# Tunneling amplitudes in QFT

## a Lorentzian worldline perspective


Karthik Rajeev
with
Anton Ilderton & William Lindved

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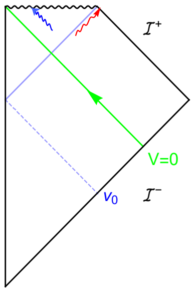

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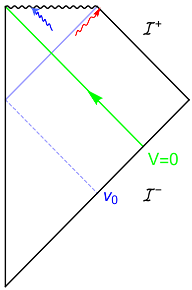


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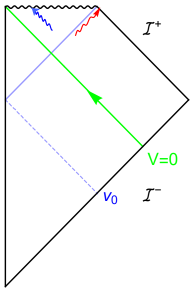


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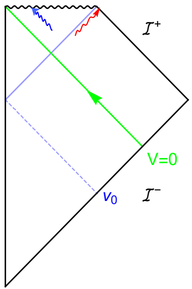


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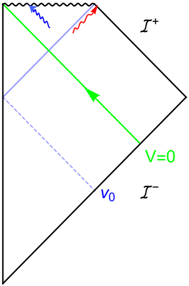


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- Analytic in the upper-half of the complex-$v$ plane.
- Implies $\tilde{\mathcal{M}}(v \to \mathcal{E}')$ cannot vanish for all of $v > v_0$: Tunneling!

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- Explore their similarities deeper using the worldline path integral formalism.
- **From here on:** scalar QED for Schwinger effect and massless scalar for Hawking radiation.

# Quick intro

**Schwinger effect**

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- Number of pairs $\sim e^{-\frac{(p_\perp^2+m^2)\pi}{|eE|}}$ $\rightarrow$ Non-perturbative.

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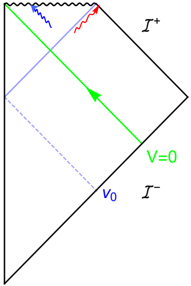

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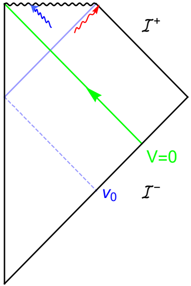


- $|\text{Pair creation amplitude}|^2 \sim e^{-8\pi GM\mathcal{E}} \rightarrow$ Thermal distribution.

# Classical physics: worldlines in constant electric field

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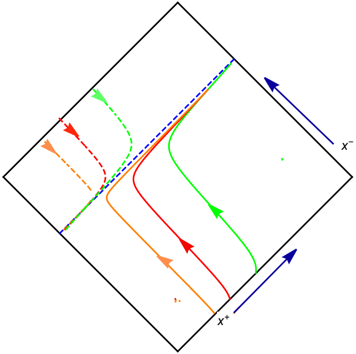
Horizon? [...,Ilderton & Lindved (2023)]



## Schwinger effect

$$\varphi_p(x) = e^{-ip_+x^+ - ip_\perp x^\perp} e^{-\left(\frac{1}{2} - i\frac{m^2+p_\perp^2}{2eE}\right)\log\left(1 - \frac{x^-}{x_h^-} + i0^+\right)} \quad (1)$$

- Positive-energy w.r.t $x^-$.

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- Describes 'tunneling'.

# Semiclassical physics: Tunneling wavefunction

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- Tunneling exponential

$$\left|\frac{\text{Pre-exp factor in forbidden regn.}}{\text{Pre-exp factor in allowed regn.}}\right|^2 = \left|\frac{\mathcal{M}_{0\to 2}}{\mathcal{M}_{1\to 1}}\right|^2 = e^{-\frac{\pi(p_\perp^2+m^2)}{|eE|}} \quad (2)$$

# Lorentzian worldline PI approach to Schwinger effect [1]

- Tunneling mode-function can be written as

$$\varphi_p^+(x) = \int_{-\infty}^{\infty} dT \int^{z(\tau_2)=x} \mathcal{D}[z] e^{iS_{\mathrm{WL}}[z,T]} ,$$

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- Integral over $T \Rightarrow$ on-shell condition.
- QM analogue: fixed-energy wavefunction from PI, introduce a Lagrange multiplier $T$.

