Real-Time Nucleation and Off-Equilibrium Effects in High-Temperature Quantum Field Theories

arXiv:2403.07987

Joonas Hirvonen
University of Nottingham

12/12/2024

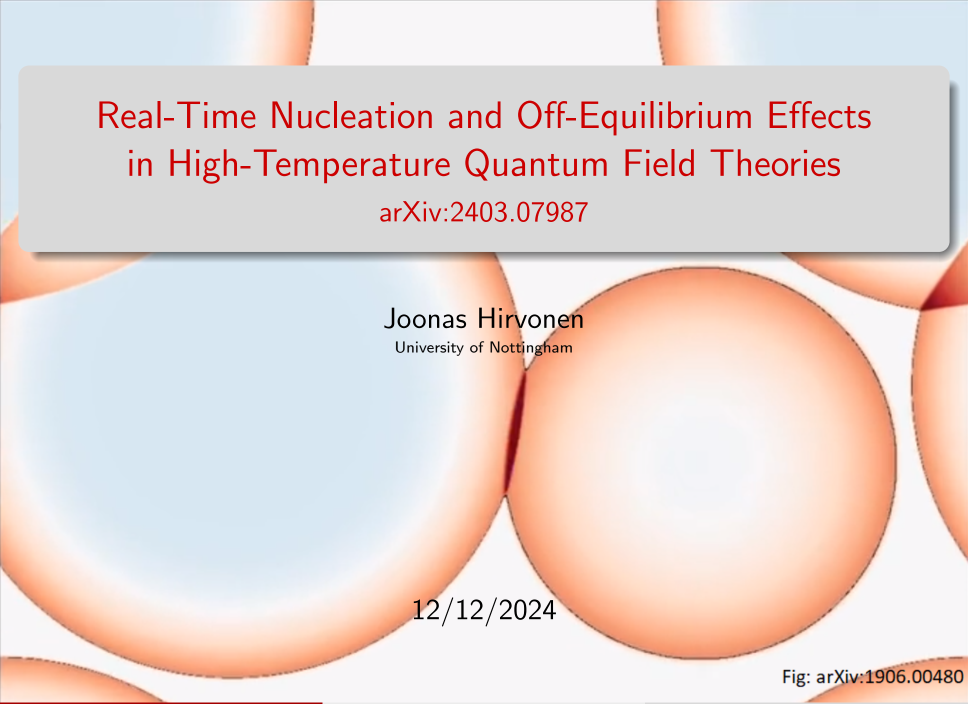


Fig: arXiv:1906.00480

Overview of the talk

- Introduction
- High-temperature QFTs
  - Particles
  - Classical fields
- Classical nucleation theory
- Rate for high-temperature QFTs

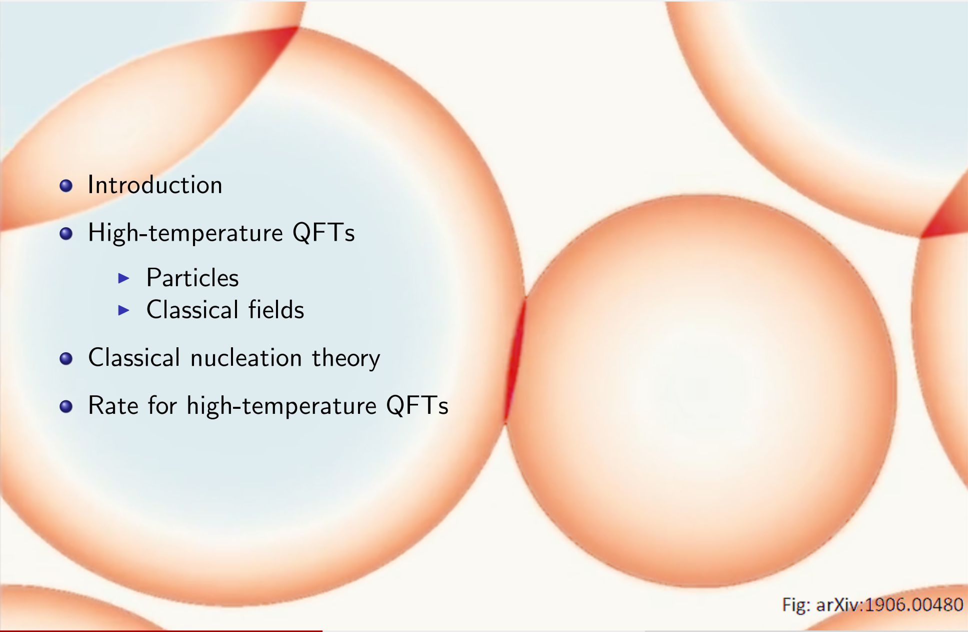


Fig: arXiv:1906.00480

Introduction

- Cosmological first-order phase transitions
- Nucleation rates and gravitational-wave signals
- Overview of aims
- General physics picture

