Non-invertible symmetries and Axions

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January 25, 2024

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Introduction

Non-invertible symmetries:

- Old mantra: ‘Symmetries are generated by groups’.
- Topological operators don’t form a group structure.
- In $d > 2$ recent development
  - 2021 [Choi, Cordova, Hsin, Lam, Shao]
  - 2021 [Kaidi, Ohmori, Zheng]
- Categorical symmetries
  - 2022 [Bhardwaj, Schafer-Nameki, Wu]
  - 2022 [Bhardwaj, Bottini, Schafer-Nameki, Tiwari]
  - 2022 [Bhardwaj, Schafer-Nameki,Tiwari]

Outline

1. Non-invertible symmetries

2. Axions

3. Applications

Non-invertible symmetries

Non-invertible symmetries

Previous lectures higher-form symmetries:

1. Generated by topological operators
2. Group-law composition

Group law composition:

1. $\forall g_1, g_2, \quad U_{g_1} \circ U_{g_2} = U_{g_3} \in G$
2. $\exists U_{g^{-1}}, \quad U_g \circ U_{g^{-1}} = e \in G$

Uses: Ward identities, selection rules, anomalies

Non-invertible symmetries

Relax second constraint, topological operators satisfy a fusion algebra:

$$U_a \circ U_b = \sum_c N_{ab}^c U_c \tag{1}$$

Group becomes an algebra! (Fusion category)

Representations $R_1 \otimes R_2 = \sum_{12}^i \oplus R_i$.

Non-invertible symmetries

Example: $U(1)$ Abelian gauge theory.

Gauge invariant extended operators:

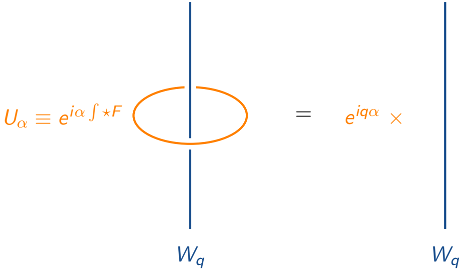
$$W_q = e^{iq \int A} \qquad U_\alpha = e^{i\alpha \int \star F}$$

$$q \in \mathbb{Z} \qquad \alpha \in \mathbb{R}/\mathbb{Z}$$

Non-invertible symmetries

Example: $U(1)$ Abelian gauge theory.

Symmetry structure: $U(1)_e^{(1)} \times U(1)_m^{(1)}$.



Interaction with other symmetries like charge conjugation C?

Non-invertible symmetries

Example: $U(1)$ Abelian gauge theory.

Under charge conjugation $A_\mu \to -A_\mu$.

This acts on Wilson lines and charges as

$$W_q \equiv e^{iq\int A} \to W_{-q}\,, \tag{2}$$

$$U_\alpha \equiv e^{i\alpha\int \star F} \to U_{-\alpha}\,. \tag{3}$$

Operators U_0 and U_π are gauge invariant and also

$$\widetilde{U}_\alpha = U_\alpha + U_{-\alpha} \quad \alpha \in (0,\pi)\,. \tag{4}$$

Non-invertible symmetries

Example: $U(1)$ Abelian gauge theory.

What is the algebra of the new symmetry operators $\widetilde{U}$?
For $\alpha \neq \beta$

$$\begin{aligned}\widetilde{U}_\alpha \circ \widetilde{U}_\beta &= (U_\alpha + U_{-\alpha})(U_\beta + U_{-\beta}) \\ &= \widetilde{U}_{\alpha+\beta} + \widetilde{U}_{\alpha-\beta}\end{aligned}$$

Unable to find an inverse $\to$ non-invertible

Resulting gauge theory is $O(2) = U(1) \rtimes \mathbb{Z}_2$,

2021 [Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela]

Non-invertible symmetries

Example: 2D Ising CFT.

Three topological line operators id, η, D with fusion:

$$\eta \otimes \eta = id\,, \tag{5}$$

$$\eta \otimes D = D \otimes \eta = D\,, \tag{6}$$

$$D \otimes D = \eta + id\,. \tag{7}$$

Non-invertible symmetries

Example: 2D Ising CFT.

$$\eta \otimes \eta = id\,. \tag{8}$$

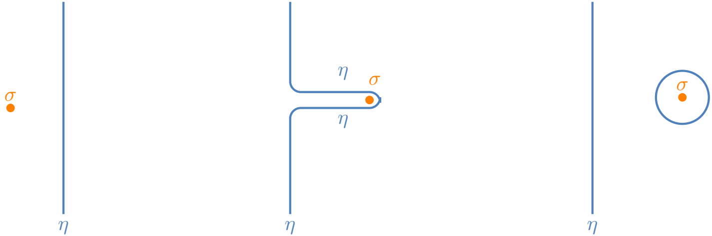


Figure: η defect.

