False vacuum decay from thin to thick walls

Lorenzo Ubaldi

Jožef Stefan Institut, Ljubljana

[arXiv:2202.04498]   [arXiv:2404.17632]

Ivanov, Matteini, Nemevšek, Shoji, LU

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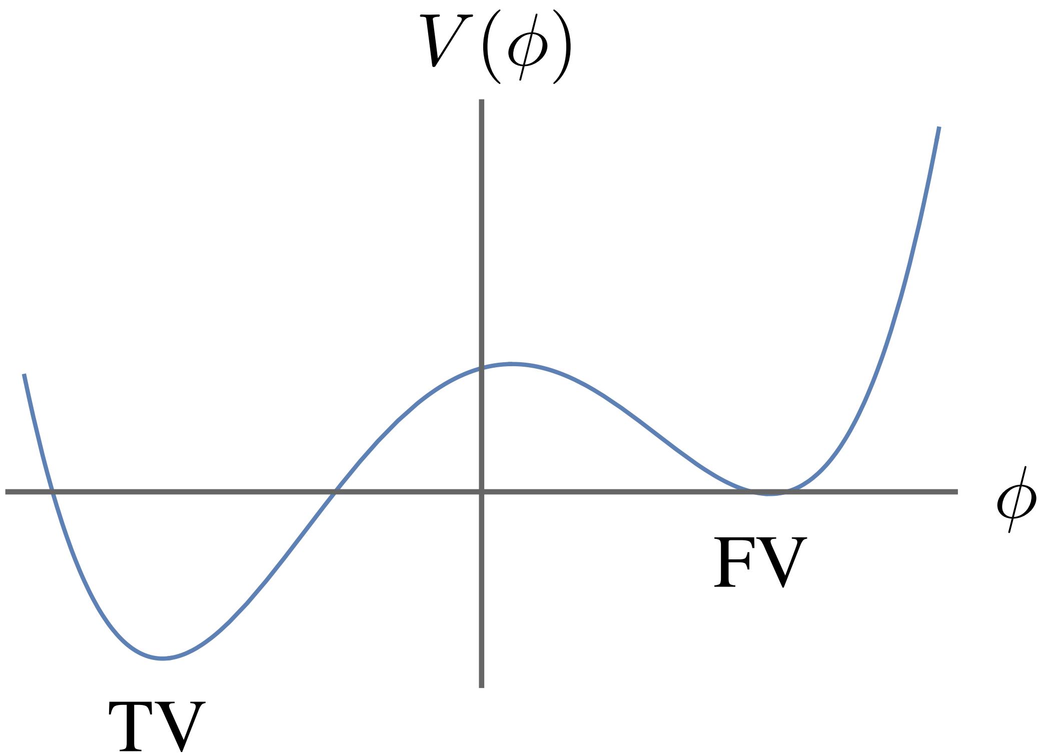
Why study vacuum decay?

- The electroweak vacuum is metastable.

- Beyond the Standard Model it is possible to have first order phase transitions in the early universe. They could lead to detectable gravitational waves' signals.

- It is an interesting and challenging problem to tackle from the theoretical point of view.

In this talk the focus is on computing the full one-loop decay rate in a theory with only one real scalar field in flat spacetime.



$$V(\phi) = \frac{\lambda}{8}(\phi^2 - v^2)^2 + \lambda v^3 \Delta(\phi - v)$$

$$S[\phi] = \int \mathrm{d}^D x \left(\frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right)$$

Euclidean spacetime

$$\frac{\Gamma}{\mathcal{V}} \approx A e^{-S[\phi_b]}$$

$$\frac{\Gamma}{\mathcal{V}} = \lim_{T\to\infty} \frac{1}{T\mathcal{V}} \frac{\mathrm{Im} \int \mathcal{D}\phi e^{-S[\phi_b]-\frac{1}{2}\phi S''[\phi_b]\phi}}{\int \mathcal{D}\phi e^{-S[\phi_{\mathrm{FV}}]-\frac{1}{2}\phi S''[\phi_{\mathrm{FV}}]\phi}}$$

$$= \left(\frac{S_R}{2\pi\hbar}\right)^{\frac{D}{2}} \left| \frac{\det'\left(-\partial^2 + \left.\frac{d^2V}{d\phi^2}\right|_{\phi_b}\right)}{\det\left(-\partial^2 + \left.\frac{d^2V}{d\phi^2}\right|_{\phi_{\mathrm{FV}}}\right)} \right|^{-\frac{1}{2}} e^{-\frac{S_R}{\hbar} - S^{(1)}_{\mathrm{ct}} + S^{(1)}_{\mathrm{ctFV}}} \left(1 + \mathcal{O}(\hbar)\right)$$

Coleman, Callan 1977

Classical bounce action

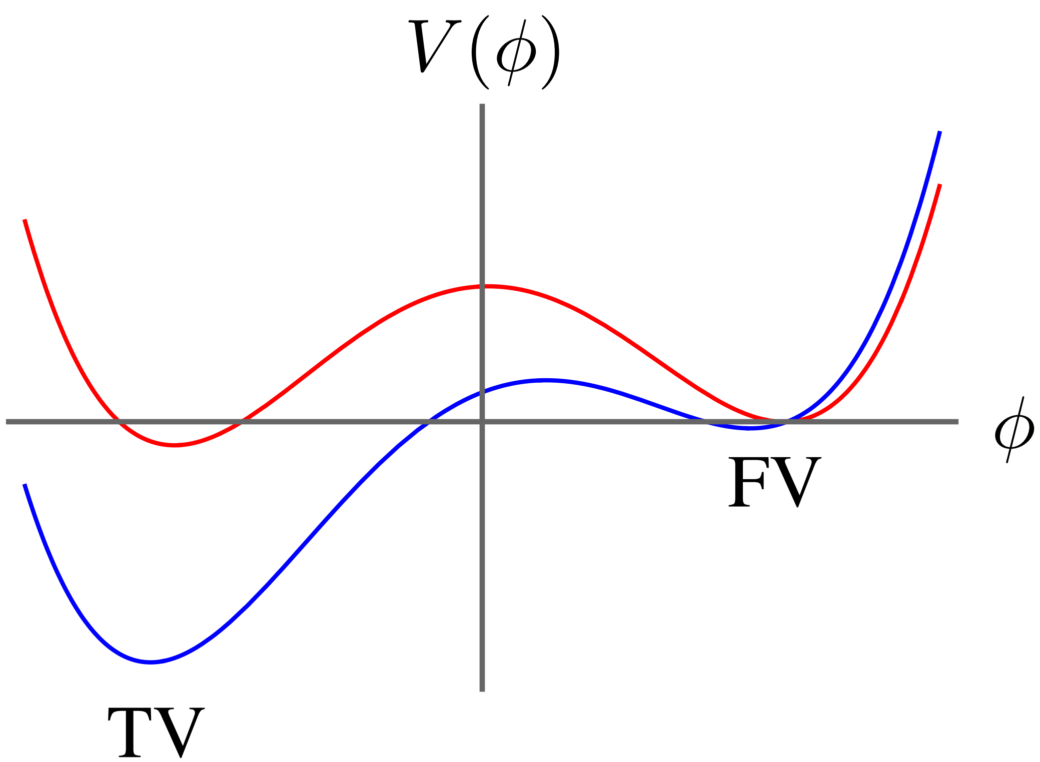
ϕ_b is the bounce: a field configuration that extremizes the action, $S'[\phi_b] = 0$.

