False Vacuum Decay beyond the quadratic approximation

Based on 2411.18421

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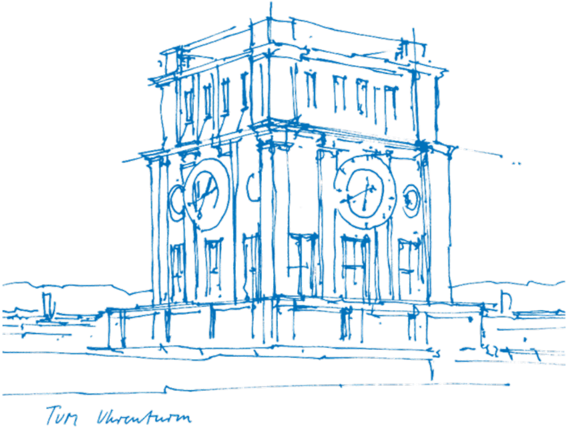
Tunnelling Seminar



Outline

Stability of the Standard Model

The Standard Model contains one scalar: the Higgs field. We must then compute its effective potential to assess whether the SM is

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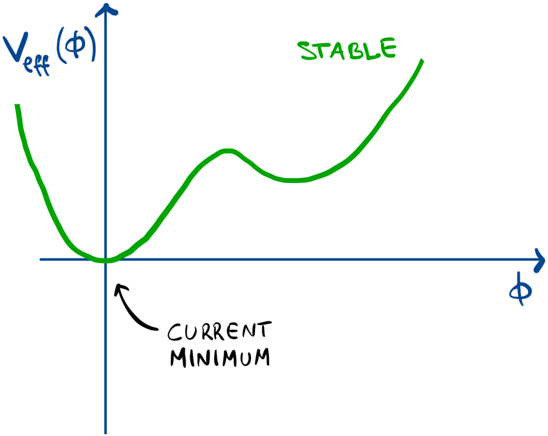

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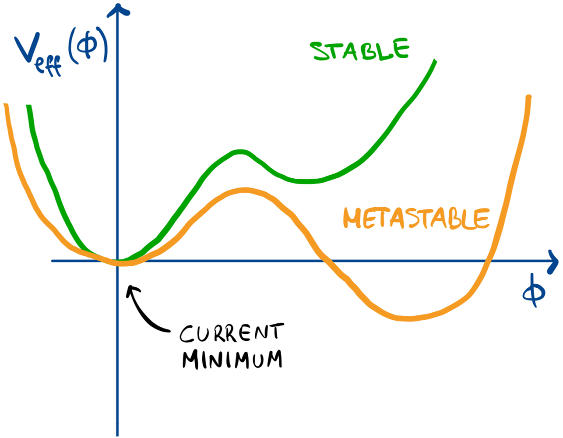

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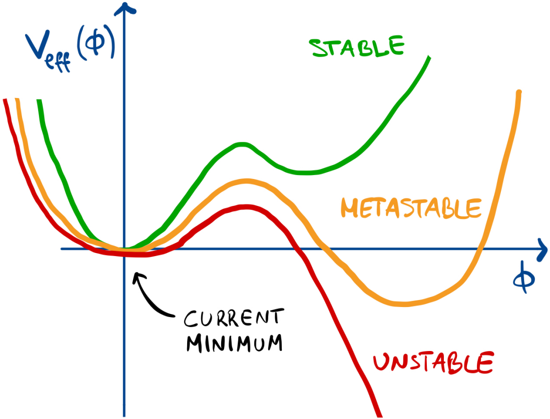

The Standard Model contains one scalar: the Higgs field. We must then compute its effective potential to assess whether the SM is

- **Stable**: the current minimum is the stable minimum.
- **Metastable**: the current minimum is unstable but very long-lived.
- **Unstable**: the current minimum is unstable and very short-lived. Incompatible with our existence!