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- $\int dT \,(\ldots) \leftrightarrow$ sum over WLs parametrised by $T$.

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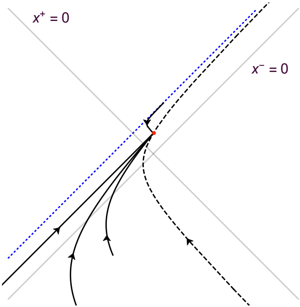


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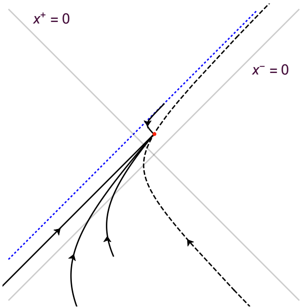


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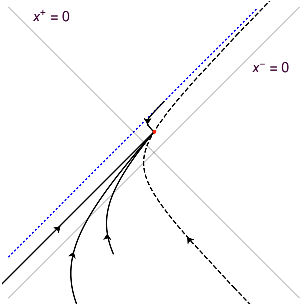


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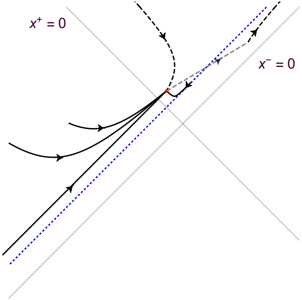


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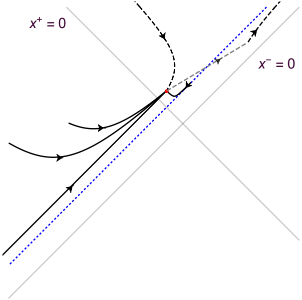


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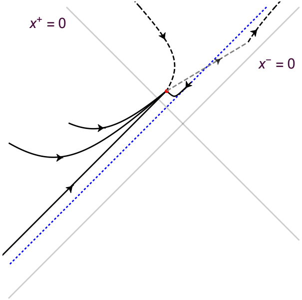


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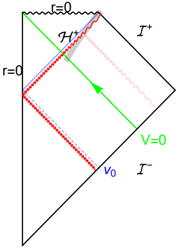
# Hawking radiation: standard approach

# Out-modes

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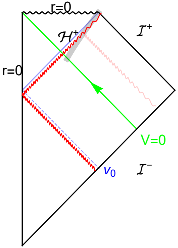


- Near $\mathcal{I}^+$, outmodes take the form $\varphi^{(\text{out})} \sim \frac{1}{r} e^{-i\mathcal{E}u}$
- Near $v \sim v_0$ in $\mathcal{I}^-$, we get

$$\varphi^{(\text{out})} \sim \frac{1}{r} e^{-i\mathcal{E}v + 4GMi \log[(v_0 - v)/|v_0|]} \Theta[v_0 - v] \tag{5}$$

$$= \sum_{\mathcal{E}'} \alpha_{\mathcal{E}\mathcal{E}'} e^{-i\mathcal{E}' v} + \sum_{\mathcal{E}'} \beta_{\mathcal{E}\mathcal{E}'} e^{i\mathcal{E}' v} \tag{6}$$

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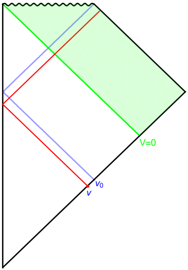
- The Feynman-modes encode amplitudes

$$\bar{\varphi}^{(F)}(x;p) \approx \begin{cases} \int_{p'} \mathcal{M}_{1\to1}(p' \to p) e^{ip'\cdot x} & ; \quad x \overset{\to}{\in} \text{Asym. past} \\ e^{ip\cdot x} + \int_{p'} \mathcal{M}_{0\to2}(p,p') e^{-ip'\cdot x} & ; \quad x \overset{\to}{\in} \text{Asym. future} \end{cases} \tag{7}$$