Non-invertible symmetries

Example: 2D Ising CFT.

$$D \otimes D = \eta + id \,. \tag{9}$$

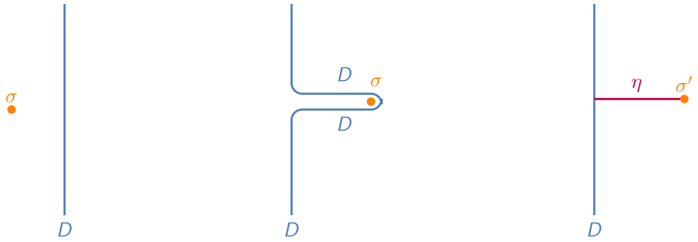


Figure: Figure taken from [2305.18296].

Witten effect!

Axions

Axion

The axion:

- Non-observation of CP violation in strong sector.
- Excellent DM or DE candidate.
- Test simple GUTs
- Strong interplay with QG and ST

Rely on (approximate) shift symmetry!

Axion EFT

The axion a is a compact field $a \equiv a + 2\pi$

(Approximate) shift symmetry $a \to a + \alpha$.

Axion-Maxwell Lagrangian

$$\mathcal{L}_{\rm axMax} \supset \frac{F_a^2}{2}\partial_\mu a \partial^\mu a - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \tag{10}$$

Axion-photon

Axion-photon coupling

$$\mathcal{L}_{\text{axMax}} \supset \frac{a}{16\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}\,, \qquad \widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{11}$$

Chern-Pontryagin density

$$F_{\mu\nu}\widetilde{F}^{\mu\nu} = \partial_\alpha\left(\epsilon^{\alpha\beta\mu\nu}A_\beta F^{\mu\nu}\right) \tag{12}$$

No perturbative effect!

What about the shift symmetry?

What is protecting the axion mass?

Axion-photon

Current

$$j^{\mu}_{\mathrm{PQ}} = \partial^{\mu} a \tag{13}$$

Not conserved!

$$\boxed{\partial_{\mu} j^{\mu}_{\mathrm{PQ}} + \frac{1}{16\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} = 0} \tag{14}$$

Improved current

$$j^{\mu} = j^{\mu}_{\mathrm{PQ}} + \frac{1}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} A_{\nu} F_{\alpha\beta} \tag{15}$$

This current is conserved!

$$\partial_{\mu} j^{\mu} = 0 \tag{16}$$

Axion-Non inv.

The 'improved' operator

$$"D_\alpha" = e^{i\alpha \int j^0} \tag{17}$$

generates shift $a \to a + \alpha$.

For a transformation $\alpha = \frac{2\pi}{N}$ with $N \in \mathbb{Z}$

$$"D_{\frac{2\pi}{N}}" = \exp\left[i\frac{2\pi}{N}\int j^0_{\mathrm{PQ}}\right]\exp\left[i\frac{1}{8\pi N}\int \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\right] \tag{18}$$

What about gauge invariance of A^μ?

What about non-perturbative effects?

Axion-Non inv.

Can we make it well defined for $\alpha = \frac{2\pi}{N}$?

Trick: attach additional topological field theory

$$D_{\frac{2\pi}{N}} = \exp\left[i\frac{2\pi}{N}\int j^0_{\mathrm{PQ}}\right]\exp\left[i\Gamma[N, A^\mu]\right] \tag{19}$$

Such that integrating out d.o.f.

$$\exp\left[i\Gamma[N, A^\mu]\right] \text{ " } = \text{ " } \exp\left[i\frac{1}{8\pi N}\int \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\right] \tag{20}$$

Axion-Non inv.