Stability of the Standard Model

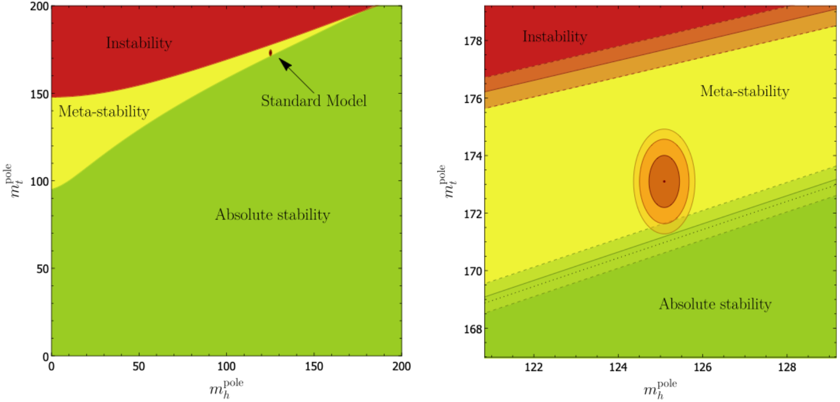

Figure 1 Stability diagram of the SM. Ellipses showing 68%, 95% and 99% confidence regions based on experimental errors on the pole masses.
From A. Andreassen, W. Frost, M. Schwartz. *Phys. Rev. D 97, 056006*.

What is the lifetime of the Universe?

- At high field values the Higgs potential is

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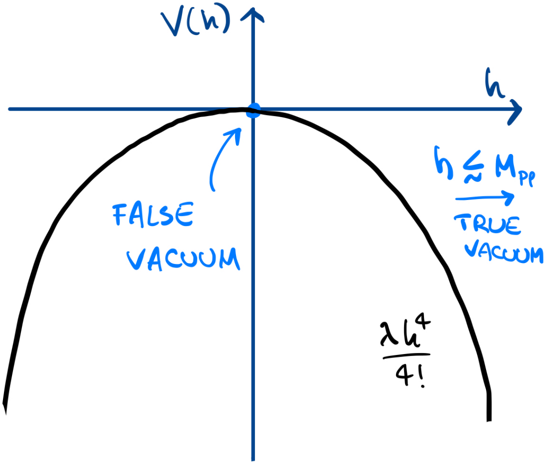

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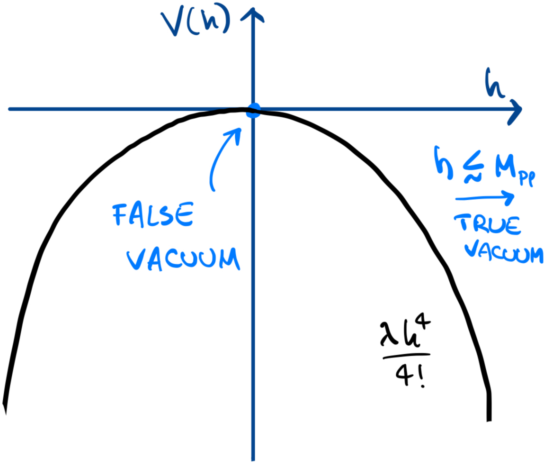

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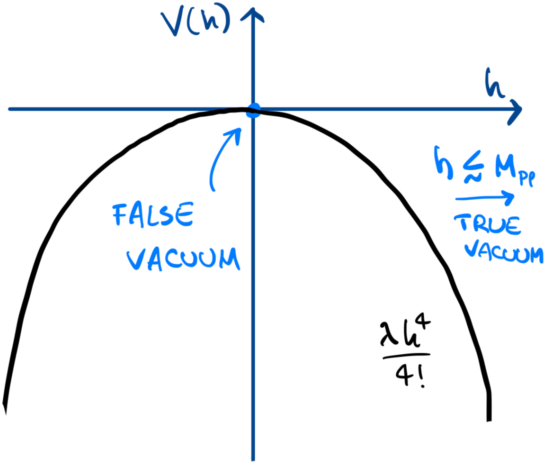
- Long history: Cabibbo et al. '79, Isidori et al. '01, Andreassen et al. '18, and many others



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Decay rate and the effective action