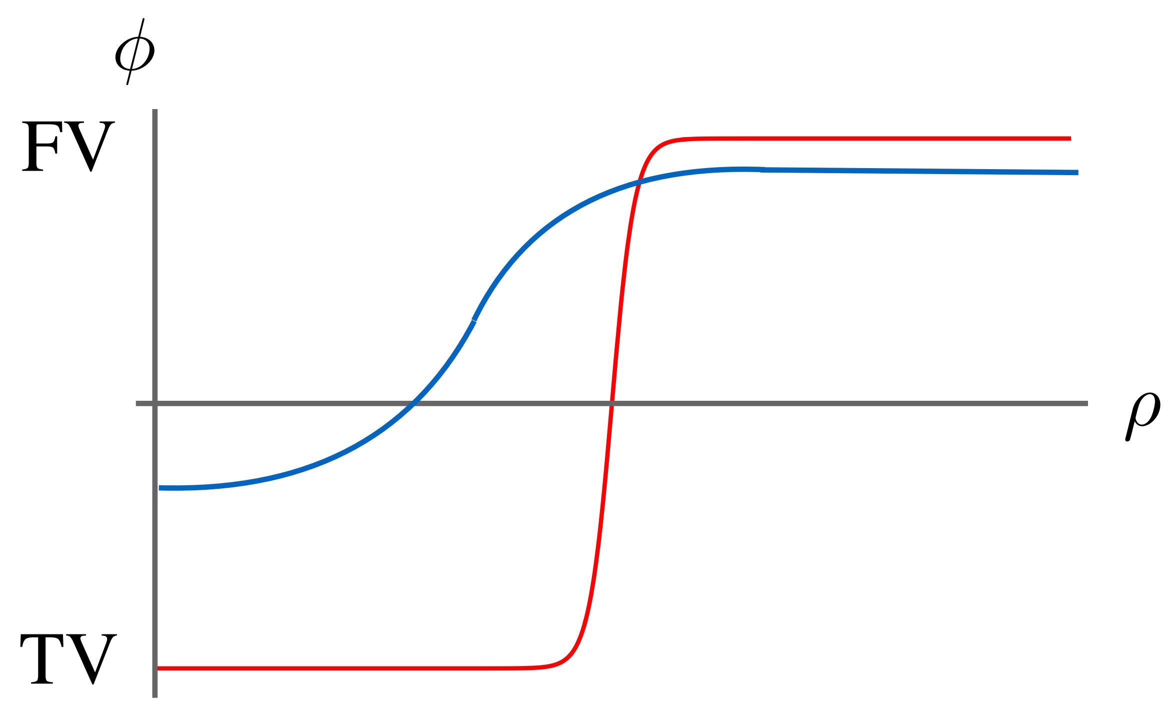
The bounce is O(4) spherically symmetric.

Coleman, Glaser, Martin 1977

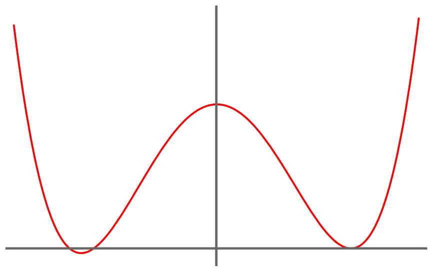
$$S[\phi] = \Omega \int_0^\infty \mathrm{d}\rho \; \rho^{D-1} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right)$$

$$\ddot{\phi} + \frac{D-1}{\rho}\dot{\phi} = \frac{\mathrm{d}V}{\mathrm{d}\phi} \qquad \phi(\rho \to \infty) = \phi_{\mathrm{FV}} \qquad \dot{\phi}(\rho = 0) = 0$$





Thin wall approximation



$$V(\phi) = \frac{\lambda}{8}(\phi^2 - v^2)^2 + \lambda v^3 \Delta(\phi - v) \qquad \Delta \ll 1$$

$$\sqrt{\lambda v^2}\rho = z + r \qquad r = \frac{1}{\Delta}(r_0 + r_1 \Delta + r_2 \Delta^2 + \cdots)$$

Munster, Rotsch 1999

$$\varphi = \phi/v \qquad \varphi(z) = \varphi_0(z) + \varphi_1(z)\Delta + \varphi_2(z)\Delta^2 + \cdots$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2} + \frac{D-1}{z+r}\frac{\mathrm{d}\varphi}{\mathrm{d}z} - \frac{1}{2}\varphi(\varphi^2 - 1) - \Delta = 0$$

Solve the bounce equation order by order in Δ

We found analytic solutions up to Δ^2

Ivanov, Matteini, Nemevšek, LU 2022

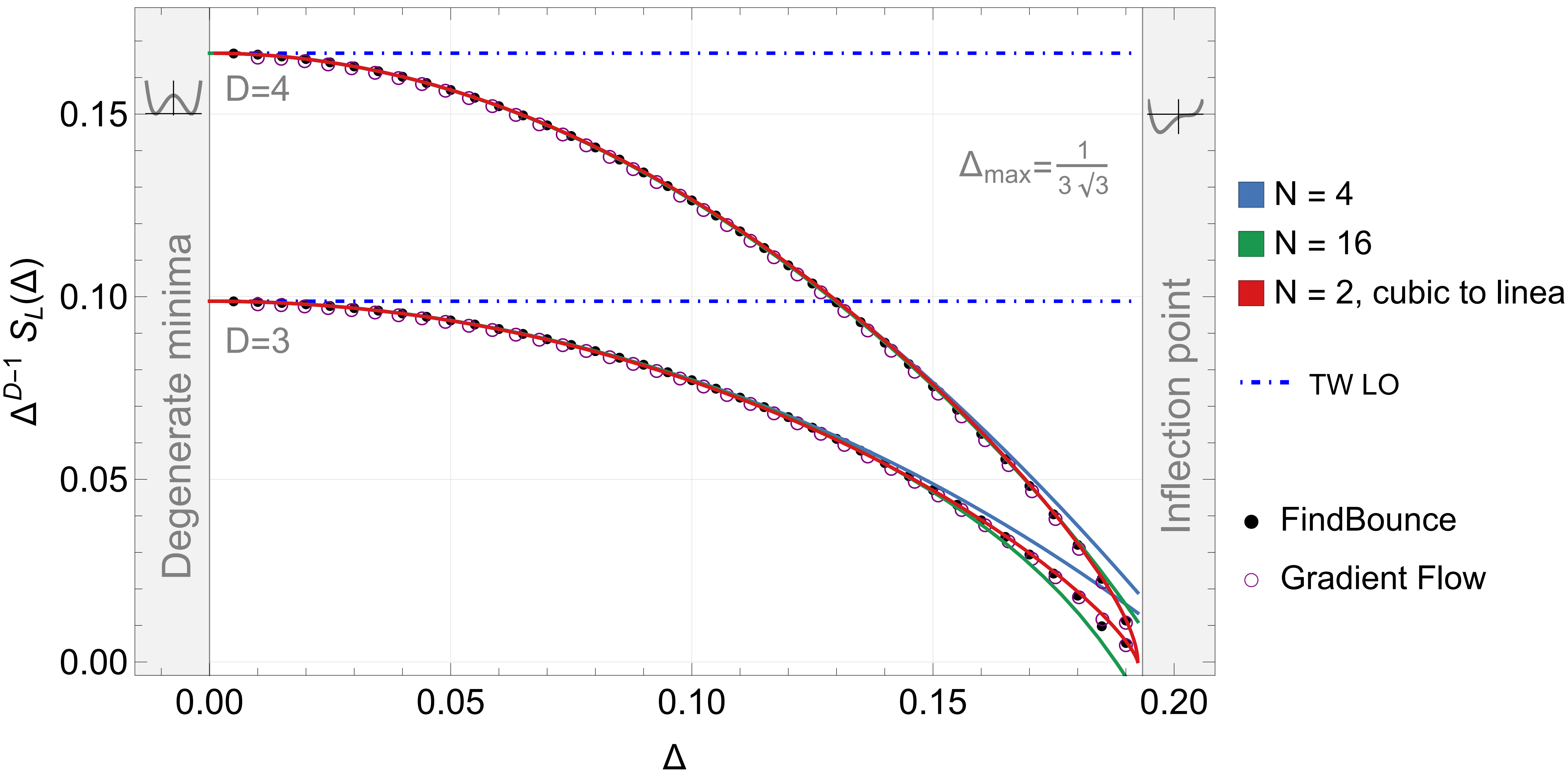
$$
\begin{aligned}
S[\phi_b] &= \Omega \frac{v^{4-D}}{\lambda^{D/2-1}} S_L(\Delta) \\
&= \Omega \frac{v^{4-D}}{\lambda^{D/2-1}} \int_{-r}^{\infty} \mathrm{d}z \; (z+r)^{D-1} \left(\frac{1}{2} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}z} \right)^2 + \tilde{V}(\varphi) - \tilde{V}(\varphi_{\mathrm{FV}}) \right)
\end{aligned}
$$

