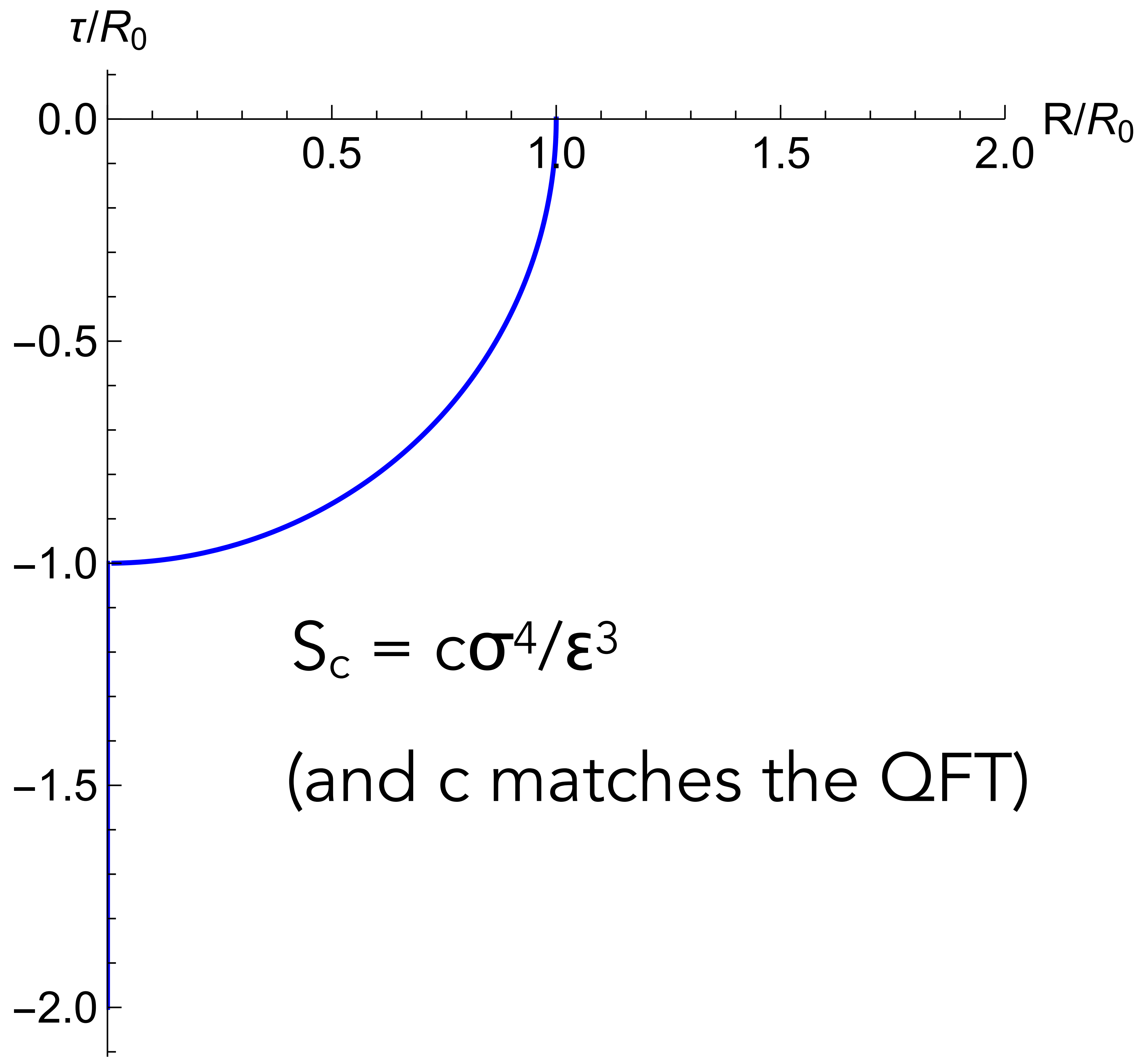

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Euclidean for $\gamma=-\pi/2$

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$$\boxed{R = \Theta(\tau - R_0 \csc\gamma)\sqrt{R_0^2 + (e^{i\gamma}\tau - R_0\cot\gamma)^2}}$$

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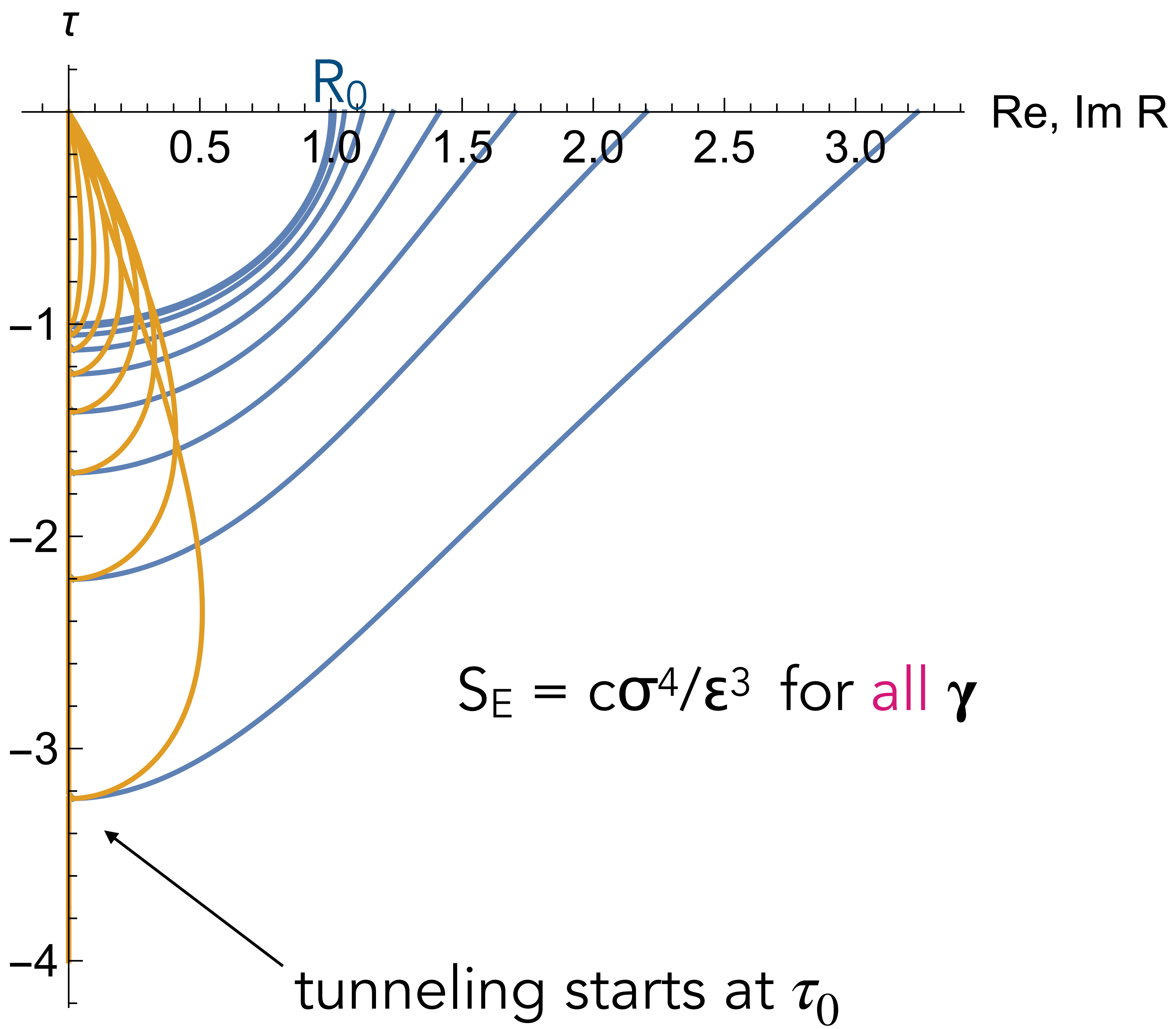