- We find the decay rate per unit volume

$$\frac{\Gamma}{\mathcal{V}} \approx \left|J_{\text{tr}}\right|^d \left|e^{-S_{\text{eff}}[\varphi_B]+S_{\text{eff}}[\varphi_{\text{FV}}]}\right| \tag{2.1}$$

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- Computing the 1-loop 1PI effective action, we recover the known result

$$S_{\text{eff}}[\varphi_B] - S_{\text{eff}}[\varphi_{\text{FV}}] = S[\varphi_B] - S[\varphi_{\text{FV}}] + \frac{1}{2}\operatorname{Tr}'\log G_{B,0}^{-1} - \frac{1}{2}\operatorname{Tr}\log G_{\text{FV},0}^{-1} + \dots$$

$$\Rightarrow \frac{\Gamma}{\mathcal{V}} \approx \left(\frac{\det' S''[\varphi_B]}{\det S''[\varphi_{\text{FV}}]}\right)^{-\frac{1}{2}} e^{-S[\varphi_B]+S[\varphi_{\text{FV}}]} \tag{2.2}$$

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- However, we have to treat **zero-modes** more systematically!

Translational modes

- The bounce can be centred at any point in (Euclidean) spacetime

$$S'[\varphi_B^{(x_0)}] = 0$$
$$\rightarrow S''[\varphi_B^{(x_0)}]\partial_{x_0}\varphi_B^{(x_0)} = 0 \quad (2.3)$$

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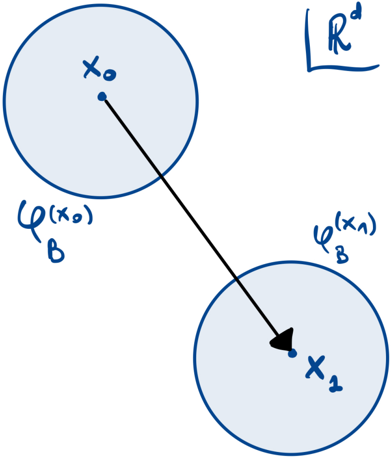

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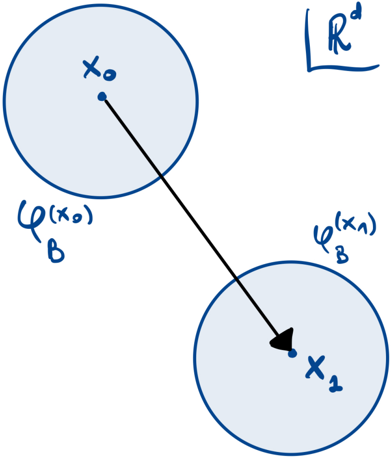
- We must integrate over all posible locations of the bounce

$$\int \mathrm{d}^d x_0 \, |J_{\mathrm{tr}}|^d = \beta\mathcal{V}|J_{\mathrm{tr}}|^d \qquad (2.4)$$



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- Bounces of any size solve the equation of motion because of *scale invariance*

$$\begin{aligned} S'[\varphi_B^{(x_0,R)}] = &\ 0 \\ &\to S''[\varphi_B^{(x_0,R)}]\partial_R \varphi_B^{(x_0,R)} = 0 \end{aligned} \tag{2.5}$$

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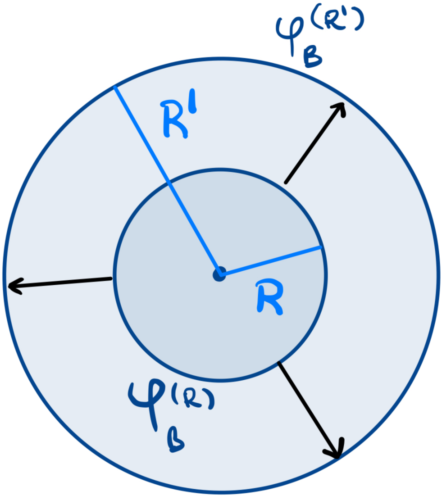