# 2. Hawking radiation and tunneling

[Hartle & Hawking (1976), Damour & Ruffini (1976), Srinivasan & Padmanabhan (1999), Parikh & Wilczek (2000),... ]

# Classical physics




$$\mathcal{E}' - \mathcal{E} \equiv \Delta\mathcal{E} = - \left. \frac{2G\,\tilde{p}\cdot p}{r} \right|_{v\to 0^-} = \frac{4GM\mathcal{E}}{\mathrm{v}} \quad (8)$$

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$$\mathcal{S}_{\mathrm{WL}}(\mathcal{E}', \mathrm{v}) = \mathcal{E}'\mathrm{v} - 4GM\mathcal{E}' \log\left(\frac{\mathrm{v}_0 - \mathrm{v} - i0^+}{r_0}\right) \tag{10}$$

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- Imaginary part of $\mathcal{S}_{\mathrm{WL}}$ for $\mathrm{v} > \mathrm{v}_0$ leads to $\mathcal{P}_{\mathcal{E}'} = e^{-8\pi GM\mathcal{E}'}$

# Path integral formalism for Hawking radiation

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- **This work:** Systematic treatment of the dynamical background in *Lorentizian* WL approach.

# Lorentzian worldline PI approach for Hawking radiation

- Feynman mode for Hawking radiation

$$\varphi_{\mathcal{E}'}(x) \stackrel{?}{=} \int_{-\infty}^{\infty} dT \int^{z(\lambda_2)=x} \mathcal{D}[z] e^{iS_{\mathrm{WL}}[z,T]} \,,$$

$$S_{\mathrm{WL}} = -\int_{\lambda_1}^{\lambda_2} \frac{1}{4} g_{\mu\nu}(z) \dot{z}^\mu \dot{z}^\nu d\lambda + \underbrace{S_{\mathrm{bdy.}}}_{\text{Boundary term}} \,,$$

$$T = \lambda_2 - \lambda_1 \,.$$

# Boundary conditions

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- Boundary term turns out to be

$$S_{\text{bdy.}} = -\mathcal{E}' \left[ V - 2R - 4GM \log \left( \frac{R}{2GM} - 1 \right) \right] \equiv -\mathcal{E}' U$$

# Geometric optics approx = semiclassical approx.

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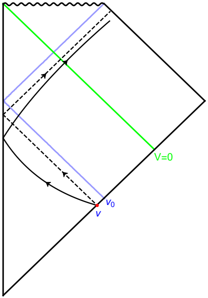
- Consistent with Hawking's ray-tracing argument.

# Case 1: Classically allowed ($v < v_0$)

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$$\varphi_{\mathcal{E}'}(x) \sim \frac{1}{r} e^{-i\mathcal{E}' v + i4GM\mathcal{E}' \log(\frac{v_0 - v}{2GM})}$$

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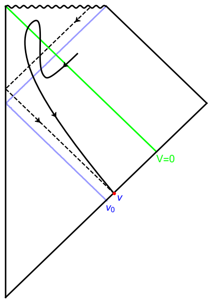

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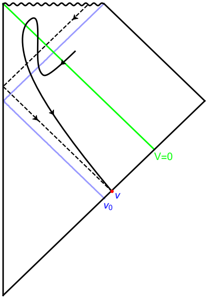


- The Feynman mode takes the form

$$\varphi_{\mathcal{E}'}(x) \sim e^{-4\pi GM\mathcal{E}'} \frac{1}{r} e^{-i\mathcal{E}' v + i4GM\mathcal{E}' \log(\frac{v-v_0}{2GM})}$$