$S_E = c\sigma^4/\varepsilon^3$ for all $\boldsymbol{\gamma}$

tunneling starts at τ_0

goes to -∞ in the real-time limit

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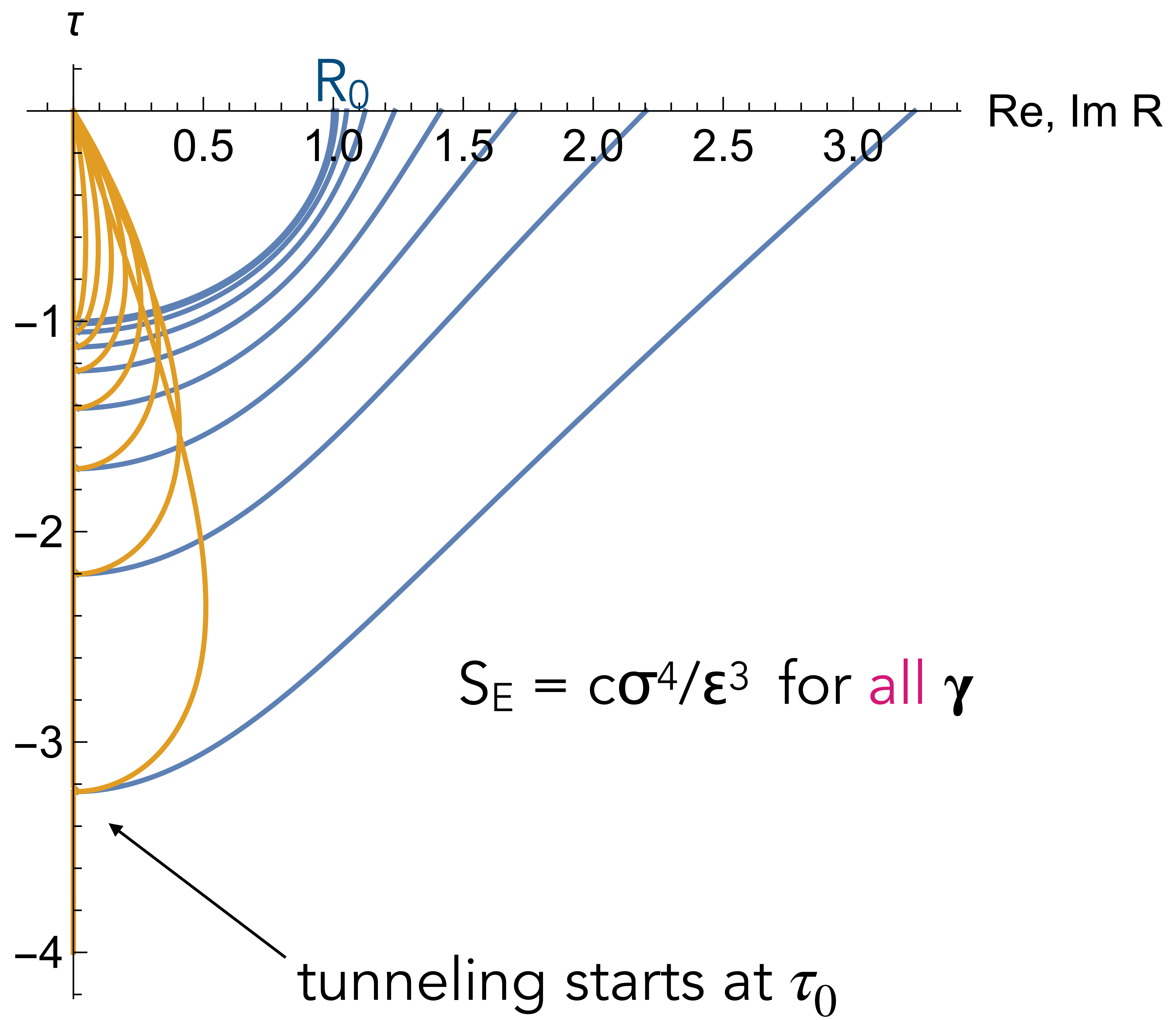
Return to real R at $\tau = 0$,
but beyond the classical turning point R_0,
with nonzero momentum

What do they mean?



$S_E = c\sigma^4/\varepsilon^3$ for all γ

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amplitudes with semiclassical wavepackets:

$$\langle\psi_f(R,T_f)|\psi_i(R,T_i)\rangle$$
$$\psi(R) = e^{iS_{i,f}(R)/\hbar}$$

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$$\psi(R) = e^{iS_{i,f}(R)/\hbar}$$

$$S_c = -iS_i(R_i) + iS_f(R_f) - i\int_{T_i}^{T_f} d\tau\, L_c(\partial_\tau R, R, \tau)$$

$$L_c(\partial_\tau R, R, \tau) = e^{i\gamma} L(e^{-i\gamma}\partial_\tau R, R, e^{i\gamma}\tau).$$

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same bulk EOM,
boundary variations relate initial and final semiclassical R,p to features of the states:

$$\boxed{\begin{aligned} p_i &\equiv \left.\frac{\delta L_c}{\delta\partial_\tau R}\right|_{\tau=T_i} = S_i'(R_i) \\ p_f &\equiv \left.\frac{\delta L_c}{\delta\partial_\tau R}\right|_{\tau=T_f} = S_f'(R_f). \end{aligned}}$$