$$
S_L^{(0)} = \frac{1}{\Delta^{D-1}} \left(\frac{D-1}{3} \right)^{D-1} \frac{2}{3D}
$$

$$
S_L^{(N)}(\Delta) = S_L^{(0)} \left(1 + \sum_{n=1}^{N/2} \Delta^{2n} s_{2n}^L \right)
$$

$$
\begin{aligned}
s_2^L &= \frac{-8D^2 + (25 - 3\pi^2)D + 1}{2(D-1)}, \\
s_4^L &= \frac{1}{40(D-1)^3} \Big(320D^5 + 80D^4 \left(3\pi^2 - 49\right) - 3D^3 \left(550\pi^2 + 3\pi^4 - 6185\right) \\
&\qquad + 5D^2 \left(426\pi^2 + 45\pi^4 - 648\zeta(3) - 7843\right) \\
&\qquad + D \left(3240\zeta(3) + 30635 + 360\pi^2 - 414\pi^4\right) + 105 \Big)
\end{aligned}
$$

From thin to thick



Cubic parametrization

$$V(\phi) = \frac{\lambda}{8}(\phi^2 - v^2)^2 + \lambda v^3 \Delta(\phi - v)$$

shift the field

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \eta\phi^3 + \frac{\lambda}{8}\phi^4$$

introduce dimensionless variables

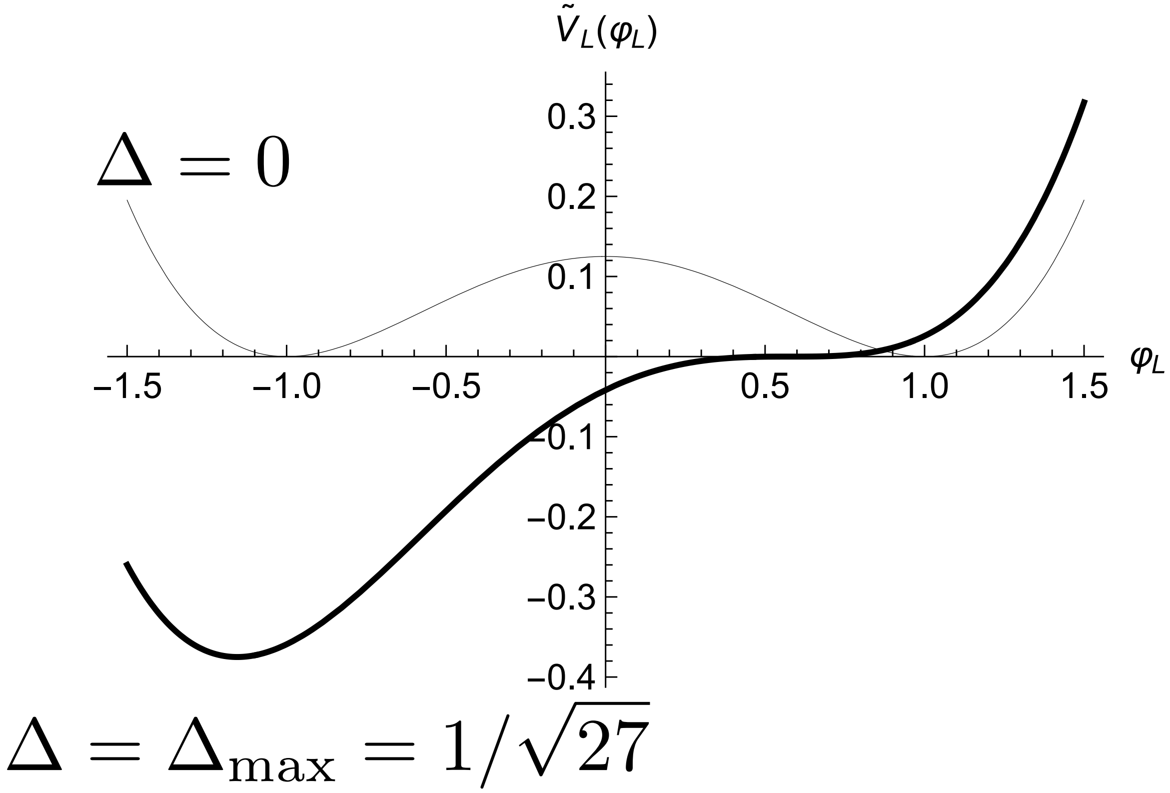
$$\varphi \equiv \frac{2\eta}{m^2}\phi\,, \qquad \tilde{\rho} \equiv m\rho\,, \qquad \varepsilon_\alpha \equiv 1 - \lambda\frac{m^2}{4\eta^2}$$