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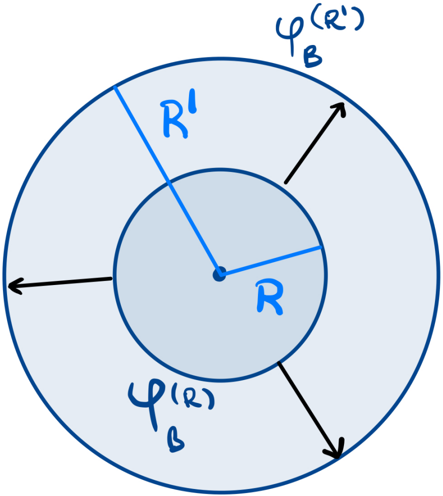
$$\int_0^\infty \mathrm{d}R \, |J_{\mathrm{dil}}| = \infty \qquad (2.6)$$



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  - [Bergner, Bettencourt '03] in the Hartree approximation
  - [Garbrecht, Millington '18] local approximation beyond Hartree
  - [**MC**, Garbrecht '24] full 2-loop action

Our work

What we do

- develop a language to include systematically the quantum corrections to the background *and* to the fluctuations to the desired order in perturbation theory
- compute and renormalise the one-loop corrections to the propagator in the bounce background
- obtain numerical results for the 1- and 2-point functions for a two-dimensional toy model

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What we do *not* do (**yet**)

- obtain numerical results for the four-dimensional model of interest
- compute the decay rate

Outline

The 2PI effective action

- Define a generating functional for the connected 1- and 2-point functions [see e.g. Berges '04, Introduction to Nonequilibrium QFT]

$$e^{W[J,R]} = \mathcal{N} \int [\mathcal{D}\phi] \; e^{-S_E[\phi] - \int_x J_x \phi_y - \frac{1}{2} \int_{x,y} \phi_x R_{xy} \phi_y} \tag{3.1}$$

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- Perform a Legendre transform to obtain the 2PI effective action

$$\Gamma_{2PI}[\varphi, G] = W[J,R] - \int_x J_x \varphi_x - \frac{1}{2} \int_{x,y} G_{xy} R_{xy} \tag{3.2}$$

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- Do a perturbative expansion in loops

$$\Gamma_{2PI}[\varphi, G] = S_E[\varphi] + \frac{1}{2} \operatorname{Tr} G_0^{-1} G - \frac{1}{2} \operatorname{Tr} \log G^{-1} + \Gamma_2[\varphi, G] \tag{3.3}$$

The bounce equation of motion

The EoM for the 1pt function is easily obtained

$$\frac{\delta\Gamma_{2PI}}{\delta\varphi(x)} = 0$$

$$\Longrightarrow \boxed{\frac{\delta S_E[\varphi]}{\delta\varphi(x)} + \Pi_G(x)\varphi(x) + \frac{\delta\Gamma_2}{\delta\varphi} = 0} \tag{3.4}$$

where

$$\Pi_G(x)\varphi(x) = \frac{1}{2}V'''(\varphi(x))G(x,x) + \text{c.t.} \tag{3.5}$$

The 2pt function equation of motion

The EoM for the connected 2pt function is obtained analogously

$$\frac{\delta \Gamma_{2PI}}{\delta G(x,y)} = 0$$

$$\Longrightarrow \boxed{G_0^{-1}(x,y) - G^{-1}(x,y) - \Sigma(x,y) = 0} \tag{3.6}$$

having defined

$$\Sigma(x,y) = -2\frac{\delta \Gamma_2}{\delta G(x,y)} \tag{3.7}$$

The 2pt function equation of motion

- The operator equation reads (we suppress indices for simplicity)

$$G = G_0 + G_0 \Sigma G \tag{3.8}$$

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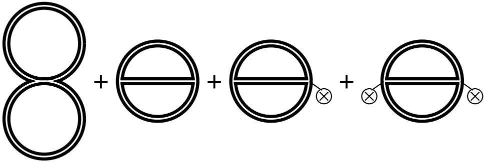
- Representing it diagrammatically makes it clear that G satisfying Eq. (3.8) is the resummed propagator

Expanding the self-energy

- The term Γ_2 contains all 2PI vacuum diagrams

$$-\Gamma_2[\varphi, G] \supset$$



(3.9)