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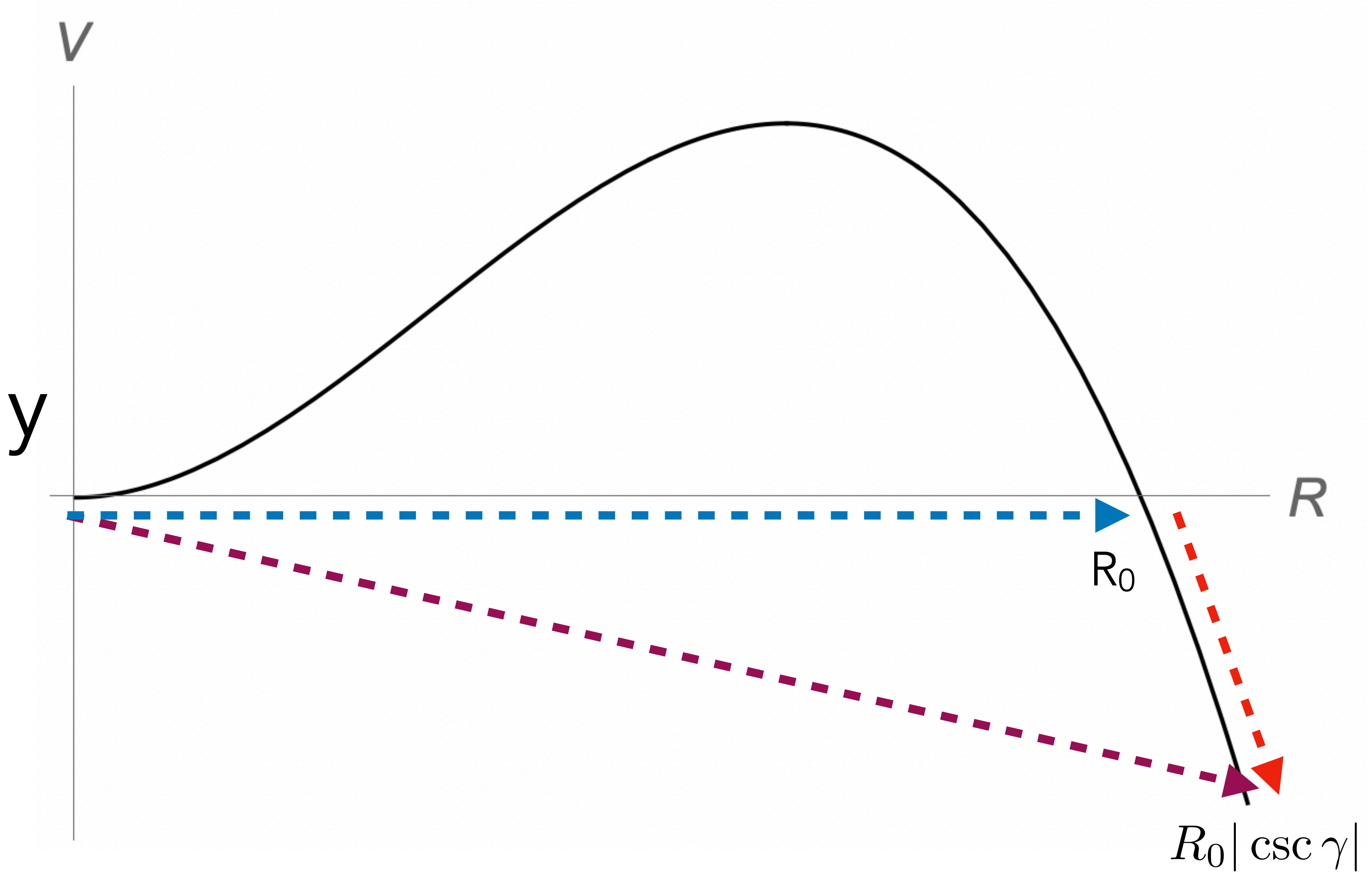
semiclassical trajectories also
appx certain packet amplitudes!

e.g. $\mathrm{Re}(p_f)$ = $\langle p \rangle$ for packets: $\psi_f = Ne^{-(R_f-R_0)^2/2\sigma^2 + ipR_f} \Rightarrow \mathrm{Re}\, S_f'(R_f) = p$ = $\mathrm{Re}(p_f)$

Interpretation:

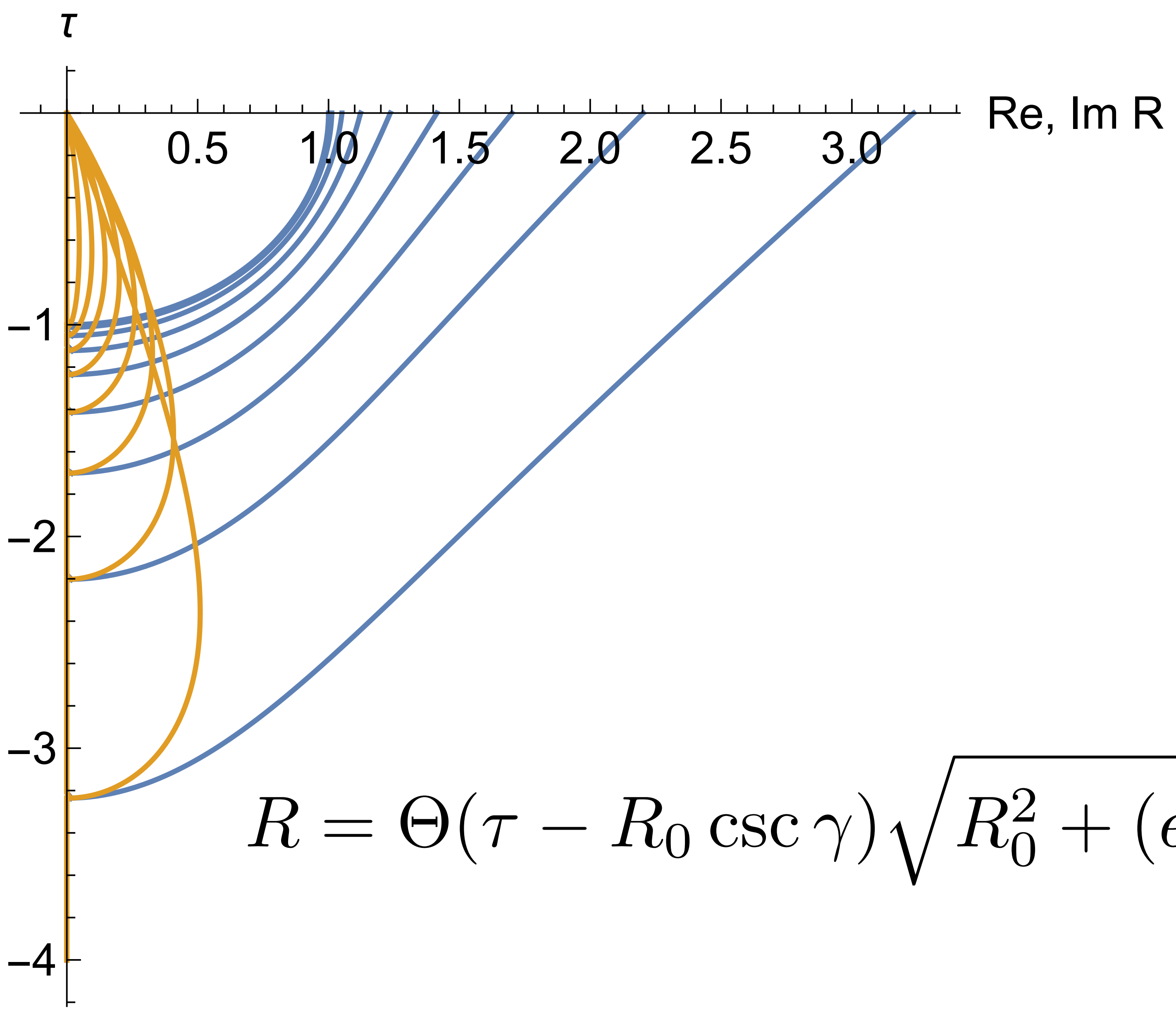


- The solutions for general **γ** connect the false vacuum
  to wavepackets outside the barrier with zero classical energy

  connected by classical, real time evolution
  to the nucleation point of the critical bubble ($R=R_0$, at rest)