The effective action is

$$\int DB_\mu \exp\left[\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho} B_\mu \left(\frac{N}{2} F^{(B)}_{\nu\rho} + F^{(A)}_{\nu\rho}\right)\right] \tag{21}$$

Integrating out B_μ generates

$$\exp\left[i\frac{1}{8\pi N}\int \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}\right] \tag{22}$$

One way of seeing this is E.O.M.

$$N F^{(B)}_{\nu\rho} = F^{(A)}_{\nu\rho} \tag{23}$$

Axion-Non inv.

Redefine our operators

$$D_{\frac{2\pi}{N}} \equiv \exp\left[i\frac{2\pi}{N}\int j^0_{\mathrm{PQ}}\right]\exp\left[i\Gamma[N, A^\mu]\right] \tag{24}$$

This operator is gauge invariant and conserved!

2022 [Choi, Lam, Shao]

What's the physics?

Axion-Non inv.

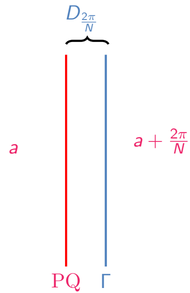


Figure: Axion domain wall

TQFT on axion domain walls!

Axion-Non inv.

Fractional Wilson line $N \in \mathbb{Z}$

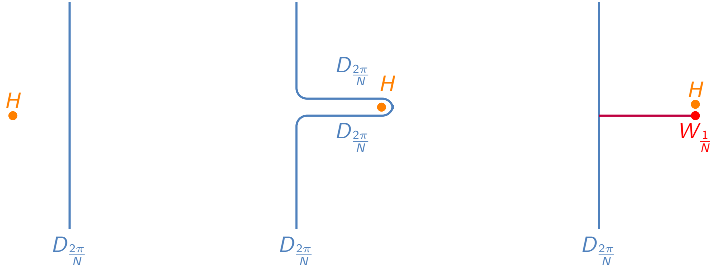
$e^{i\frac{1}{N}\int A}$ $\qquad$ $W_{\frac{1}{N}} = e^{i\frac{1}{N}\int F}$

By $F = dA$ we have

$$e^{i\frac{1}{N}\int F} \text{ " } = \text{ " } e^{i\frac{1}{N}\int A} \tag{25}$$

Axion-Non inv.

What's the physics?



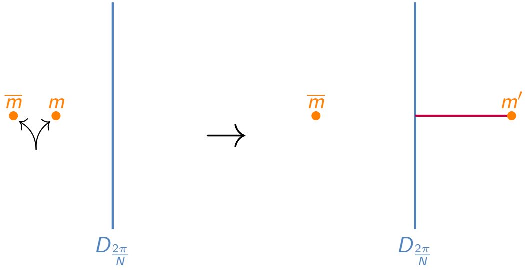
Fractionally charged particles on domain wall!

Applications

Application I

What about non-perturbative physics?

Magnetic monopoles with mass m_M



Break non-invertible symmetry!

Application I

Axion mass from loops magnetic monopoles
2021 [Fan, Fraser, Reece, Stout]

Monopole of mass m_M with rotor size ℓ_σ

$$F_a^2 m_a^2 \sim \frac{m_M^3}{\ell_\sigma} e^{-2\pi\sqrt{m_M \ell_\sigma}} \tag{26}$$

For 't Hooft-Polyakov monopoles

$$e^{-2\pi\sqrt{m_M \ell_\sigma}} \to e^{-\frac{2\pi}{\alpha_W}} \tag{27}$$

Application I

Massless charged Dirac fermion (2022 [Cordova, Ohmori])

$$\mathcal{L} \supset i\overline{\Psi}\not{D}\Psi \tag{28}$$

Chiral symmetry

$$j^{\mu}_{A} = i\overline{\Psi}\gamma^{5}\gamma^{\mu}\Psi\,, \qquad \partial_{\mu}j^{\mu}_{A} = \frac{1}{16\pi^{2}}F\widetilde{F} \tag{29}$$

Magnetic monopoles scale $\sim v \implies$ fermion mass

$$m_f \sim \exp\left(-\frac{2\pi}{\alpha'(v)}\right) \tag{30}$$

Application II

Axion coupled to Non-Abelian group

$$G = \frac{SU(N)}{\mathbb{Z}_N} \, . \tag{31}$$

Interesting because SM gauge group

$$G = \frac{SU(3)_C \times SU(2)_L \times U(1)_Y}{\Gamma} \, , \qquad \Gamma = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6 \, . \tag{32}$$

2017 [Tong]

Example: $G = \frac{SU(N)}{\mathbb{Z}_N}$.