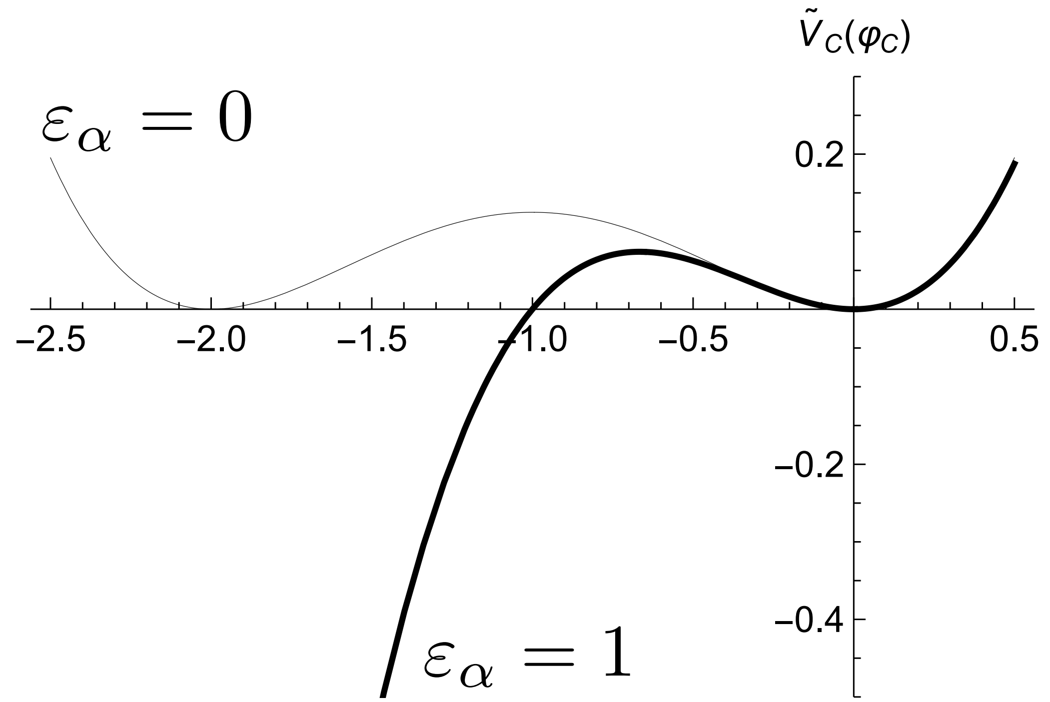
$$S = \Omega\frac{m^{6-D}}{4\eta^2}S_C(\varepsilon_\alpha) = \Omega\frac{m^{6-D}}{4\eta^2}\int_0^\infty \mathrm{d}\tilde{\rho}\ \tilde{\rho}^{D-1}\left(\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{\rho}}\right)^2 + \tilde{V}(\varphi)\right)$$

$$\tilde{V}(\varphi) = \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi^3 + \frac{1-\varepsilon_\alpha}{8}\varphi^4$$

There is an exact analytic map

$$\{\lambda, v, \Delta\} \;\longleftrightarrow\; \{m, \eta, \varepsilon_\alpha\}$$





$$\tilde{V}(\varphi) = \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi^3 + \frac{1-\varepsilon_\alpha}{8}\varphi^4$$

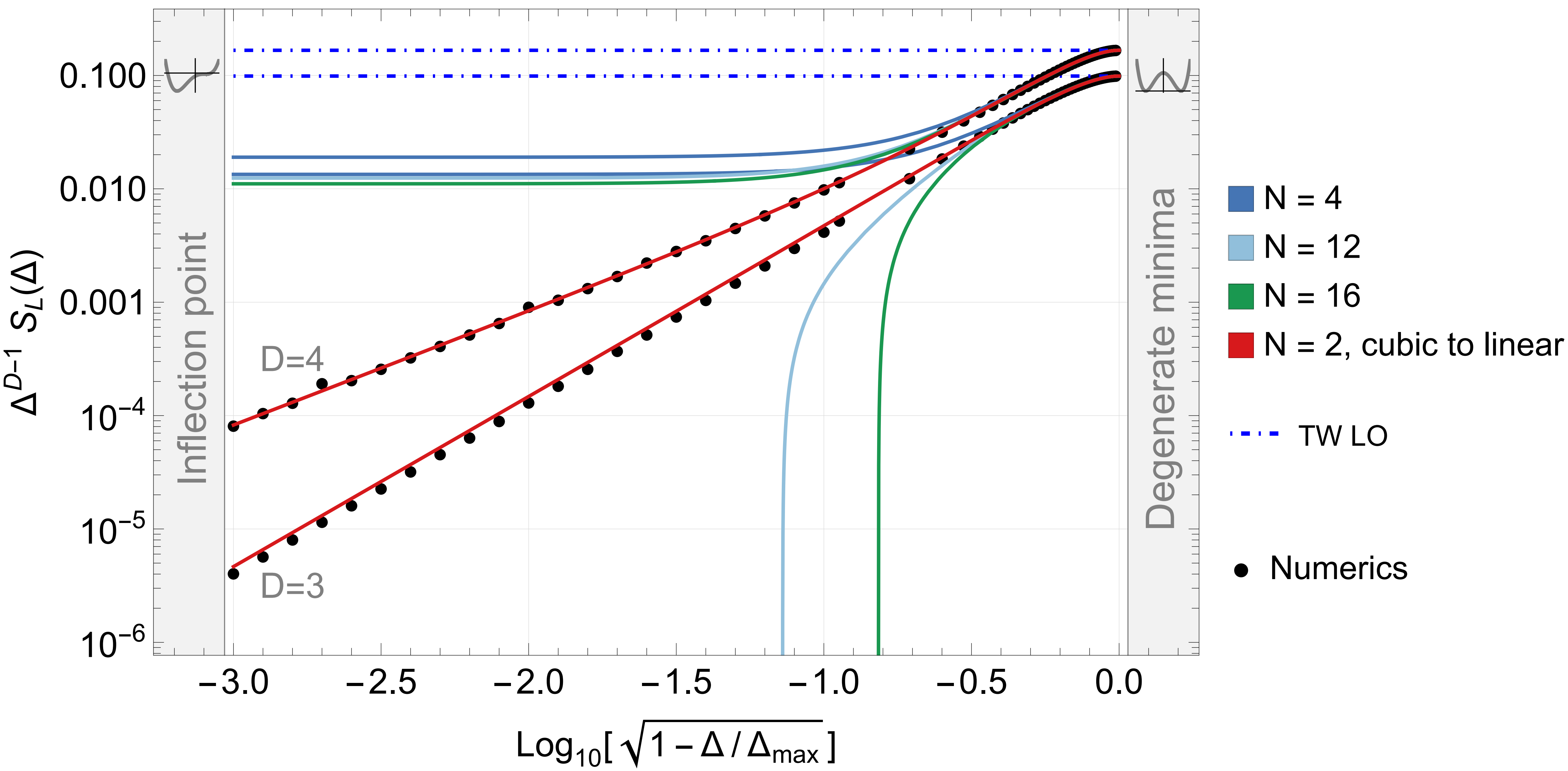
$$S_C^{(N)}(\varepsilon_\alpha) = S_C^{(0)}\left(1 + \sum_{n=1}^{N} \varepsilon_\alpha^n s_n^C\right)$$