- tunneling starts earlier
  (only makes sense if $T_i < R_0 \csc \boldsymbol{\gamma}$)

$$\langle \psi_f(R,T_f)|\psi_i(R,T_i)\rangle$$

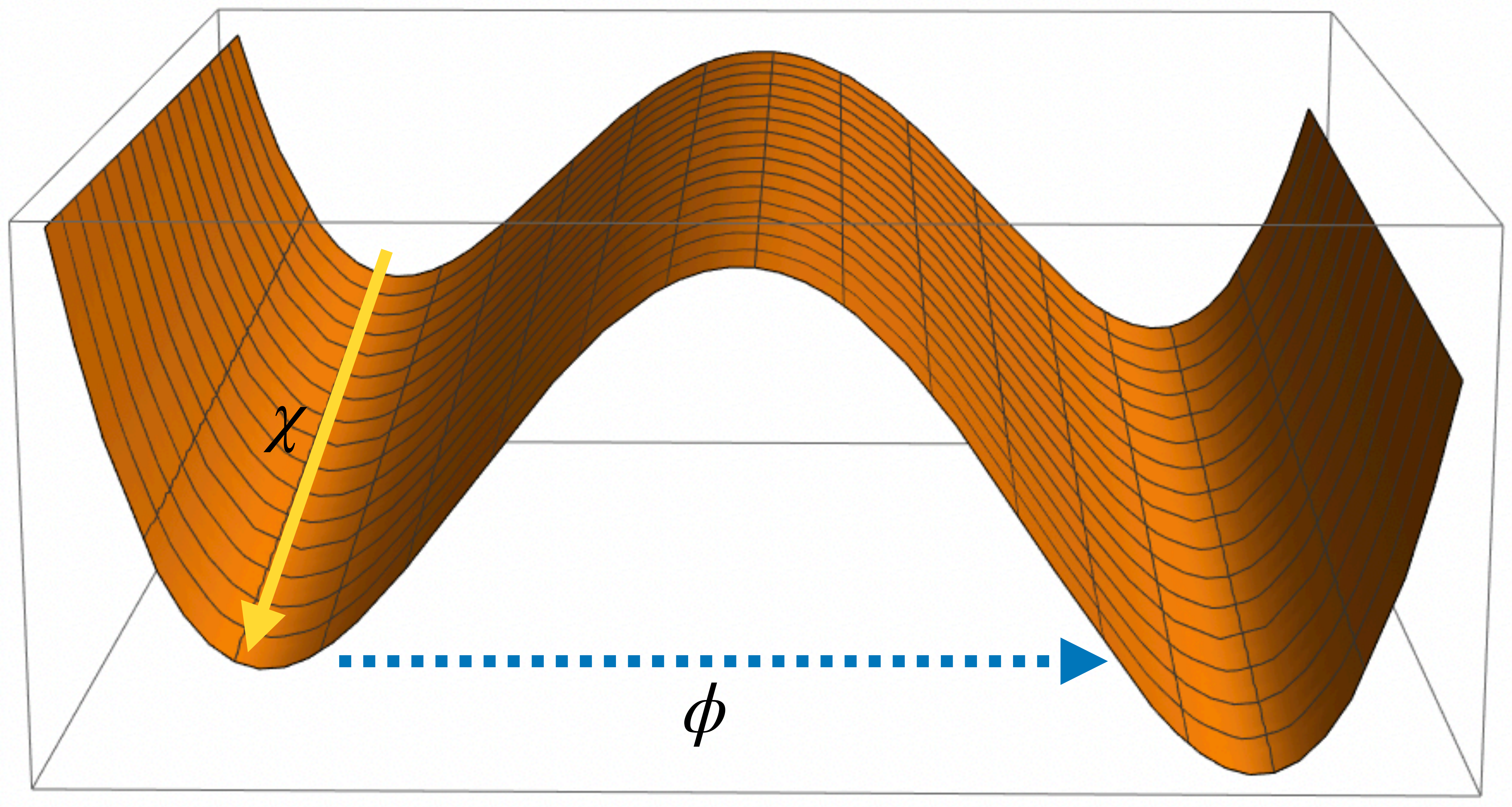


$$R = \Theta(\tau - R_0 \csc\gamma)\sqrt{R_0^2 + (e^{i\gamma}\tau - R_0\cot\gamma)^2}$$

γ is accounting for the time translation collective coordinate

Now let us add time dependence.

This can arise, for example, if the tunneling field ϕ is weakly coupled to another field χ slowly rolling in a shallow potential.



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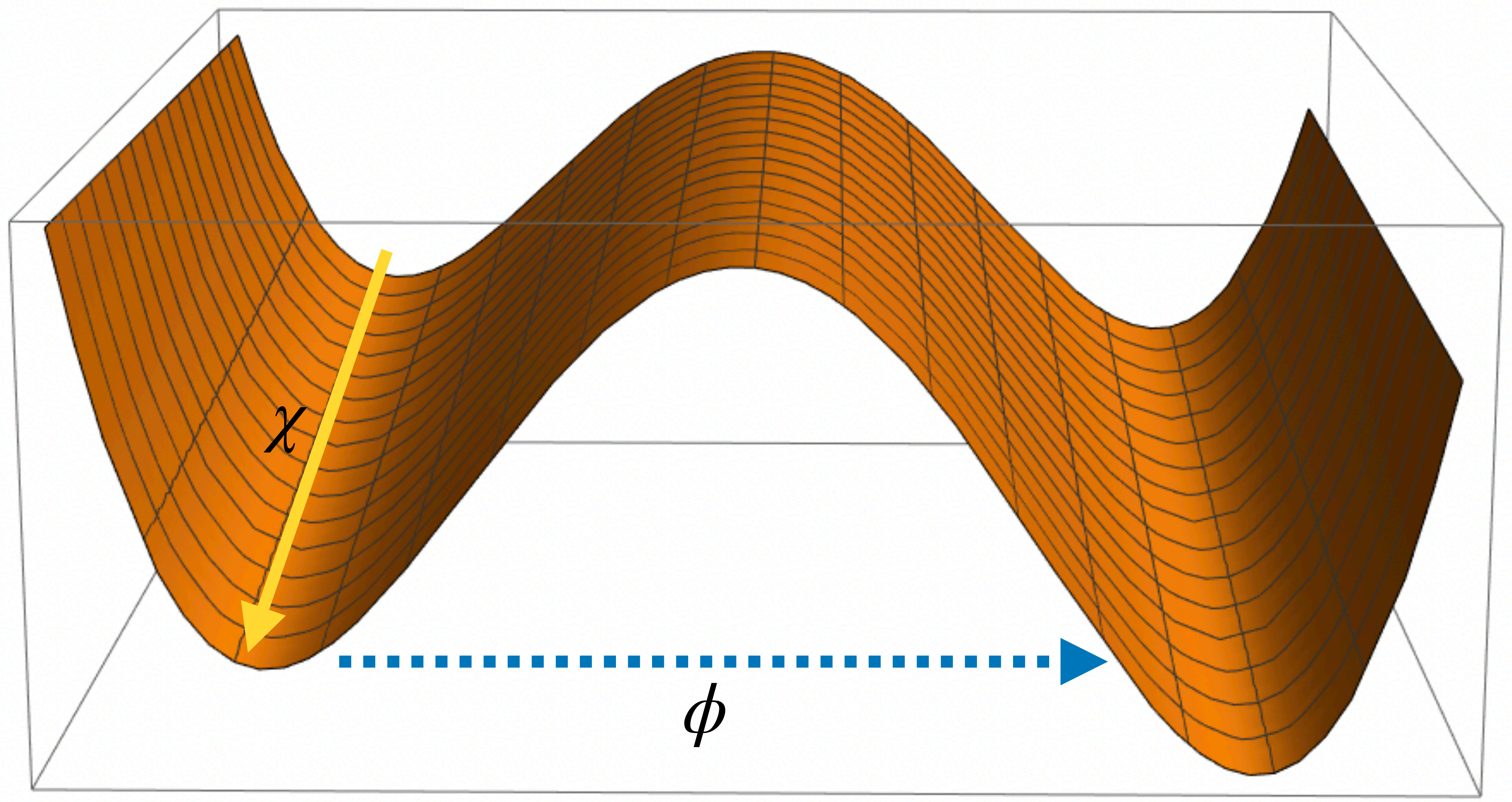