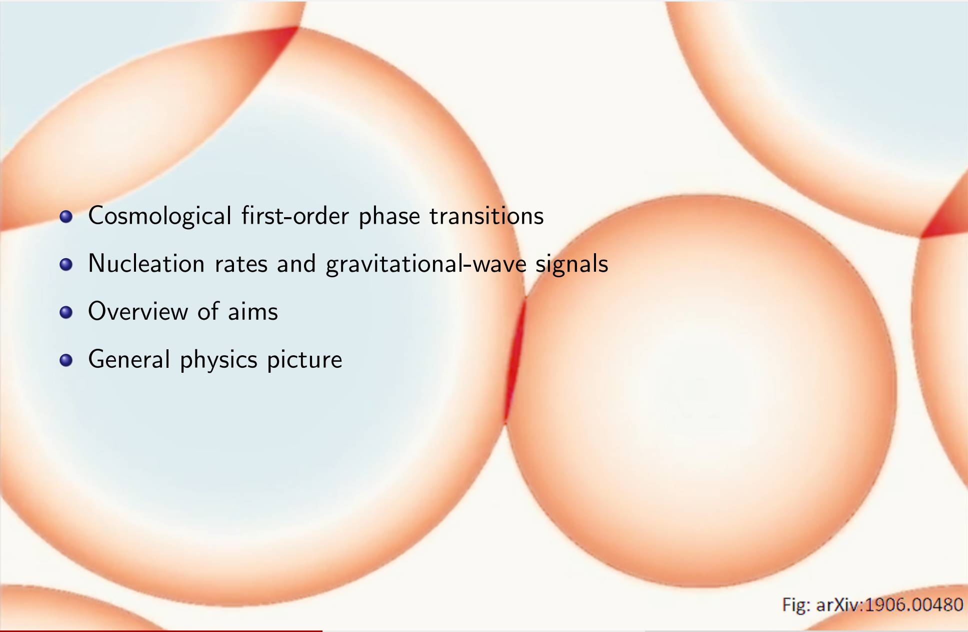


Fig: arXiv:1906.00480

Cosmological first-order phase transitions

- Bubbles nucleating in primordial plasma
- None in Standard Model
- Mechanism for matter production
  - Electroweak baryogenesis
- Gravitational-wave signal
  - Release of energy
  - Observable in the future?

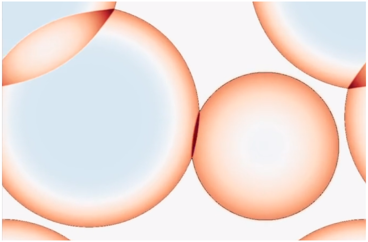


Figure: Growing bubbles (arXiv:1906.00480)

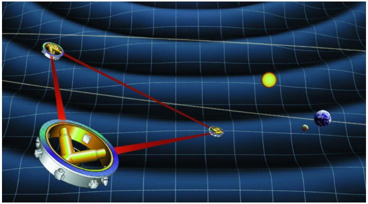


Figure: Gravitational waves and LISA (ESA)

Gravitational waves and nucleation rates [Enqvist, Ignatius, Kajantie, Rummukainen '92]

$$\Delta t = \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\Gamma\right)^{-1}$$

$$8\pi v_{\mathrm{w}}^3\,\Delta t^4\,\Gamma \approx 1 \Rightarrow T_{\mathrm{n}}$$

- Setting of a transition
  - Duration of a transition, Δt
  - Transition temperature, T_{n}
    - Affected also by bubble expansion velocity, v_{w}
- Large uncertainties from nucleation rate

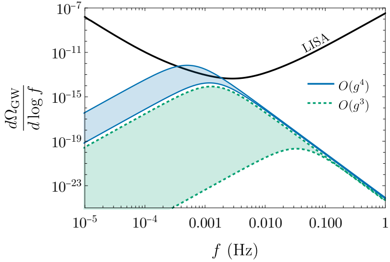


Figure: GW uncertainties from nucleation rates [Gould, Tenkanen '21]

Towards precision nucleation

- First analytical study of off-equilibrium plasma directly affecting bubbles
  - Corroborates equilibrium methods up to a certain validity
  - Future studies with SM extensions
  - Tools for assessing if current nucleation theory is applicable

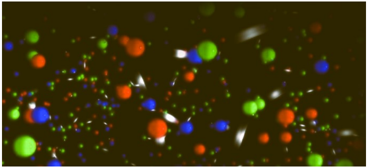


Figure: Plasma particles (Dr Rene Bellwied)

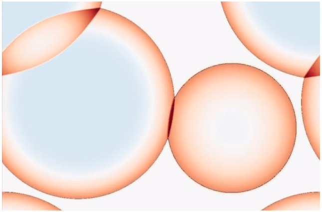


Figure: Growing bubbles (arXiv:1906.00480)

General physics picture

1. Bubble nucleating
2. Thermal-particle shower
3. Bubble growth pushes them off equilibrium
4. Off-equilibrium particles slow the growth, and nucleation rate

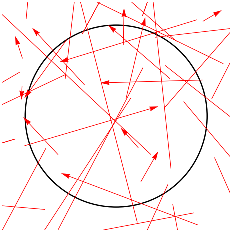


Figure: Bubble in a particle shower

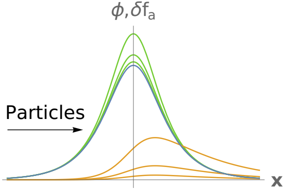


Figure: A growing bubble and off-equilibrium particles

Weakly coupled QFTs at high temperatures

- Example QFT model
- Thermal system
  - Classical fields
  - Particles
- Equations of motion