Axion coupled as

$$\mathcal{L} \supset \frac{La}{16\pi^2} \mathrm{Tr}\left(G_{\mu\nu} \widetilde{G}^{\mu\nu} \right) , \quad L \in \mathbb{Z} \tag{33}$$

Instantons break $U(1) \to \mathbb{Z}_L$.

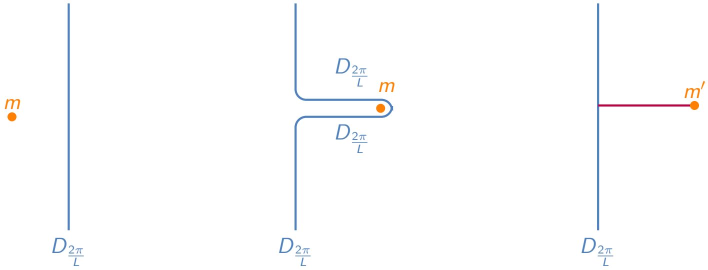
But! Less electric reps $\implies$ more magnetic reps.

Electric reps only adjoint $\implies$ magnetic monopoles fundamental.

Application II

Example: $G = \frac{SU(N)}{\mathbb{Z}_N}$.

Electric reps only adjoint $\implies$ magnetic monopoles fundamental.



Thus only $D_{\frac{2\pi N}{L}}$ invertible!

Application II

Example: $G = \frac{SU(N)}{\mathbb{Z}_N}$.

Embedding

$$SU(N^2-1) \to \frac{SU(N)}{\mathbb{Z}_N} \tag{34}$$

Breaking with fundamental $\implies$ finite mass monopoles.

Non-invertible shift symmetry broken

$$\mathbb{Z}_L \to \mathbb{Z}_{\frac{L}{N}} \tag{35}$$

Application II

(Partially) alleviating axion domain wall problem.

Axion with shift symmetry $\mathbb{Z}_L$.

Break to invertible subgroup $\mathbb{Z}_{\frac{L}{N}} \supset \mathbb{Z}_L$.

2023 [Cordova, Hong, Wang]

Application III

Example: SM with $\Gamma = \mathbb{Z}_6$.

SM gauge group

$$G = \frac{SU(3)_C \times SU(2)_L \times U(1)_Y}{\mathbb{Z}_6} . \tag{36}$$

Below EW breaking scale[1]

$$G = \frac{SU(3)_C \times U(1)_{\widetilde{EM}}}{\mathbb{Z}_3} \tag{37}$$

Coloured magnetic monopoles

[1] Here $3A_{\widetilde{EM}} = A_{EM}$

QCD axion couples as

$$\mathcal{L} \supset \frac{Na}{8\pi^2}\mathrm{Tr}\left(G_{\mu\nu}\widetilde{G}^{\mu\nu}\right) + \frac{Ea}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}, \tag{38}$$

where $N \in \frac{1}{2}\mathbb{Z}$ is the primordial anomaly with QCD and E is the primordial anomaly $U(1)_{\mathrm{PQ}}$ with $U(1)_{\mathrm{em}}$.

What E and N are allowed?

Axion $a \to a + 2\pi$ domain wall has to be invertible.

Application III

QCD axion couples as

$$\mathcal{L} \supset \frac{Na}{8\pi^2}\mathrm{Tr}\left(G_{\mu\nu}\widetilde{G}^{\mu\nu}\right) + \frac{Ea}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}, \tag{39}$$

Implication, only valid representation if

$$\frac{4N}{3} + E \in \mathbb{Z} \tag{40}$$

2023 [Reece]
2023 [Choi, Forslund, Lam, Shao]

Application III

For SM with $\Gamma = \mathbb{Z}_6$

$$\frac{4N}{3} + E \in \mathbb{Z} \tag{41}$$

QCD axion-photon coupling

$$g_{a\gamma\gamma} = \frac{N\alpha_{EM}}{\pi F_a}\left(\frac{E}{N} - 1.92(4)\right) \tag{42}$$

If no domain wall problem $N = \frac{1}{2}$, then smallest value $|g_{a\gamma\gamma}|$ for $\frac{E}{N} = \frac{8}{3}$.

Assumption: no other axion mixing!
2022 [Agrawal, Nee, Reig]

Discussion

- Non-invertible symmetries.
- Connection with axion physics.
- Applications.