at zeroth order in the coupling, same thin-wall profile for ϕ + rolling solution for χ

integrate out space to get $L = -\sigma(t)R^2\sqrt{1-(\partial_t R)^2} + \epsilon(t)R^3$ $\qquad \chi(t) \to \sigma(t), \epsilon(t)$

could also include backreaction of ϕ profile on χ,
but higher order and may be subleading to corrections to the thin-wall limit

$$L = -\sigma(t) R^2 \sqrt{1-(\partial_t R)^2} + \epsilon(t) R^3 \quad + \quad t \to e^{i\gamma}\tau$$

Look for nontrivial small-R solutions that can connect to the R=0 "vacuum"

Relevant solutions behave as

$$\boxed{\begin{aligned} &R = c\sqrt{\tau-\tau_0}\,(1+\mathcal{O}(\tau-\tau_0))\ , \\ &c = ie^{i\gamma/2}\sqrt{\frac{6(\sigma_0+e^{i\gamma}\tau_0 v_\sigma)}{3i(\epsilon_0+e^{i\gamma}\tau_0 v_\epsilon)+v_\sigma}} \end{aligned}}$$

where

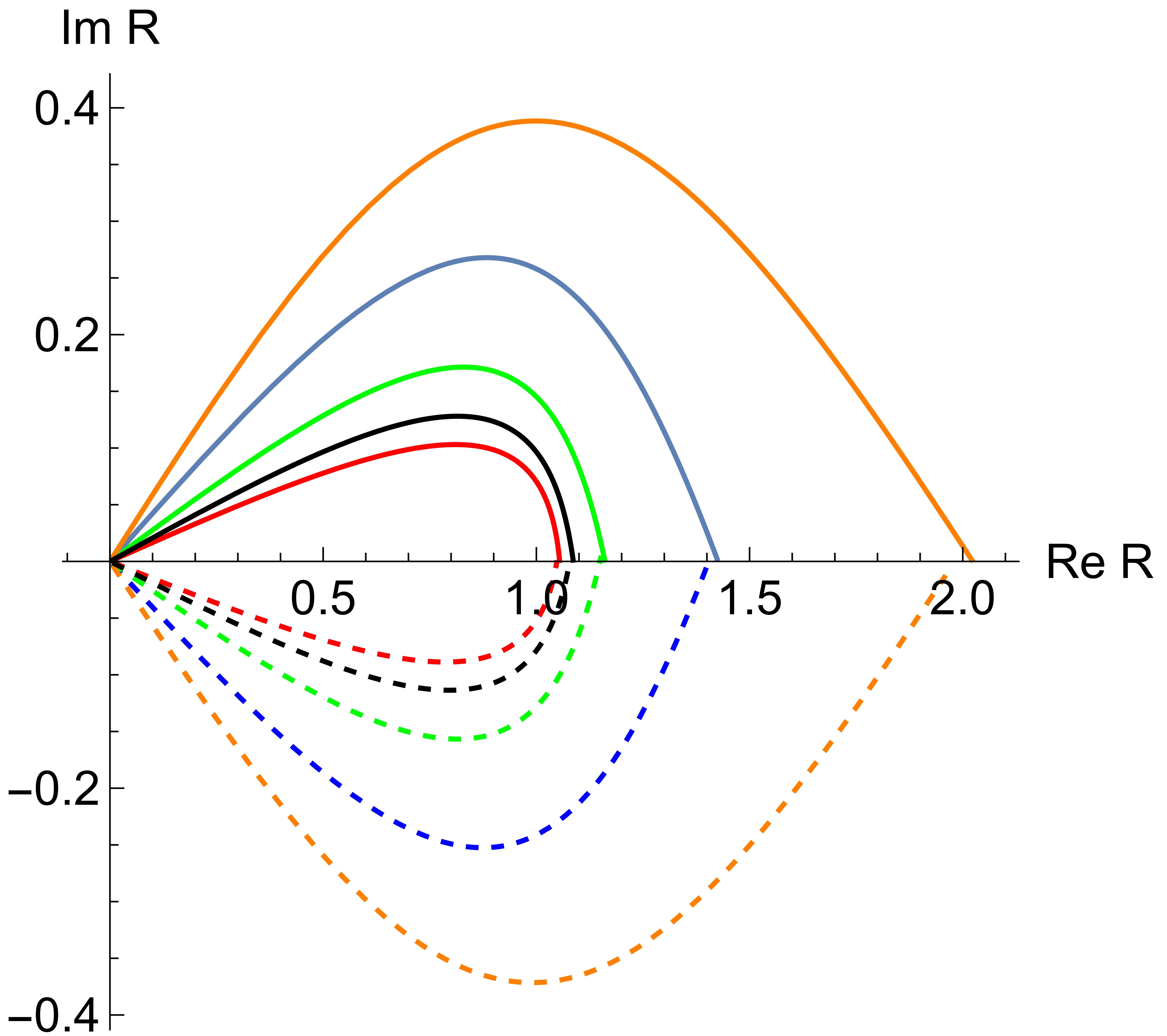
$$\sigma(e^{i\gamma}\tau) = \sigma_0 + e^{i\gamma} v_\sigma(\tau-\tau_0) + \ldots,$$
$$\epsilon(e^{i\gamma}\tau) = \epsilon_0 + e^{i\gamma} v_\epsilon(\tau-\tau_0) + \ldots$$

To determine τ_0, shoot for solutions that return to Im R=0 at τ=0 examples with slow linear evolution of **σ**, **ε** and a range of **γ**

final states generally acquire small Im p_f — slightly off-peak

$$\psi_f = N e^{-(R_f - R_0)^2/2\sigma^2 + i p R_f}$$

$$\mathrm{Re}\, p_f = -\sigma R_0^2 \csc^2 \gamma \cot \gamma + \mathcal{O}(v_{\epsilon,\sigma})$$