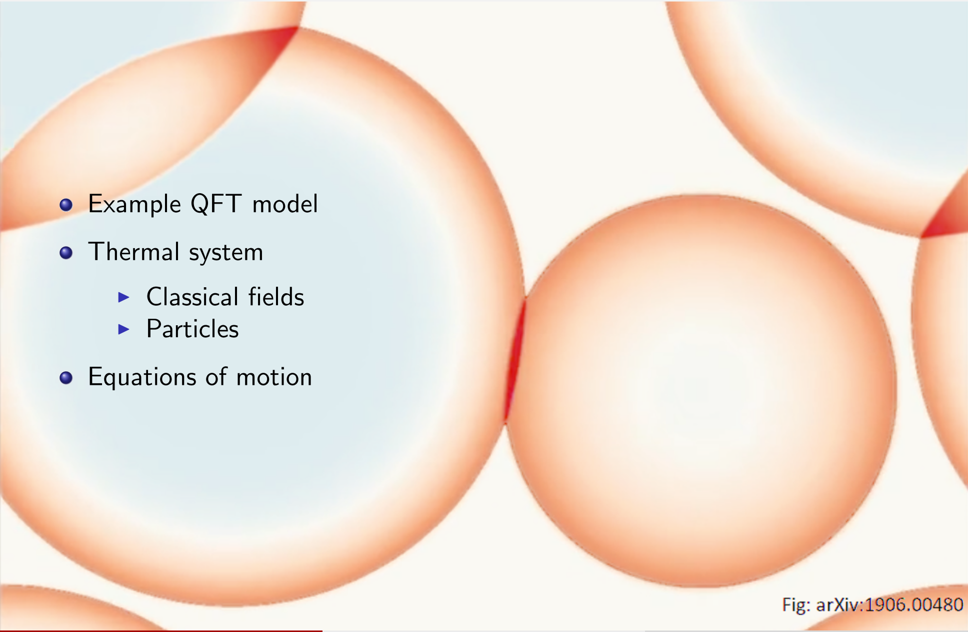


Fig: arXiv:1906.00480

Example QFT model

- Real scalar, ϕ
  - Undergoes a phase transition
- A Dirac fermion, ψ
  - Particles interact with the bubbles
  - Mass changes due to the background

$$\mathscr{L} = \mathscr{L}_\phi + \mathscr{L}_\psi - y\phi\bar{\psi}\psi,$$

$$\mathscr{L}_\phi = -\frac{1}{2}\phi\Box\phi - \frac{m_\mathrm{b}^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4,$$

$$\mathscr{L}_\psi = \bar{\psi}(i\partial\!\!\!/ - m_\mathrm{f})\psi$$

$$m_\phi^2 = V''(\phi),$$
$$m_\psi = m_\mathrm{f} + y\phi$$

Power counting

- Perturbative couplings
- No mass or coupling hierarchies in vacuum
- Leads to a high-temperature hierarchy at the phase transition [Gould & Hirvonen '21, Hirvonen '22]
  - Perturbative thermal corrections initiate the transition

$$\lambda \sim g^2/m_\phi^2 \sim y^2 \ll 1,$$

$$m_\phi^2 \sim m_\psi^2$$

$$m_\phi^2 \sim m_\psi^2 \sim \lambda T^2 \ll T^2$$

Thermal plasma in high-T QFTs

- Thermal energy and excitations: plasma system
- Two pieces
  - Thermal particles
  - Long-range, classical bosonic fields

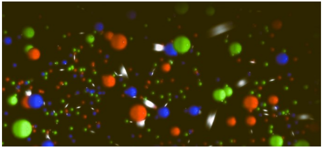


Figure: Plasma particles

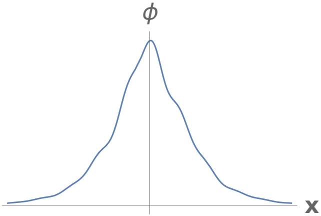


Figure: A bubble in a long-range scalar field

Bosonic classical fields

- Long wavelenght, $\gg T^{-1}$, low energy, $\ll T$
  - Classicalization due to large occupation numbers
- One corresponding to ϕ
- Thermal-equilibrium potential, V_T
- Off-equilibrium term
  - Off-equilibrium particle distribution functions, δf_a
  - Change in masses

$$f_{\text{eq,bos}} = \frac{1}{e^{\beta E} - 1} \sim \frac{T}{m_{\text{bos}}} \gg 1$$

$$\Box\phi + V_T'(\phi) = -\sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_a} \delta f_a$$

$a \in \{\text{real scalar, Dirac fermion}\}$

Effective potential for nucleation [Gould & Hirvonen '21]

- Coarse grain leaving the bubble scale

$$L_{\text{bubble}} \sim m_\phi^{-1} \sim (yT)^{-1}$$

- Thermal dependency in V_T
  - Initiates the transition
- For the example
  - $E \sim T$ integrated out
  - *High-temperature dimensional reduction*

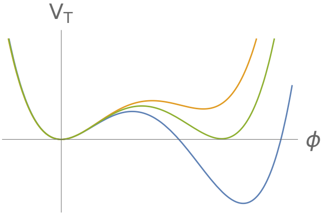


Figure: Thermal dependence and a phase transition in a potential

$$V_T = s_T\phi + \frac{1}{2}m_T^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4,$$
$$s_T = (g + 4ym_{\text{f}})T^2/24,$$
$$m_T^2 = m_{\text{b}}^2 + (\lambda^2 + 4y^2)T^2/24.$$

Thermal particles

- Regime $E \sim T$
- Described by particle distribution functions, $f_a(t, \mathbf{x}, \mathbf{p})$
- Boltzmann equation
  - LHS: collisionless evolution
  - RHS: collision term, C
  - Force, $\mathbf{F}_a$: scalar-field dependent mass

$$(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{F}_a \cdot \partial_{\mathbf{p}}) f_a = C[f],$$

$$\mathbf{F}_a = -\boldsymbol{\nabla} E_a = -\frac{\boldsymbol{\nabla} m_a^2(\phi)}{2E_a}$$