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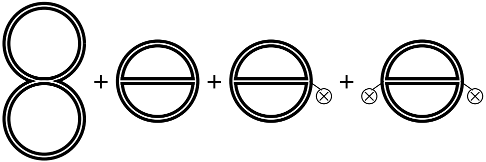

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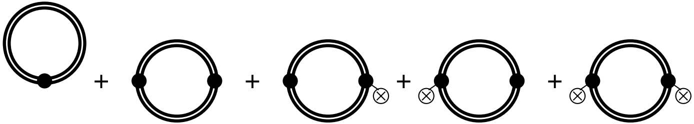
- The self-energy Σ is obtained by differentiation, i.e. cutting one leg

$\mathbf{\Sigma}_\varphi \supset$



(3.10)

Outline

A system of coupled equations

- At 1-loop order we obtain a system of coupled *non-linear integrodifferential* equations

$$-\Delta_x \varphi_x + V'(\varphi_x) + \Pi_x \varphi_x = 0 \tag{4.1}$$

$$(-\Delta_x + V''(\varphi_x))\, G_{xy} + \int_z \Sigma_{xz}\, G_{zy} = \delta^{(d)}_{xy} \tag{4.2}$$

- Let's make use of the central symmetry of the problem to simplify the equations

Angular momentum decomposition

- Introduce an angular momentum decomposition ($\kappa = d/2 - 1$)

$$G(x, y) = \frac{1}{(r_x r_y)^\kappa} \sum_{j,\{\ell\}} Y_{j,\{\ell\}}(\Omega_x) Y_{j,\{\ell\}}(\Omega_y) G_j(r_x, r_y) \tag{4.3}$$

- Split the self-energy into local and non-local contributions

$$\Sigma(x, y) = \delta^{(d)}(x - y)\Pi(x) + \Sigma_{\text{n.l.}}(x, y) \tag{4.4}$$

- Make a similar ansatz for the non-local term

$$\Sigma_{\text{n.l.}}(x, y) = \frac{1}{(r_x r_y)^\kappa} \sum_{j,\{\ell\}} Y_{j,\{\ell\}}(\Omega_x) Y_{j,\{\ell\}}(\Omega_y) \Sigma_j(r_x, r_y) \tag{4.5}$$

A system of coupled equations

- We get a system of ordinary integro-differential equations, though now we have infinitely many of them, all coupled!

$$-\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{d-1}\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r) + V'(\varphi(r)) + \Pi(r)\varphi(r) = 0$$

$$\left(-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{(j+\kappa)^2}{r^2} + V''(\varphi(r)) + \Pi(r)\right)G_j(r,r') + \int_0^\infty \mathrm{d}r''\, r''\, \Sigma_j(r,r'')\, G_j(r'',r') = \frac{1}{r}\delta(r-r')$$

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- We can only solve this **self-consistently**, and we must **truncate** at some $j_{\max}$

A system of coupled equations: Hartree approximation

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$$-\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{d-1}\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r)+V'(\varphi(r))+\Pi(r)\varphi(r)=0$$

$$\left(-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}+\frac{(j+\kappa)^2}{r^2}+V''(\varphi(r))+\Pi(r)\right)G_j(r,r')$$
$$+\int_0^\infty \mathrm{d}r''\,r''\,\Sigma_j(r,r'')\,G_j(r'',r')=\frac{1}{r}\delta(r-r')$$