v=0 on-shell action $S \sim R_0^3\sigma_0, R_0^4\epsilon_0$

first correction can be computed from time-independent solutions
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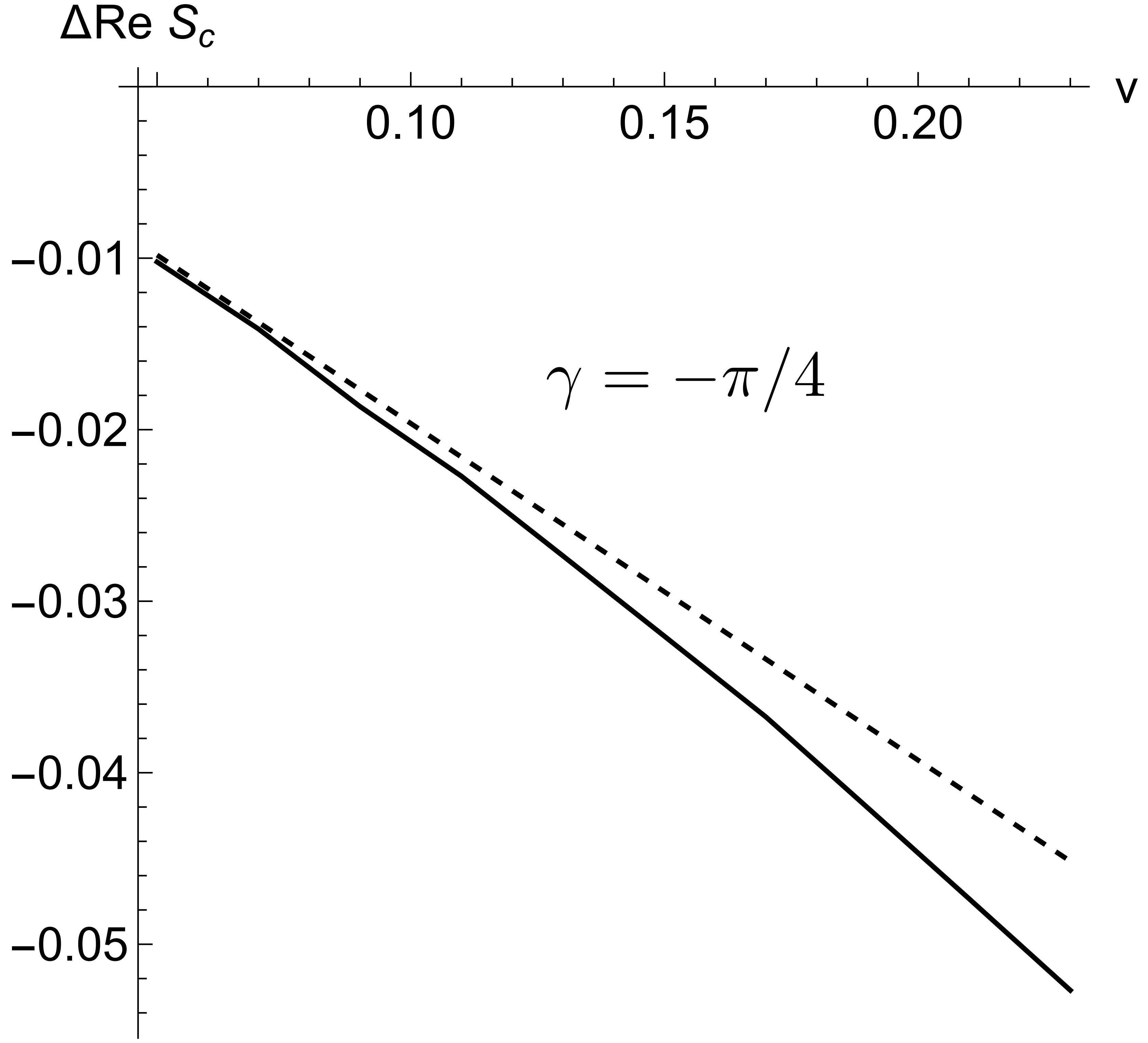
$$\Delta\mathrm{Re}\ S_c|_{\mathcal{O}(v)} = \frac{\pi}{4}\cot(\gamma)R_0^4\left(v_\sigma - \frac{3}{4}R_0 v_\epsilon\right)$$



For fixed **γ**, small cf. leading order if variations are slow.

But unbounded as **γ** approaches 0, pi… what **γ** should we use?

Recall tunneling starts at $\tau_0 = R_0 \csc\gamma$ — require > T_i

If e.g. barrier is growing monotonically, starting earlier = larger amplitudes

$$\gamma = \csc^{-1}(T_i/R_0) \quad \Rightarrow \quad \boxed{\Delta \mathrm{Re}\ S_c|_{\mathcal{O}(v),\max} = -\frac{\pi}{4}\sqrt{T_i^2 - R_0^2}R_0^3\left(v_\sigma - \frac{3}{4}R_0 v_\epsilon\right)}$$

if +ve, favors earlier decay

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Otherwise later is better, $\gamma = -\pi/2$ (Euclidean) and leading O(v) correction vanishes

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What precise question does exp(-S_c) address?

T_i is the latest time when we're sure there was no bubble
maximizing over $\boldsymbol{\gamma}$ computes the leading decay probability over times up to T_f=0

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There are two effects in this correction.

The first is just time-translating the couplings appearing in the leading time-independent action, $S \sim \sigma^4/\varepsilon^3$, back to the starting time Ti in the linearized appx

This effect dominates for $T_i << R_0$, but it is an incomplete accounting of the O(v) effects which include also barrier evolution during tunneling

In the Euclidean case $T_i = R_0$ the linear correction is purely imaginary

It is not straightforward to generalize this treatment beyond the thin-wall effective QM

Need $\boldsymbol{\phi}$(r,t) . May be possible to set up numerically.

Might be able to find few-parameter ansatze that work well.

Coupling to gravity

Three approaches to keep the problem tractable (not exhaustive):

- Rigid Minkowski => rigid something time-dep e.g. FRW
- Effective action for a membrane coupled to gravity
- Exact solutions from analytic continuation

Rigid Minkowski => rigid FRW:

$$ds^2 = -dt^2 + a(t)^2 dx_i^2$$

$$a \approx 1 + Ht \qquad (HR_0 \ll 1)$$

$$S \approx \int d^4x \left[\mathcal{L}_{flat} + Ht(3\mathcal{L}_{flat} + (\partial_i \phi)^2)\right]$$

background spacetime provides the time dependence

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$$\Rightarrow L_{eff} = \underbrace{-(1+3Ht)\sigma R^2 \sqrt{1-(\partial_t R)^2} + (1+3Ht)\epsilon R^3}_{\text{same as previous}} + \underbrace{\frac{\sigma H t R^2}{\sqrt{1-(\partial_t R)^2}}}_{\text{new term}}$$

O(H) contribution to the on-shell action vanishes

Membrane effective action:

in the static case, use a small generalization of an effective action found by Visser (1992)

EH + spherical brane:

(1) $$S = \frac{1}{16\pi G_N} \int_{\mathcal{M}_{1,2}} \sqrt{|g|} d^4x \left(R_{1,2} - 2\Lambda_{1,2}\right) + \frac{1}{8\pi G_N} \int_{\mathcal{T}} \sqrt{|h|} d^3y K_{1,2} - \mu \int_{\mathcal{T}} \sqrt{|h|} d^3y$$

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Effective action for R_{brane}:

(2) $$S^{\text{eff}}_{\text{brane}} = \frac{1}{2G_N}\int d\lambda \Bigg[-2R\dot{R}\sinh^{-1}\left(\frac{\dot{R}}{\sqrt{N^2 f_{\text{dS}_T}}}\right) + 2R\sqrt{f_{\text{dS}_T}N^2 + \dot{R}^2} + 2R\dot{R}\sinh^{-1}\left(\frac{\dot{R}}{\sqrt{N^2 f_{\text{dS}_F}}}\right) - 2R\sqrt{f_{\text{dS}_F}N^2 + \dot{R}^2} - 8\pi G_N \mu N R^2\Bigg]$$

$$\boxed{f_{T,F} = 1 - R^2/L^2_{1,2}}$$

- λ is an arbitrary parametrization and N is the lapse on the world"line" $-N^2 d\lambda^2 = -f_{T,F}dt^2_{T,F} + f^{-1}_{T,F}dR^2$
- The arcsinh is Visser's trick to rewrite in 1st order form
- Looks rather different, but actually equivalent to previous for $G_N \to 0$

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dS/dN=0 in (2) = junction condition from (1):

$$R\sqrt{f_{dS_T}+\dot{R}^2}-R\sqrt{f_{dS_F}+\dot{R}^2}-4\pi\mu G_NR^2=0$$ gauge fix N=1, λ=proper time

N provides reparam invariance => equiv to H=0

This is the only independent equation (dS/dR=0 gives the proper time derivative of it)

Now we continue $\lambda \to e^{i\gamma}\tau$

The continued energy is

$$E = e^{i\gamma}\frac{\pi}{G_N}\left[R\sqrt{f_{dS_F} + e^{-2i\gamma}\dot{R}^2} - R\sqrt{f_{dS_T} + e^{-2i\gamma}\dot{R}^2} + 4\pi G_N \mu R^2\right] = 0$$

One solution is R=0.