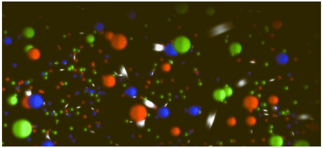


Figure: Plasma particles

Vlasov equation instead of Boltzmann

- $C[f]$: Dissipation without noise
  - No fluctuations, no nucleation
- In our example $C[f] \approx 0$
  - Collisions important: $L_{\text{coll}} \sim (y^4 \log y^{-1} T)^{-1}$
  - Bubble size: $L_{\text{CB}} \sim m_\phi^{-1} \sim (yT)^{-1}$
  - Hierarchy of couplings (e.g. QCD coupling)
- Probably not a limitation of our results

$$(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{F}_a \cdot \partial_{\mathbf{p}})\, f_a = C[f]$$

$$\Downarrow$$

$$(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla} + \mathbf{F}_a \cdot \partial_{\mathbf{p}})\, f_a = 0$$

Equation for off-equilibrium particles, δf_a

- Subtracting off equilibrium distribution, $f_{\mathrm{eq},a}$
- $f_{\mathrm{eq},a}$ time-dependent
  - Due to field-dependent masses, $m_a^2(\phi)$
  - Sources δf_a
- Force-term negligible for light particles, $m_a^2 \ll T^2$

$$\delta f_a = f_a - \overbrace{\frac{1}{e^{\beta E_a} \mp_a 1}}^{\equiv f_{\mathrm{eq},a}},$$

$$E_a = \sqrt{\mathbf{p}^2 + m_a^2(\phi)}$$

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\delta f_a = -\frac{f'_{\mathrm{eq},a}(E)}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\partial_t\phi$$

Summary

$$\Box\phi + V_T'(\phi) = -\sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}\delta f_a$$

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\delta f_a = -\frac{f'_{\mathrm{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\partial_t\phi$$

- System splits into fields and particles
- Off-equilibrium particles and fields coupled
  - System conservative
  - Still large, collective field changes *effectively* dissipated into off-equilibrium particles

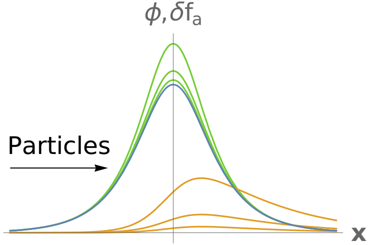
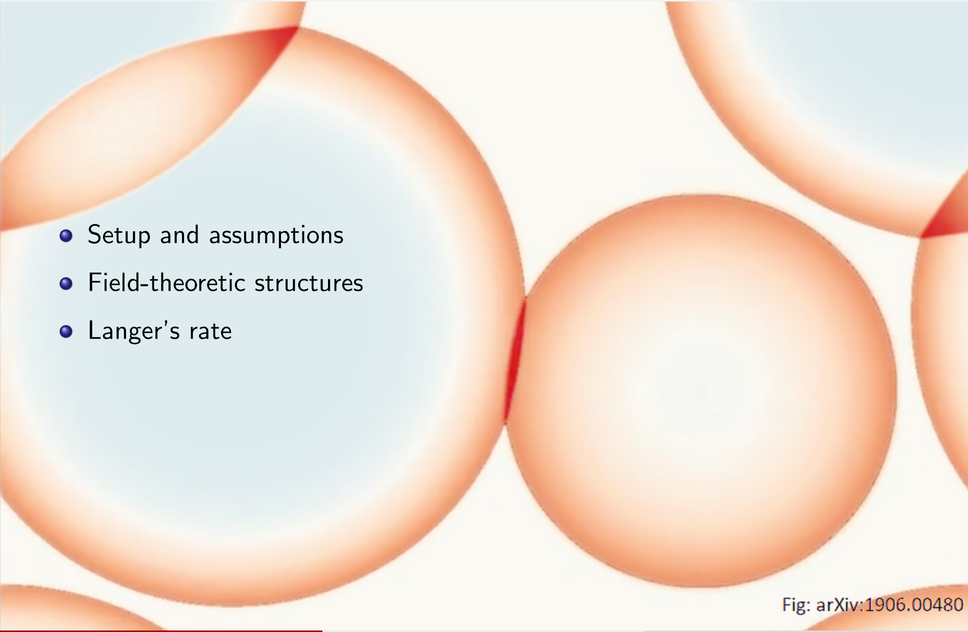


Figure: Large field changes dissipated into off-equilibrium particles

Classical nucleation theory



- Setup and assumptions
- Field-theoretic structures
- Langer's rate

Fig: arXiv:1906.00480

Setup for classical nucleation theory [Kramers '40]

- Two phases
  - Metastable: thermal population
  - Stable: empty
- Nucleation rate
  - Leak from meta to stable

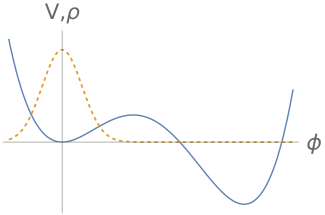


Figure: Schematic figure of an equilibrium distribution in the metastable state

Assumptions of classical nucleation theory [Kramers '40]