# Amplitudes from Feynman-modes

- Recall

$$\bar{\varphi}^{(F)}(x;p) \approx \begin{cases} \int_{p'} \mathcal{M}_{1\to 1}(p' \to p) e^{ip'\cdot x} & ; \quad x \overset{\to}{\in} \text{Asym. past} \\ e^{ip\cdot x} + \int_{p'} \mathcal{M}_{0\to 2}(p,p') e^{-ip'\cdot x} & ; \quad x \overset{\to}{\in} \text{Asym. future} \end{cases} \tag{11}$$

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- We have

$$\bar{\varphi}_{\mathcal{E}'}(x) = \begin{cases} \frac{e^{i\mathcal{E}' v - i4GM\mathcal{E}' \log\left(\frac{v_0 - v - i0^+}{\mu}\right)}}{\mathcal{E}' r} & ; \quad x \overset{\to}{\in} \mathcal{I}^- \\ \frac{e^{i\mathcal{E}' u}}{\mathcal{E}' r} \quad ; & ; \quad x \overset{\to}{\in} \mathcal{I}^+ \\ e^{-4\pi GM\mathcal{E}'} \phi^{\mathcal{H}^+}(x;\mathcal{E}') \quad ; & ; \quad x \overset{\to}{\in} \mathcal{H}^+ \end{cases} \tag{12}$$

# Summary

**Schwinger Effect**

**Hawking radiation**

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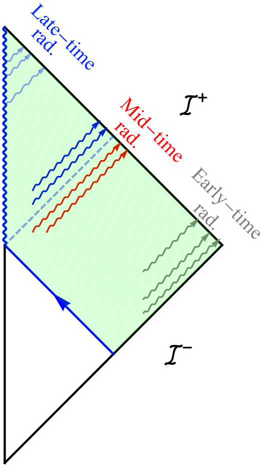
- Non-radial null-geodesics $\leftrightarrow$ 'tunneling'?
- Complex geodesic $\rightarrow$ not really complex!
- Picard-Lefschetz analysis? (However, $T-$integral is still just a normalization.)

THANK YOU![2]

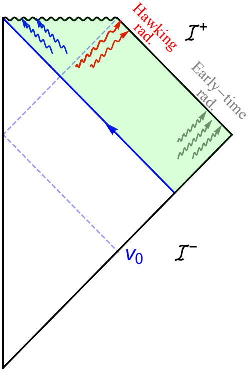
[2]Bonus slides below!

# Hawking radiation from the double copy

[A. Ilderton, W. Lindved, KR, [PRL. 136 (2026)]




(a) $\sqrt{\text{Vaidya}}$




(b) Vaidya

# Classical Double Copy of Geodesics

- For $\sqrt{\text{Vaidya}}$:

$$\Delta\mathcal{E} = \frac{2g^2 qQ}{\mathrm{v}}. \tag{13}$$

[3] In the spirit of [Goldberger and Ridgway, (2017)],[Gonzo and Shi, (2021)],...

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- Double-copy rule![3]

$$c^a \tilde{c}^a \to -p \cdot \tilde{p} \qquad ; \qquad g^2 \to 2G \tag{14}$$

where $\tilde{p} = Mdt$ is the BH and $p$ is the *initial* probe momenta.

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# Worldline Representation of Modes

- The wavefunctions, for $x \to \mathcal{I}^-$, in $\sqrt{\mathrm{Vaidya}}$ obtained from WL formalism:

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- Under the double copy rules, we get the correct out-modes in Vaidya

$$\varphi_{\mathcal{E}'}(x) \to \frac{1}{r} e^{-i\mathcal{E}' v + 4iGM\mathcal{E}' \log\left(\frac{v_0 - v}{r_0}\right)} . \tag{15}$$

# Pair-creation amplitude

- For $\sqrt{\text{Vaidya}}$:

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- Under the double-copy rule

$$|\alpha^{-1}\beta| \to e^{-4\pi GME'}.$$

This is exactly Hawking's result!

THANK YOU!