To find the other, rearrange: $\boxed{e^{-2i\gamma}\left(\frac{dR}{d\tau}\right)^2 + 1 - \alpha^2 R^2 = 0}$

$$\alpha^2 = L_F^{-2} + \frac{\left(L_F^{-2} - L_T^{-2} - 16\pi^2 G_N^2 \mu^2\right)^2}{64\pi^2 G_N^2 \mu^2}$$
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$$\alpha^2 = L_F^{-2} + \frac{\left(L_F^{-2} - L_T^{-2} - 16\pi^2 G_N^2 \mu^2\right)^2}{64\pi^2 G_N^2 \mu^2}$$
$$\approx R_0^{-2} + (2L_F)^{-2} + (2L_T)^{-2} + \mathcal{O}(G_N^2)$$

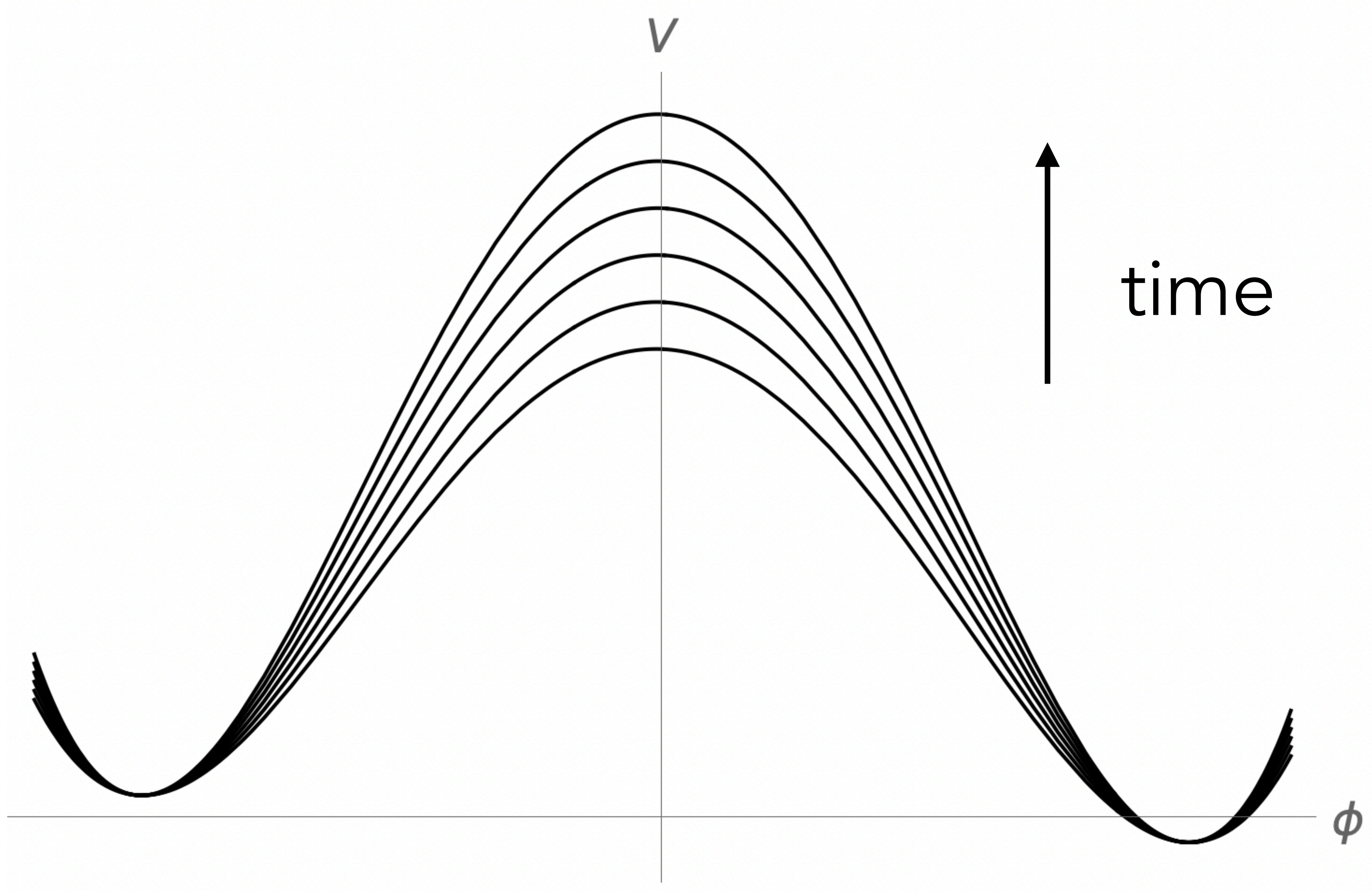
Relevant solutions: $R(\tau) = \alpha^{-1}\cosh\left[\alpha\, e^{i\gamma}(\tau - \tau_0) + i(\pi/2)\right]$ $\quad \tau_0 = \frac{\pi}{2}\alpha^{-1}\csc(\gamma)$

Nucleation point: $R(0) = \alpha^{-1}\cosh((\pi/2)\cot\gamma)$ real

Similar to previous, real-time <E> of the final wavepacket vanishes

Simplest (space)time-dependent modification:
μ=>μ(t,r)

other modifications quite nontrivial…



This can again arise from certain (rather artificial) weak couplings to slowly-evolving spectators.