- Full system not analytically treatable
- Zooming onto the barrier
  - System linear on the barrier
- Steady state
  - Rate sourced by equilibrium

$$\rho_{\text{eq}} \propto e^{-\beta H}$$

  - Static distribution

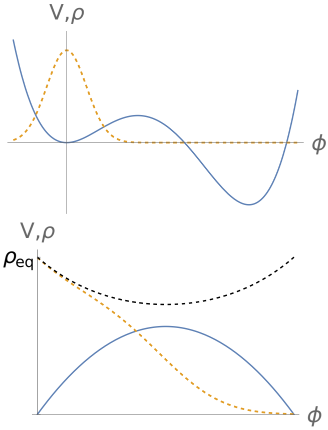


Figure: Zooming onto the barrier

Langer's theory focus [Langer '67, '69, '74]

- Local in space and time
- Explicit
  - Dissipation, η
  - Noise, ξ
- Second-order equations of motion

$$\Box\phi + V_T'(\phi) = -\eta\,\partial_t\phi + \xi$$

$$\Box\phi + V_T'(\phi) = -\sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}\delta f_a$$

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\delta f_a = -\frac{f_{\mathrm{eq},a}'}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\partial_t\phi$$

Field-theoretic structures [Langer '67, '69, '74]

- Configuration space
- Transition surface
  - Separates the phases
  - Rate: flow over the surface
- Important configurations
  - Critical bubble, ϕ_{CB}
  - Exponential growth, $\overline{\delta\phi}$

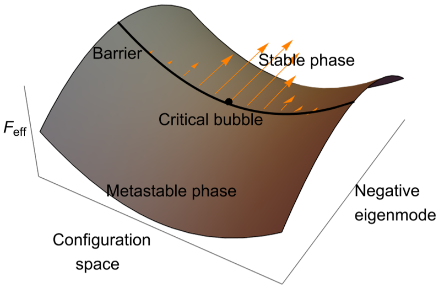


Figure: Transition surface

$$\nabla^2\phi_{\text{CB}} = V_T'(\phi_{\text{CB}})$$

$$\phi = \phi_{\text{CB}} + \overline{\delta\phi}\, e^{\kappa t}$$

Langer's rate, Γ [Langer '67, '69, '74]

- Rate: Total over-barrier flow
- Statistical part [Gould & Hirvonen '21]
  - Exponential suppression from critical bubble
  - Fluctuations around critical bubble
- Dynamical part, κ
  - Exponential growth rate

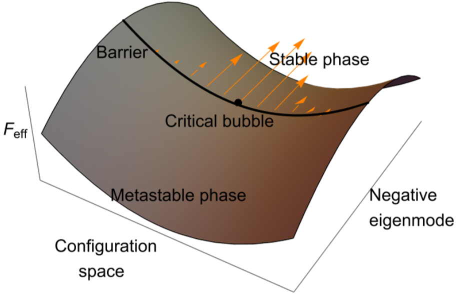


Figure: Over-barrier flow

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi} \underbrace{\left(\frac{H_{\mathrm{CB}}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\nabla^2 + V''_{\mathrm{meta}}\right)}{\det'\left(-\nabla^2 + V''_{\mathrm{CB}}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\mathrm{CB}}}}_{\text{Statistical part}}$$

Comparison to vacuum decay

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi}\left(\frac{H_{\mathrm{CB}}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\nabla^2 + V''_{\mathrm{meta}}\right)}{\det'\left(-\nabla^2 + V''_{\mathrm{CB}}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\mathrm{CB}}}$$

- Formula similar, physics different
- No analytic continuation over the instability
- All configurations physical
- Time-evolution affects rate, κ

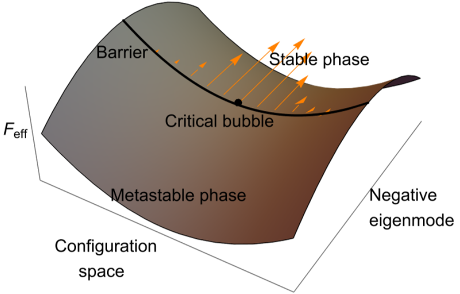
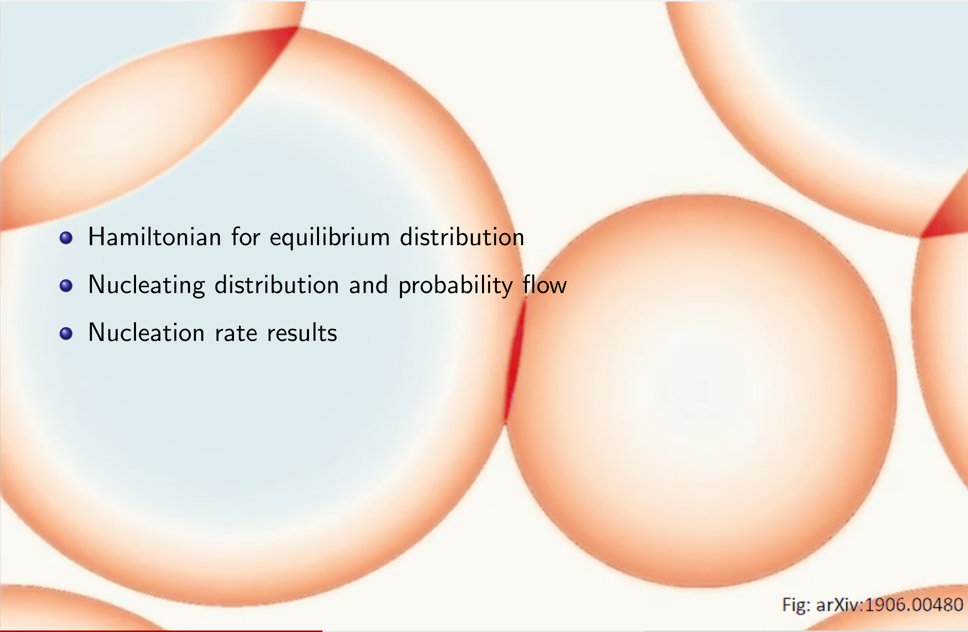