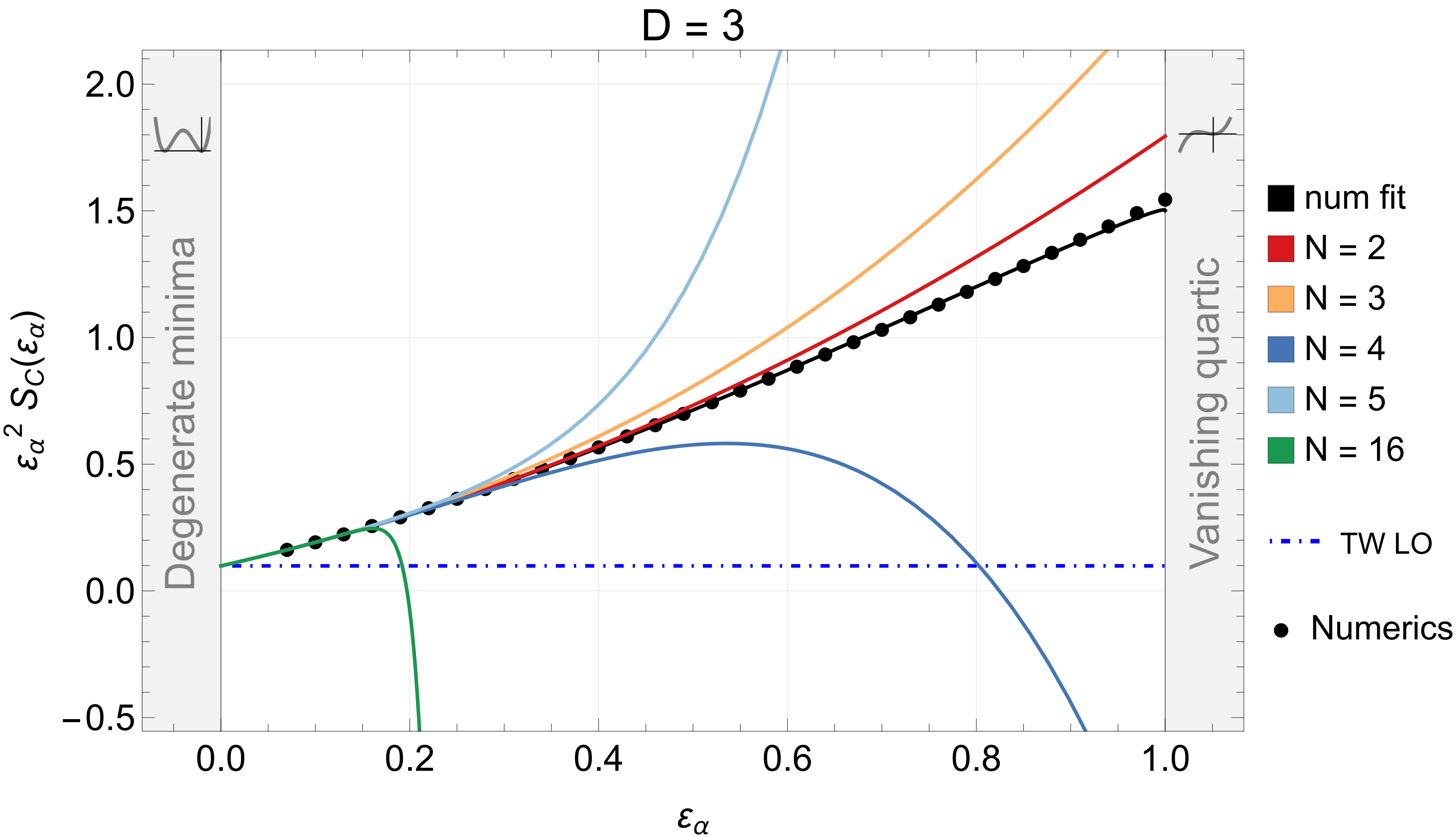
$$S_C^{(0)} = \frac{1}{\varepsilon_\alpha^{D-1}}\left(\frac{D-1}{3}\right)^{D-1}\frac{2}{3D}$$

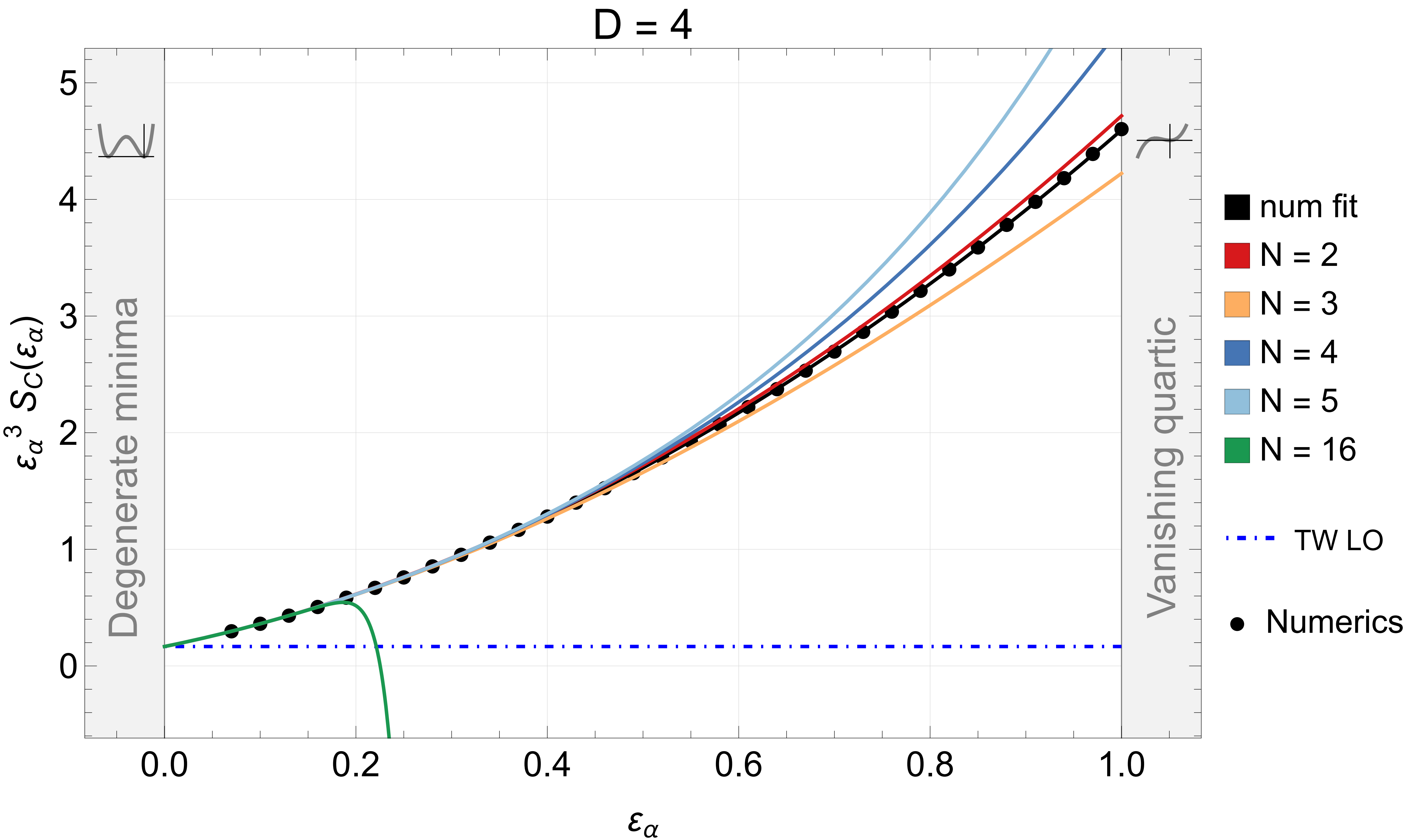
$$S = \Omega\frac{v^{4-D}}{\lambda^{D/2-1}}S_L(\Delta) = \Omega\frac{m^{6-D}}{4\eta^2}S_C(\varepsilon_\alpha)$$

$$m^2 = \lambda v^2 f_{m^2}(\Delta) \qquad \eta = \frac{1}{2}\lambda v f_\eta(\Delta)$$

$$S_L(\Delta) = \frac{\lambda^{D/2-1}}{v^{4-D}}\frac{m^{6-D}}{4\eta^2}S_C(\varepsilon_\alpha) \simeq \frac{f_{m^2}^{3-D/2}(\Delta)}{f_\eta^2(\Delta)}S_C^{(2)}(\varepsilon_\alpha(\Delta))$$