EOM:

$$\boxed{\frac{d}{d\lambda}\left[\frac{2R}{\left(\frac{\partial q}{\partial N}\right)}\left(\sqrt{f_{dS_T}+N^{-2}\dot{R}^2}-\sqrt{f_{dS_F}+N^{-2}\dot{R}^2}-4\pi G_N \mu R\right)\right]=-8\pi G_N N R^2\frac{\partial \mu}{\partial t_{dS_F}}}$$

$q \equiv f_{dS_F}^{-1}\sqrt{f_{dS_F}N^2+\dot{R}^2}$

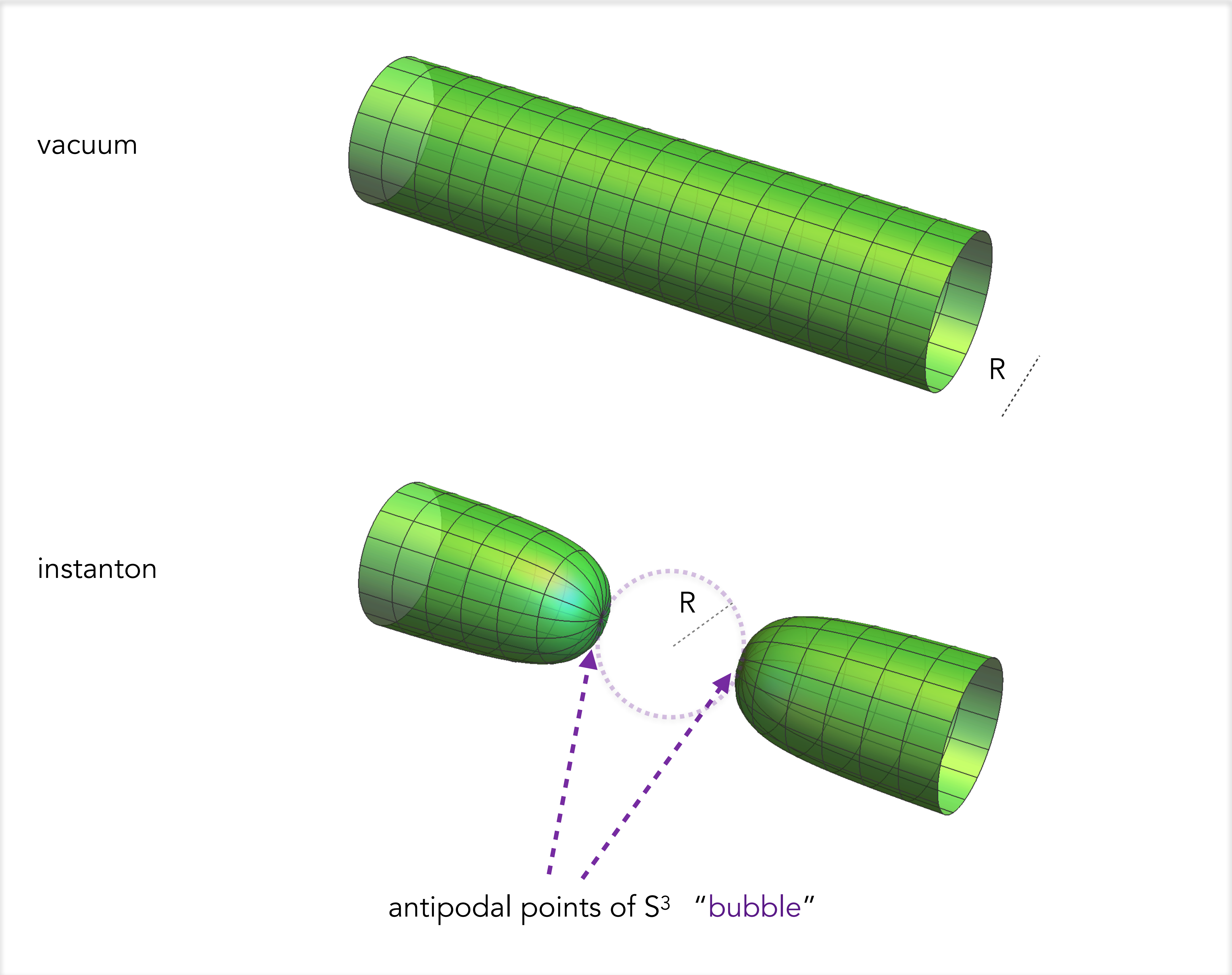
gauge fix N=1 at the end

Can follow same procedure: continue, solve dμ/dt=0 case, compute dμ/dt correction to S

An exact solution

Lorentzian KK cosmology: $$ds^2 = -dy^2 + y^2 d\phi^2 + dt^2 + dx^2 + x^2 d\psi^2$$

If the KK cicle is $t \sim t + 2\pi$, this is ordinary KK theory in a funny "Milne"-type coordinate system.



$$ds_5^2 = f dt^2 + f^{-1} dr^2 + r^2 d\Omega_3^2$$

$$f = 1 - (R/r)^2$$

$$r \geq R$$

Smooth at r=R if $t \sim t + 2\pi R$

"wall" or "core" at r=R

Identify t with KK circle x_5

Euclidean action:

$$S = -\frac{1}{8\pi G_5} \int d^4x \sqrt{h}(K - K_0)$$
$$= \pi^2 R^2 M_p^2$$

An exact solution

Lorentzian KK cosmology: $$ds^2 = -dy^2 + y^2 d\phi^2 + dt^2 + dx^2 + x^2 d\psi^2$$

If instead the KK circle is $\phi \sim \phi + 2\pi$,
then it is a KK cosmology with the circle shrinking/growing in time y

An exact solution

Lorentzian KK cosmology: $$ds^2 = -dy^2 + y^2 d\phi^2 + dt^2 + dx^2 + x^2 d\psi^2$$

If instead the KK circle is $\phi \sim \phi + 2\pi$,
then it is a KK cosmology with the circle shrinking/growing in time y

The singularity at y=0 is an annoyance which we can regulate by twisting the periodicity:

take $(t, \phi) \sim (t + 2\pi n \frac{\mu}{\sqrt{\mu - a^2}}, \phi + 2\pi n \frac{a}{\sqrt{\mu - a^2}})$ with $0 \leq a^2 \leq \mu$

The minimum circle radius in Milne patch is now $\mu/\sqrt{\mu - a^2}$. Also extend spacetime Milne->Mink (not completely innocuous)

Is there a bubble of nothing?

An exact solution

This “vacuum” cosmology is unstable. There is a candidate “decay product”:

$$ds^2 = \frac{r^2 + a^2\cosh^2\theta}{r^2 + a^2 - \mu}dr^2 + dt^2 - \frac{\mu}{r^2 + a^2\cosh^2\theta}(dt - a\sinh^2\theta d\phi)^2$$
$$+r^2\left(-\left[1 + \frac{a^2\cosh^2\theta}{r^2}\right]d\theta^2 + \sinh^2\theta\left[1 + \frac{a^2}{r^2}\right]d\phi^2 + \cosh^2\theta d\psi^2\right)$$

This is a real Lorentzian manifold given by the continuation $t \to it, \theta \to i\theta, \phi \to i\phi$ of the 5D Myers-Perry black hole with one angular momentum ($a = J/M, \mu = r_S$)

Geometry caps off smoothly at $r = r_H = \sqrt{\mu - a^2}$: a bubble of nothing

Asymptotics match the vacuum spacetime+periodicities ($\theta \to \tanh^{-1}(y/x),\ r \to \sqrt{x^2 - y^2}$)

The induced metric on the bubble wall is that of a spheroid which expands in time

The “instanton” (MP: $dt \to idt$) is quasi-Euclidean because MP is stationary rather than static,

$dtd\phi \to idtd\phi$

It can be joined smoothly to the nucleated bubble on a hypersurface of zero momentum

$S = \dfrac{\pi^2\mu^2}{G_5\sqrt{\mu - a^2}}$ recovers Witten’s result for a=0, diverges in the extremal limit where the minimum KK circle size diverges

Since the background is time dependent, there is no time translation symmetry and the instanton computes an amplitude rather than a rate.

Questions:

application to time-dependent Schwinger process? cosmological settings?

generalization to multiple collective coordinate QM, full QFT?

membrane approach to other cases with dynamical gravity?

other exact instantons from other MP/black ring continuations?