Figure: Over-barrier flow

Rate in high-temperature QFTs



- Hamiltonian for equilibrium distribution
- Nucleating distribution and probability flow
- Nucleation rate results

Fig: arXiv:1906.00480

Towards the Hamiltonian

- Introducing the conjugate momentum field,

$$\pi \equiv \partial_t \phi$$

  - Sources off-equilibrium particles

$$\dot{\phi} = \pi$$

$$\dot{\pi} = \nabla^2 \phi - V_T'(\phi) - \sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E} \delta f_a$$

$$\dot{\delta f_a} = -\mathbf{v} \cdot \nabla \delta f_a - \frac{f'_{\mathrm{eq},a}}{2E} \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi} \pi$$

Effective Hamiltonian, H_{eff}

- Structure
  - Basic field terms
  - Particle-term related to statistics
- Conserved under time evolution

$$H_{\text{eff}} = \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V_T(\phi) + \sum_a \frac{1}{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\delta f_a^2}{-f'_{\text{eq},a}} \right)$$

- Time evolution from Poisson brackets
  - Not relevant for our analysis

$$\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}),$$

$$\{\delta f_a(\mathbf{x}, \mathbf{p}), \delta f_a(\mathbf{y}, \mathbf{q})\} = (2\pi)^3 f'_{\text{eq},a} \delta(\mathbf{p} - \mathbf{q}) \times \mathbf{v} \cdot \nabla\delta(\mathbf{x} - \mathbf{y}),$$

$$\{\pi(\mathbf{x}), \delta f_a(\mathbf{y}, \mathbf{q})\} = \frac{f'_{\text{eq},a}}{2E} \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi} \delta(\mathbf{x} - \mathbf{y})$$

Equilibrium distribution function

- Given by the Hamiltonian
- Factorization allows for sanity checks:
- Equilibirum distribution for ϕ
  - *High-temperature dimensional reduction*
  - Connection to equilibrium methods
    [Gould & Hirvonen '21]
- Bose-Einstein and Fermi-Dirac statistic

$$
\begin{aligned}
\rho_{\text{eq}} &\propto e^{-\beta H_{\text{eff}}} \\
&= \exp\left(-\beta \int \mathrm{d}^3\mathbf{x} \frac{1}{2}\pi^2\right) \\
&\quad \times \exp\left(-\beta \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)\right) \\
&\quad \times \exp\left(-\beta \int \mathrm{d}^3\mathbf{x} \sum_a \frac{1}{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\delta f_a^2}{-f'_{\text{eq},a}}\right)
\end{aligned}
$$

Zooming onto nucleation barrier

- Linearizing around the critical bubble, ϕ_{CB}:

$$\nabla^2 \phi_{\text{CB}} = V_T'(\phi_{\text{CB}}),$$
$$\phi = \phi_{\text{CB}} + \delta\phi$$

- Analytically solvable!

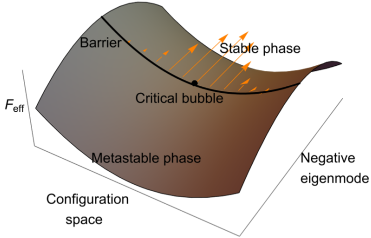


Figure: Barrier between the phases

$$\dot{\pi} = (\nabla^2 - \underbrace{V''_{\text{CB}}}_{\equiv V_T''(\phi_{\text{CB}})})\delta\phi - \sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E} \delta f_a$$

$$\frac{1}{2}(\nabla\phi)^2 + V_T(\phi)$$
$$= \underbrace{\frac{1}{2}(\nabla\phi_{\text{CB}})^2 + V_T(\phi_{\text{CB}})}_{\text{Constant term}} + \underbrace{\frac{1}{2}(\nabla\delta\phi)^2 + V''_{\text{CB}}\delta\phi^2}_{\text{Quadratic in fluctuations}}$$

Obtaining the nucleating distribution

- Solves the Liouville equation
- Time independent
- Equilibrium in metastable phase
    - Sources nucleation

$$\int \mathrm{d}^3\mathbf{x}\Big(\dot{\delta\phi}\frac{\delta}{\delta\delta\phi} + \dot{\pi}\frac{\delta}{\delta\pi} + \sum_a \int \mathrm{d}^3\mathbf{p}\, \dot{\delta f}_a \frac{\delta}{\delta\delta f_a}\Big)\rho_{\text{nucl}} = 0$$

$$\rho_{\text{nucl}}[\phi_{\text{meta}}] = \frac{1}{Z_{\text{meta}}}\, e^{-\beta H_{\text{eff}}},$$

$$\rho_{\text{nucl}}[\phi_{\text{stable}}] = 0$$

Nucleating distribution

- Thermal flow from metastable phase

$$\rho_{\text{nucl}} = \underbrace{\theta(u)}_{\substack{\text{flow from}\\\text{metastable}\\\text{phase}}} \; \underbrace{Z_{\text{meta}}^{-1}\, e^{-\beta H_{\text{eff}}^{\text{lin}}}}_{\substack{\text{in thermal}\\\text{equilibrium}}}$$