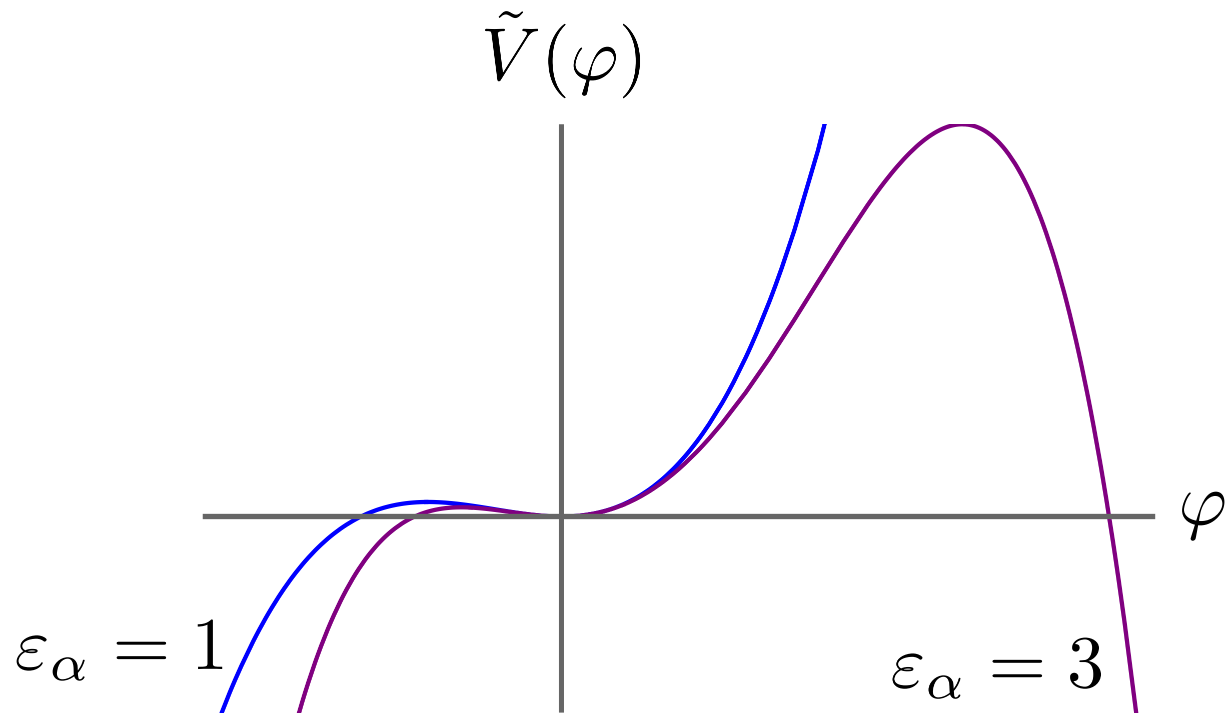

0.100
0.010
0.001
10^{-4}
10^{-5}
10^{-6}
−3.0
−2.5
−2.0
−1.5
−1.0
−0.5
0.0
$\Delta^{D-1}\, S_L(\Delta)$
$\mathrm{Log}_{10}[\sqrt{1-\Delta/\Delta_{\max}}\,]$
Inflection point
Degenerate minima
D=4
D=3
N = 4
N = 12
N = 16
N = 2, cubic to linear
TW LO
Numerics


D = 3
2.0
1.5
1.0
0.5
0.0
−0.5
0.0
0.2
0.4
0.6
0.8
1.0
$\varepsilon_\alpha{}^2\, S_C(\varepsilon_\alpha)$
ε_α
Degenerate minima
Vanishing quartic
num fit
N = 2
N = 3
N = 4
N = 5
N = 16
TW LO
Numerics