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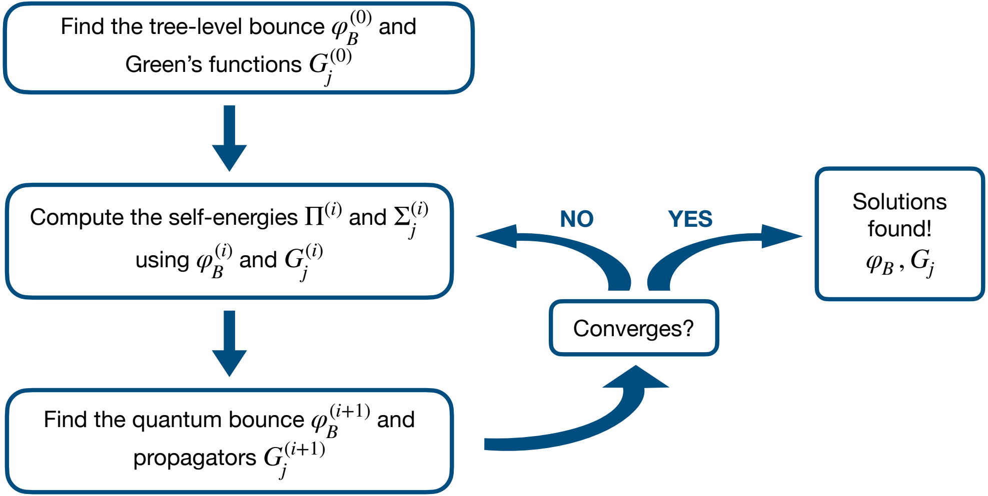
The self-consistent procedure



Renormalising the tadpole

- Take the potential

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \tag{4.8}$$

Renormalising the tadpole

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- We must renormalise the coincident Green's function

$$\begin{aligned} G(x,x) &= \frac{1}{r^{2\kappa}} \sum_{j,\{\ell\}} Y_{j,\{\ell\}}(\Omega)\, Y_{j,\{\ell\}}(\Omega)\, G_j(r,r) \\ &= \frac{2}{(4\pi)^{\kappa+\frac{1}{2}}} \frac{1}{\Gamma\left(\kappa+\frac{1}{2}\right)} \frac{1}{r_x^{2\kappa}} \sum_{j=0}^{\infty} (j+\kappa) \frac{\Gamma(j+2\kappa)}{\Gamma(j+1)} G_j(r,r) \end{aligned} \tag{4.10}$$

Renormalising the tadpole

- The UV divergence is due to the large angular momentum modes: use WKB to obtain an expression for these

$$G_j^{\mathrm{WKB}}(r,r) = \frac{1}{2(j+\kappa)}\left(1 - \frac{m_\phi^2(r)r^2}{2(j+\kappa)^2} + \mathcal{O}\left((j+\kappa)^{-4}\right)\right) \tag{4.11}$$

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- We can use dim. reg. to regularise the sum

$$\begin{aligned}
[G(x,x)]_{\kappa=1-\epsilon} &= \frac{1}{2\pi^2 r^2}\sum_{j=0}^{\infty}(j+1)^2\left(G_j(r,r) - G_j^{\mathrm{WKB}}(r,r)\right) + \left[G^{\mathrm{WKB}}(x,x)\right]_{\kappa=1-\epsilon} \\
&= \frac{1}{2\pi^2 r^2}\sum_{j=0}^{\infty}(j+1)^2\left(G_j(r,r) - G_j^{\mathrm{WKB}}(r,r)\right) \\
&\quad - \frac{1}{16\pi^2 r^2}\left[\frac{m_\phi^2(r)r^2}{\epsilon} + \frac{1}{3} + m_\phi^2(r)r^2\log\frac{1}{4}e^2r^2\mu^2\right]
\end{aligned} \tag{4.12}$$

Renormalising the tadpole

- We can then define the $\overline{\text{MS}}$-renormalised coincident Green's function

$$[G(x,x)]^{\overline{\text{MS}}} = \frac{1}{2\pi^2 r^2} \sum_{j=0}^{\infty} (j+1)^2 \left(G_j(r,r) - G_j^{\text{WKB}}(r,r) \right) - \frac{1}{32\pi^2 r^2} \left[\frac{1}{3} + m_\phi^2(r) r^2 \log \frac{1}{4} e^2 r^2 \mu^2 \right] \tag{4.13}$$