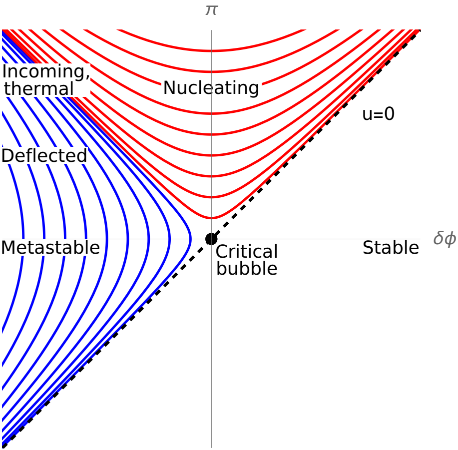


Figure: 2D slice of the phase space

Nucleation rate from the distribution

- Rate: nucleating probability flux
- Probability current:

$$J = \begin{pmatrix} \dot{\delta\phi} & \dot{\pi} & \dot{\delta f_a} \end{pmatrix}^{\mathsf{T}} \rho_{\text{nucl}}$$

- Integrate over the surface

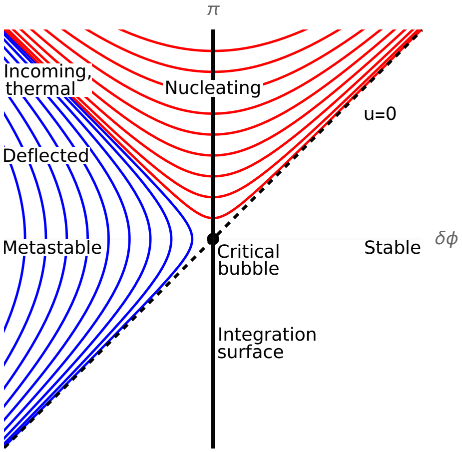


Figure: Integration surface

Form of the results

- Matches Langer
  - Suppression form ϕ_{CB}
  - Fluctuations around ϕ_{CB}
  - Exponential growth rate, κ
- Rate formula seems universal
- Equilibrium part unchanged!
  [Gould & Hirvonen '21]
- Particles affect exponential growth
  - QFT effects in κ

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi}\left(\frac{H_{\text{CB}}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\boldsymbol{\nabla}^2 + V''_{\text{meta}}\right)}{\det'\left(-\boldsymbol{\nabla}^2 + V''_{\text{CB}}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\text{CB}}},$$

$$H_{\text{CB}} = H_{\text{eff}}[\phi_{\text{CB}}] - H_{\text{eff}}[\phi_{\text{meta}}]$$

Exponential growth

- Time derivatives yield the growth rate, κ:

$$\partial_t\left(\overline{\delta\phi}\, e^{\kappa t}\right) = \kappa\left(\overline{\delta\phi}\, e^{\kappa t}\right)$$

- Incoming particles
  - Initially equilibrium
  - Bubble growth $\rightarrow$off equilibrium
  - Inhibit the growth
- Not exponential suppression
  - Leading equilibrium $\log\Gamma$ correct

$$\kappa^2\overline{\delta\phi} = (\boldsymbol{\nabla}^2 - V''_{\mathrm{CB}})\overline{\delta\phi} - \sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}\overline{\delta f_a}$$

$$\kappa\overline{\delta f_a} = -\mathbf{v}\cdot\boldsymbol{\nabla}\overline{\delta f_a} - \frac{f'_{\mathrm{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\kappa\overline{\delta\phi}$$

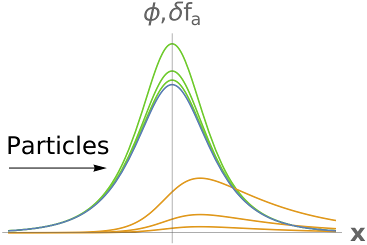


Figure: Exponentially growing configuration, δf_a comes from the left

Bosonic IR divergence

- Source of off equilibrium has $f'_{\mathrm{eq},a}$
  - Dominant fermionic: $E \sim T$
  - Dominant bosonic: $E \ll T$
- Bosonic fields damp more than the particles
- Separation of bosonic $E \sim T$ easy in dim reg

$$\kappa\overline{\delta f_a} = -\mathbf{v}\cdot\nabla\overline{\delta f_a} - \frac{f'_{\mathrm{eq},a}}{2E}\frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\kappa\overline{\delta\phi}$$

$$\Rightarrow \overline{\delta f_a} \propto \frac{f'_{\mathrm{eq},a}}{E}$$

$$\int_{\Lambda_{\mathrm{IR}}}^{\infty}\mathrm{d}E\, f'_{\mathrm{eq,ferm}} = -\frac{1}{2} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{IR}}}{T}\right)$$

$$\int_{\Lambda_{\mathrm{IR}}}^{\infty}\mathrm{d}E\, f'_{\mathrm{eq,bos}} = -\frac{T}{\Lambda_{\mathrm{IR}}} + \frac{1}{2} + \mathcal{O}\left(\frac{\Lambda_{\mathrm{IR}}}{T}\right)$$

Boltzmann equations are OK!

$$\kappa^2\overline{\delta\phi} = (\nabla^2 - V''_{\rm CB})\overline{\delta\phi} - \sum_a \frac{dm_a^2}{d\phi}\int \frac{d^3\mathbf{p}}{(2\pi)^3 2E}\overline{\delta f_a}$$

$$\kappa\overline{\delta f_a} = -\mathbf{v}\cdot\nabla\overline{\delta f_a} - \frac{f'_{{\rm eq},a}}{2E}\frac{dm_a^2}{d\phi}\kappa\overline{\delta\phi} + C_{\rm lin}[\overline{\delta f_a}]$$