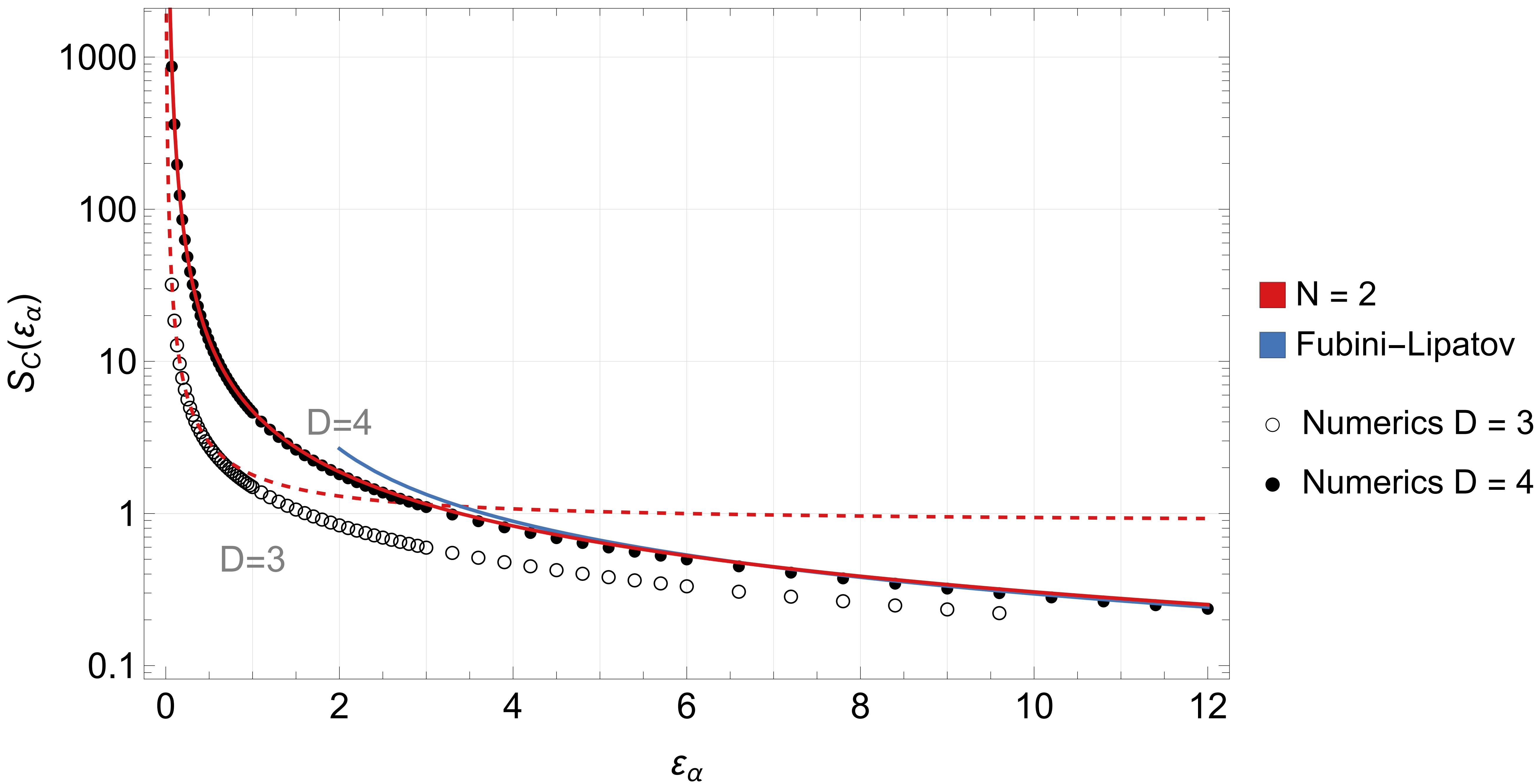


D = 4
$\epsilon_\alpha^{\ 3}\, S_C(\epsilon_\alpha)$
ϵ_α
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
Degenerate minima
Vanishing quartic
num fit
N = 2
N = 3
N = 4
N = 5
N = 16
TW LO
Numerics

Unbounded potential



$$\tilde{V}(\varphi) = \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi^3 + \frac{1-\varepsilon_\alpha}{8}\varphi^4$$



1000
100
10
1
0.1
$S_C(\varepsilon_\alpha)$
D=4
D=3
0
2
4
6
8
10
12
ε_α
N = 2
Fubini–Lipatov
Numerics D = 3
Numerics D = 4

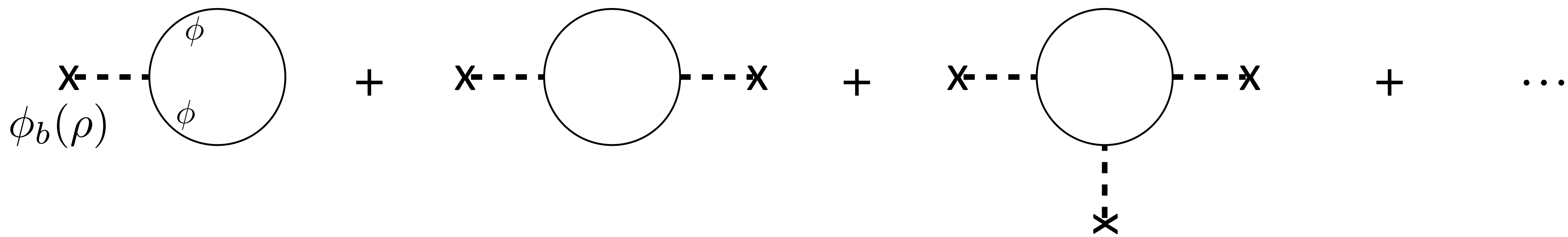
$$S_C^{\mathrm{FL}}(\varepsilon_\alpha) = \frac{8}{3}\frac{1}{\varepsilon_\alpha - 1} \approx \frac{8}{3\varepsilon_\alpha}\,, \qquad \text{for } \varepsilon_\alpha \gg 1$$

Fubini-Lipatov

$$S_C^{(2)}(\varepsilon_\alpha) = \frac{1}{6\varepsilon_\alpha^3}\left[1 + 10\varepsilon_\alpha + (37 - 2\pi^2)\varepsilon_\alpha^2\right] \xrightarrow[\varepsilon_\alpha \gg 1]{} \frac{1}{6\varepsilon_\alpha}(37 - 2\pi^2)$$

Thin-wall expansion

Fluctuations



$$\frac{\Gamma}{\mathcal{V}} = \lim_{T\to\infty} \frac{1}{T\mathcal{V}} \frac{\mathrm{Im} \int \mathcal{D}\phi e^{-S[\phi_b] - \frac{1}{2}\phi S''[\phi_b]\phi}}{\int \mathcal{D}\phi e^{-S[\phi_{\mathrm{FV}}] - \frac{1}{2}\phi S''[\phi_{\mathrm{FV}}]\phi}}$$

$$= \left(\frac{S_R}{2\pi\hbar}\right)^{\frac{D}{2}} \left| \frac{\det'\left(-\partial^2 + \left.\frac{d^2V}{d\phi^2}\right|_{\phi_b}\right)}{\det\left(-\partial^2 + \left.\frac{d^2V}{d\phi^2}\right|_{\phi_{\mathrm{FV}}}\right)} \right|^{-\frac{1}{2}} e^{-\frac{S_R}{\hbar} - S_{\mathrm{ct}}^{(1)} + S_{\mathrm{ctFV}}^{(1)}} \left(1 + \mathcal{O}(\hbar)\right)$$

Coleman, Callan 1977

$D = 4$

$$S_R + S_{\mathrm{ct}}^{(1)} - S_{\mathrm{ctFV}}^{(1)} = S\left(1 - \frac{9\lambda_0}{(4\pi)^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{\mu_0}\right)\right)$$

dimensional regularization

Functional determinant

$$\det\left(-\partial^2 + \left.\frac{d^2V}{d\phi^2}\right|_{\phi_b}\right) \equiv \det\mathcal{O} = \prod_{l=0}^{\infty} \det\mathcal{O}_l$$

$$\mathcal{O}_l = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{l\,(l+D-2)}{\rho^2} + \frac{d^2V}{d\phi^2}$$

trade l for $\nu = l + \frac{D}{2} - 1$

Gelfand-Yaglom theorem

$$\ln\frac{\det\mathcal{O}}{\det\mathcal{O}_{\rm FV}} = \sum_{\nu=D/2-1}^{\infty} d_\nu \ln R_\nu(\infty)$$

$$R_\nu(\infty) = \frac{\psi_\nu(\infty)}{\psi_{\nu\rm FV}(\infty)}$$

$$\mathcal{O}_\nu\psi_\nu(\rho) = 0$$
$$\mathcal{O}_{\nu\rm FV}\psi_{\nu\rm FV}(\rho) = 0$$

with boundary conditions at $\rho = 0$

$$d_\nu = \frac{2\nu\left(\nu + \frac{D}{2} - 2\right)!}{(D-2)!\left(\nu - \frac{D}{2} + 1\right)!} \simeq \frac{2}{(D-2)!}\nu^{D-2}$$

$$R_{l=0}(\infty) < 0\,, \quad R_{l=1}(\infty) = 0$$

There is ONE negative eigenvalue, D vanishing eigenvalues.
Gelfand-Yaglom requires a zero removal procedure.