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- We can then obtain the on-shell one by subtracting the $\phi \equiv \phi_{\mathrm{FV}}$ result

$$\begin{aligned}[G(x,x)]^{\mathrm{OS}} &= [G(x,x)]^{\overline{\mathrm{MS}}} - [G(x,x)]^{\overline{\mathrm{MS}}}_{\phi\equiv\phi_{\mathrm{FV}}} \\ &= \frac{1}{2\pi^2 r^2}\sum_{j=0}^{\infty}(j+1)^2\left[G_j(r,r) - G_{0,j}(r,r) + \frac{(m_\phi^2(r) - m^2)r^2}{4(j+1)^3}\right] \\ &\quad - \frac{1}{32\pi^2}(m_\phi^2(r) - m^2)\log\frac{1}{4}e^2 r^2\mu^2 \end{aligned} \tag{4.14}$$

Renormalising the bubble

- Renormalising the bubble is much harder. It requires finding the divergent structure of Σ_j defined by

$$G(x,y)^2 = \left(\frac{2}{(4\pi)^{\kappa+\frac{1}{2}}} \frac{\Gamma(2\kappa)}{\Gamma\left(\kappa+\frac{1}{2}\right)} \frac{1}{(r_x r_y)^\kappa} \sum_{j=0}^{\infty} (j+\kappa) C_j^\kappa(\cos\theta) G_j(r_x, r_y) \right)^2$$

$$= \frac{2}{(4\pi)^{\kappa+\frac{1}{2}}} \frac{\Gamma(2\kappa)}{\Gamma\left(\kappa+\frac{1}{2}\right)} \frac{1}{(r_x r_y)^\kappa} \sum_{j=0}^{\infty} (j+\kappa) C_j^\kappa(\cos\theta) \Sigma_j(r_x, r_y) \qquad (4.15)$$

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- After a long computation, we find

$$\Sigma_j(r_x, r_y) \approx \sum_q q^2 G_q(r_x, r_y) G_{q+j}(r_x, r_y) \approx \sum_q \left(\frac{r_<}{r_>}\right)^{2q} \left(1 + \mathcal{O}(q^{-2})\right) \tag{4.16}$$

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- The divergence can be renormalised via local counter-terms

Outline

Bounce

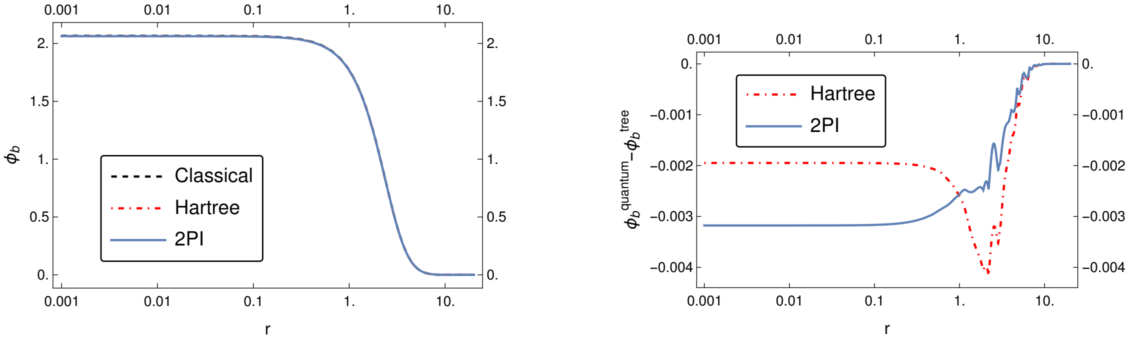

Figure 2 Classical vs. self-consistent bounce in the Hartree approximation and full 2PI.

Self-energy

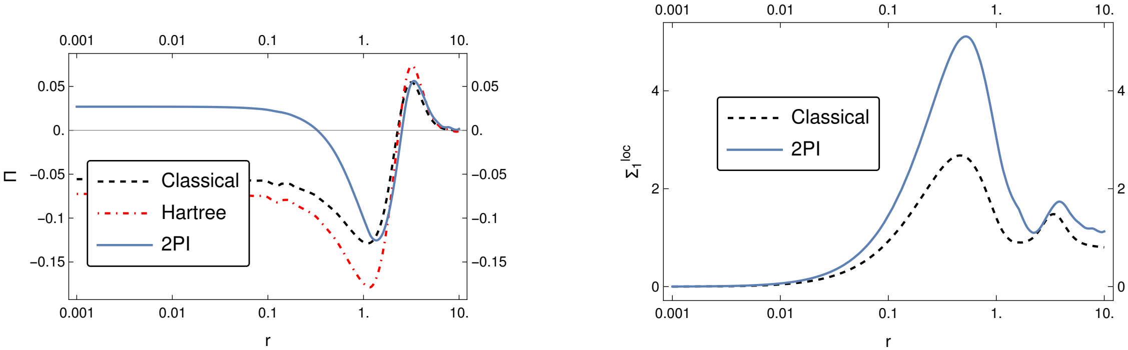

Figure 3 Tadpole and coincident bubble self-energy.