- Previous problem:
  - No fluctuations
  - No nucleation
- The exponential configuration:
  - Already nucleated
  - No (need for) fluctuations
- Langer's universality
- Fluctuation-dissipation relation

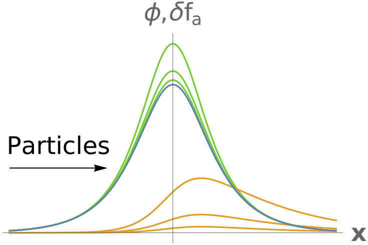


Figure: Exponentially growing configuration, δf_a comes from the left

Summary

- High-temperature nucleation with off-equilibrium-particle effects
- Found a Langerian nucleation rate
  - Particles affect the exponential growth rate, κ
  - Corroborates equilibrium computations
- Bosonic fields more important than the particles

$$\frac{\Gamma}{V} = \frac{\kappa}{2\pi}\left(\frac{H_{\rm CB}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det\left(-\boldsymbol{\nabla}^2 + V''_{\rm meta}\right)}{\det'\left(-\boldsymbol{\nabla}^2 + V''_{\rm CB}\right)}\right|^{\frac{1}{2}} e^{-\beta H_{\rm CB}}$$

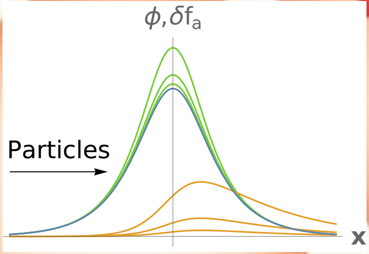


Fig: arXiv:1906.00480

Outlook

- Realistic models, and effects on
  - Duration of the transition
  - Nucleation temperature
- Langer's rate applicable at all? [Pîrvu, Shkerin Sibiryakov '24]
- Finding bosonic-field damping [Ekstedt '22]

$$\Delta t = \left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\Gamma\right)^{-1}$$

$$8\pi v_w^3 \Delta t^4 \Gamma \approx 1 \Rightarrow T_n$$

$$t_{\text{thermalization}} \lesssim \frac{H_{CB}}{\kappa T}$$

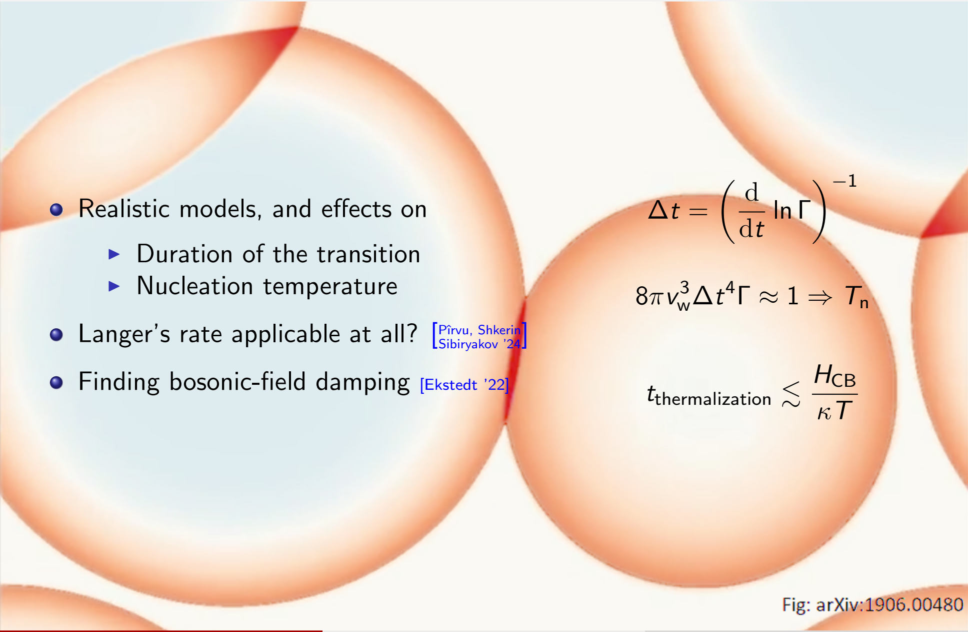


Fig: arXiv:1906.00480

$u = 0$ surface

- Not originating from either of the phases
- E.g. exponential growth:

  $$\delta\phi = \overline{\delta\phi}\, e^{\kappa t}, \quad \delta f_a = \overline{\delta f_a}\, e^{\kappa t}$$

  - Asymptotically from the critical bubble, $t \to -\infty$

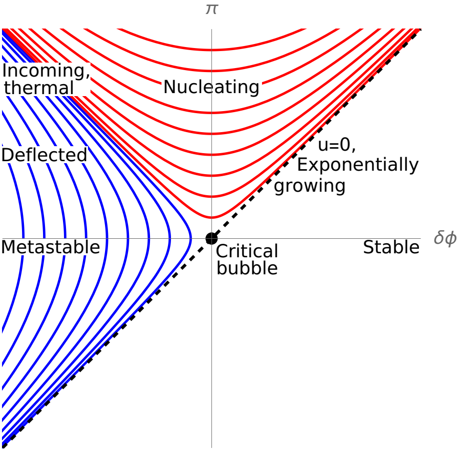


Figure: 2D slice of the phase space