Ivanov, Matteini, Nemevšek, Shoji, LU 2022 - 2024

$$\sum_\nu d_\nu \ln R_\nu \xrightarrow[\nu\to\infty]{} \sum_\nu \frac{2}{\Gamma(D-1)} \nu^{D-2} \left(\frac{c_{\nu 1}}{\nu} + \frac{c_{\nu 3}}{\nu^3} + O(\nu^{-5})\right)$$

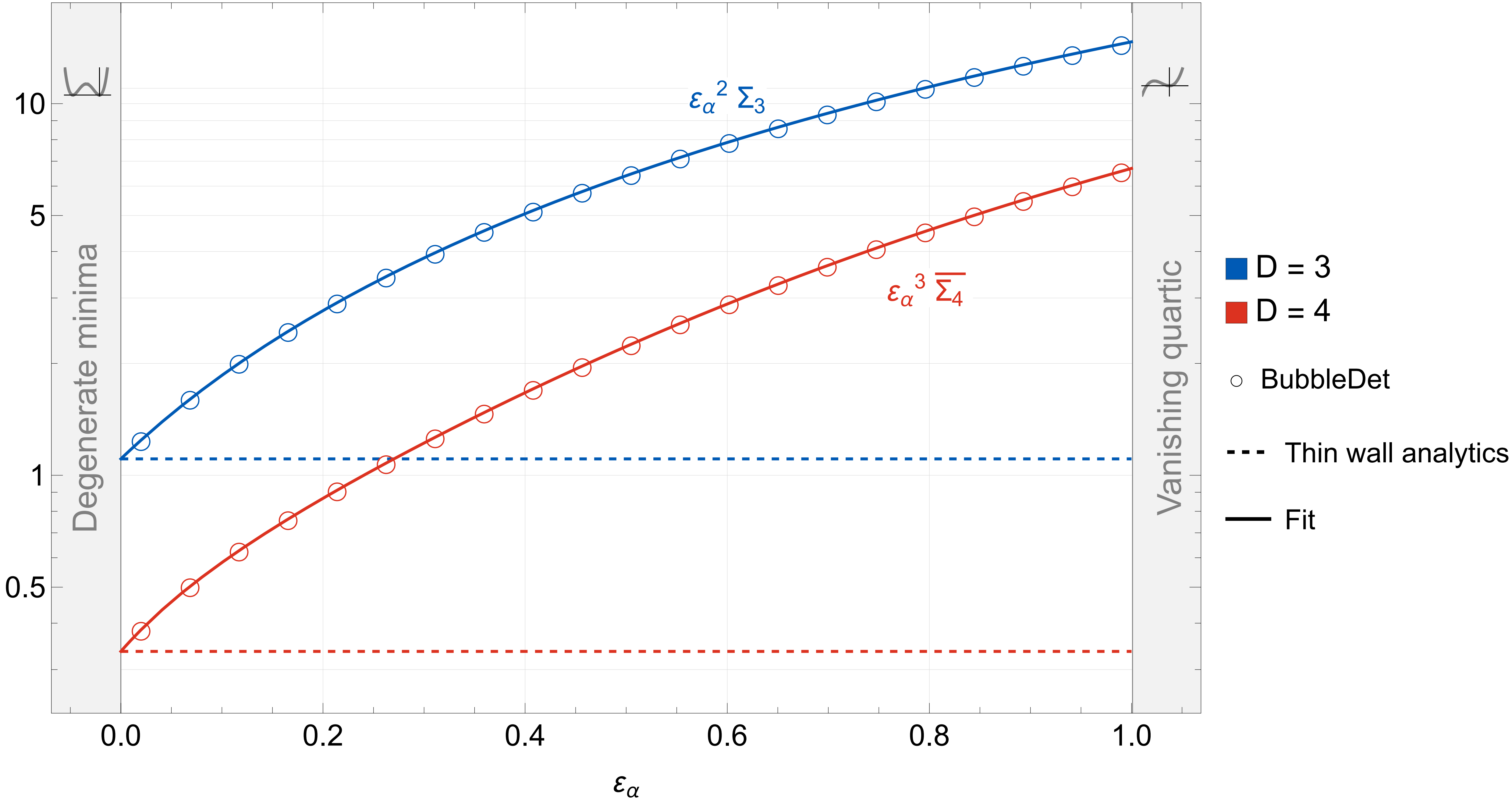
The sum is UV divergent, must be regularized.

Dunne, Min 2005
Dunne, Kirsten 2006

$$\Sigma_4 = \sum_\nu \nu^2 \left(\ln R_\nu - \frac{1}{2\nu} I_1 + \frac{1}{8\nu^3} I_2\right) - \frac{1}{8}\tilde{I}_2 \qquad D = 4$$

$$I_m = \int_0^\infty \mathrm{d}\rho\, \rho^{2m-1} \left(V^{(2)m} - V_{\mathrm{FV}}^{(2)m}\right)$$

$$\tilde{I}_2 = \int_0^\infty \mathrm{d}\rho\, \rho^3 \left(V^{(2)2} - V_{\mathrm{FV}}^{(2)2}\right) \left[\frac{1}{\epsilon} + 1 + \gamma_E + \ln\left(\frac{\mu\rho}{2}\right)\right]$$



$$\Sigma_3(\varepsilon_\alpha) = \frac{20 + 9\ln 3}{27}\frac{1}{\varepsilon_\alpha^2}\left(1 + 6.0\varepsilon_\alpha + 8.0\varepsilon_\alpha^2 - 1.8\varepsilon_\alpha^3\right)\ ,$$

$$\Sigma_4(\varepsilon_\alpha) = \frac{27 - 2\pi\sqrt{3}}{48}\frac{1}{\varepsilon_\alpha^3}\left(1 + 7.2\varepsilon_\alpha - 0.6\varepsilon_\alpha^2 + 24\varepsilon_\alpha^3 - 15\varepsilon_\alpha^4 + 3.5\varepsilon_\alpha^5\right)$$

Summary

$$V = \frac{1}{2}m^2\phi^2 + \eta\phi^3 + \frac{\lambda}{8}\phi^4\,, \qquad \varepsilon_\alpha \equiv 1 - \lambda\frac{m^2}{4\eta^2}$$

$$\frac{\Gamma}{\mathcal{V}} = \left(\frac{S_D}{2\pi}\right)^{D/2} m^D e^{-S_D - \frac{1}{2}\Sigma_D}$$

$$S_3 = \frac{32\pi}{81}\frac{m^3}{4\eta^2}\frac{1}{\varepsilon_\alpha^2}\left[1 + \frac{17}{2}\varepsilon_\alpha + \left(\frac{247}{8} - \frac{9\pi^2}{4}\right)\varepsilon_\alpha^2\right]\,,$$

$$S_4 = \frac{\pi^2}{3}\frac{m^2}{4\eta^2}\frac{1}{\varepsilon_\alpha^3}\left[1 + 10\varepsilon_\alpha + \left(37 - 2\pi^2\right)\varepsilon_\alpha^2\right]\,,$$

$$\Sigma_3 = \frac{20 + 9\ln 3}{27}\frac{1}{\varepsilon_\alpha^2}\left(1 + 6.0\varepsilon_\alpha + 8.0\varepsilon_\alpha^2 - 1.8\varepsilon_\alpha^3\right)\,,$$

$$\Sigma_4 = \frac{27 - 2\pi\sqrt{3}}{48}\frac{1}{\varepsilon_\alpha^3}\left(1 + 7.2\varepsilon_\alpha - 0.6\varepsilon_\alpha^2 + 24\varepsilon_\alpha^3 - 15\varepsilon_\alpha^4 + 3.5\varepsilon_\alpha^5\right)$$