Outline

Conclusions and outlook

Lessons learned (see 2411.18421)

- ✔ We can express the instanton method in terms of the effective action of the bounce
- ✔ We can use spherical symmetry to sum self-energy diagrams, *even non-local ones*, into the propagator
- ✔ The Hartree approximation is, in general, not justified

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Our next steps

- ➡ Run the numerics for the four-dimensional scale invariant theory and analyse the renormalisations scale dependence of the renormalised determinant
- ➡ Compute the decay rate

BACK-UP SLIDES

The translational zero-mode

- The fluctuation operator actually has a zero-mode, related to translational invariance. Starting from the tree-level EoM

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{d-1}\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r)+V'(\varphi(r))\right) &= 0\\
\Longrightarrow \left(-\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{d-1}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{d-1}{r^2}+V''(\varphi(r))\right)\dot{\varphi}(r) &= 0
\end{aligned}
\tag{7.1}
$$

- There is d-many zero-modes in the $j=1$ sector
- Quantum corrections do not break translational symmetry, thus we should find

$$
\begin{aligned}
&\left(-\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{d-1}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{d-1}{r^2}+V''(\varphi(r))+\Pi(r)\right)\phi_{\mathrm{tr}}(r)\\
&\qquad+\int_0^\infty \mathrm{d}r''\, {r''}^{d-1}\,\frac{1}{{r''}^{\frac{d}{2}-1}}\Sigma_j(r,r'')\,\phi_{\mathrm{tr}}(r'') = 0
\end{aligned}
\tag{7.2}
$$

Subtracting the zero-mode

- The zero-modes are not propagating degrees of freedom and must thus be subtracted

$$\mathcal{O}G^{\perp} = \mathbb{1}^{\perp} \tag{7.3}$$

- The operator $\mathbb{1}^{\perp}$ is the identity on the orthogonal subspace to the one spanned by the zero-modes

$$\mathbb{1}^{\perp} = \mathbb{1} - \sum_i \phi_i \phi_i^* \tag{7.4}$$

- This defines the subtracted Green's function $G^{\perp}$
- Truncations of the effective action however, are known not to reproduce the correct spectrum of zero-modes. The *symmetry improved effective action* is needed!

Green's function

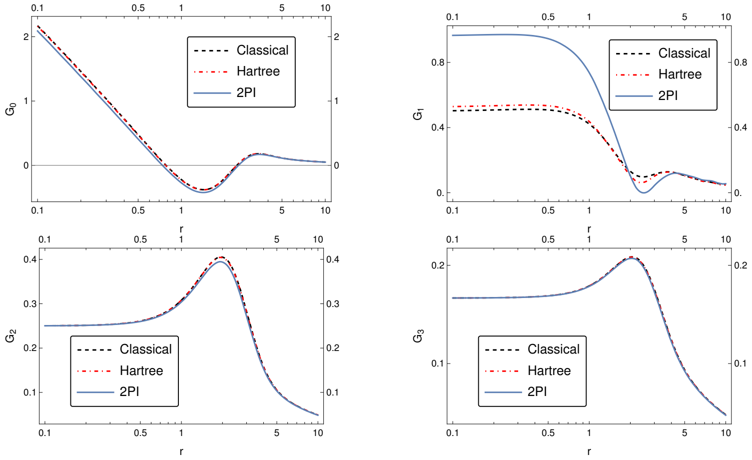

Figure 4 Coincident limit of the Green's function $G_j(r, r)$. Plots taken from 2411.18421.