# Perturbative and non-perturbative aspects of flavour oscillations

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## Summary

1. Quantum Field Theory of neutrino mixing and oscillations

2. Mixing in the interaction picture: QM toy model, boson field, neutrinos

3. Chiral oscillations

4. Other aspects

# Motivations

• CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

• Theoretical interest: origin of mixing in the Standard Model;

• Bargmann superselection rule*: coherent superposition of states with different masses is not allowed in non-relativistic QM;

• Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles†; oscillation formulas‡.

*V. Bargmann, Ann. Math. (1954); D.M. Greenberger, Phys. Rev. Lett. (2001).
†C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993). C. Giunti, J. Phys. G (2007).
‡M. Beuthe, Phys. Rept. (2003).

# Neutrino oscillations in QFT: a brief (early) history

- Pontecorvo theory[*]
- Vacuum-condensate structure and neutrino oscillations[†]
- First attempts to define flavor Fock space[‡][§]
- External wavepackets[¶]
- Flavor Fock space approach[‖].

[*] V. Gribov and B. Pontecorvo, Phys. Lett. B (1969).
[†] L.N. Chang and N.P. Chang, Phys. Rev. Lett. (1980).
[‡] P.T. Mannheim, Phys. Rev. D **37**, 1935 (1988).
[§] C. Giunti, C.W. Kim and U.W. Lee, Phys. Rev. D (1992).
[¶] C. Giunti, C.W. Kim, J.A. Lee and U.W. Lee, Phys. Rev. D (1993).
[‖] M.B. and G.Vitiello, Ann. Phys. (1995).

# Neutrino oscillations in QM *

Pontecorvo mixing relations

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle + \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta\,\sin^2\left(\frac{\Delta E}{2}t\right) = 1 - P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 + |\langle\nu_\mu|\nu_e(t)\rangle|^2 = 1$$

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*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

# QFT Lagrangian (flavor basis)

Free Lagrangian:

$$\mathcal{L}_0 \;=\; \sum_{\sigma,\rho=e,\mu} \left[\overline{\nu}_\sigma \left(i\gamma_\mu\partial^\mu - M_\nu^{\sigma\rho}\right)\nu_\rho + \bar{l}_\sigma \left(i\gamma_\mu\partial^\mu - M_l^{\sigma\rho}\right) l_\rho\right]$$

where $l_e \equiv e$, $l_\mu \equiv \mu$, and:

$$M_\nu \;=\; \begin{bmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{bmatrix} \quad ; \qquad M_l \;=\; \begin{bmatrix} \tilde{m}_e & 0 \\ 0 & \tilde{m}_\mu \end{bmatrix}$$

Interaction term (charged current):

$$\mathcal{L}_{int} \;\;=\;\; \frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \left[W_\mu^+(x)\,\overline{\nu}_\sigma\,\gamma^\mu\,(1-\gamma^5)\,l_\sigma + h.c.\right]$$

# Mixing transformation

Kinetic part diagonalized by mixing transformation

$$\nu_\sigma(x) \ = \ \sum_j U_{\sigma\, j} \nu_j(x)$$

between flavor fields $\nu_\sigma$ and mass fields $\nu_j$. $U$ is the mixing matrix

$$U \ = \ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

with

$$\tan 2\theta \ = \ 2m_{e\mu}/(m_e - m_\mu)$$

# QFT Lagrangian (mass basis)

In the mass basis

$$\mathcal{L}_0 = \sum_{j=1,2} \overline{\nu}_j \left(i\gamma_\mu \partial^\mu - m_j\right) \nu_j + \sum_{\sigma=e,\mu} \bar{l}_\sigma \left(i\gamma_\mu \partial^\mu - \tilde{m}_\sigma\right) l_\sigma$$

where

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} m_e \\ m_\mu \end{bmatrix}$$

Interaction term is no-more diagonal

$$\mathcal{L}_{int} = \frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \sum_{j=1,2} \left[W_\mu^+(x)\, \overline{\nu}_j\, U^*_{j\sigma}\, \gamma^\mu \left(1-\gamma^5\right) l_\sigma + h.c.\right]$$

In computing an amplitude $\langle \nu_\sigma l_\sigma^+ P_F | S | P_I \rangle$, what is definition of $|\nu_\sigma\rangle$?

# Pontecorvo flavor states

Pontecorvo states*:

$$|\nu^r_{\mathbf{k},e}\rangle_P = \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P = -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

Consider the amplitude of the neutrino detection process $\nu_\sigma + X_i \to e^- + X_f$:

$$\langle e^s_{\mathbf{q},-}|\bar{e}(x)\,\gamma^\mu\,(1-\gamma^5)\,\nu_e(x)|\nu^r_{\mathbf{k},\sigma}\rangle_P\,h_\mu(x) \not\propto \delta_{\sigma e}$$

where $h_\mu$ are the matrix elements of the $X$ part.

Problem: since neutrino flavor is defined by the charged-lepton, the above amplitude should be proportional to $\delta_{\sigma e}$.

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*S.M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978)

# Weak process states

Other proposal: Production (detection) states[†]:

$$|\nu_\sigma^r\rangle_{P(D)} \equiv \Big(\sum_j |\mathcal{A}_{\sigma j}^{P(D)}|^2\Big)^{-\frac{1}{2}} \sum_j \mathcal{A}_{\sigma j}^{P(D)} \, |\nu_j^r\rangle$$

where $\mathcal{A}_{\sigma j}^P = \langle \nu_j l_\sigma^+ P_F | S | P_I\rangle$ and $\mathcal{A}_{\sigma j}^D = \langle l_\sigma^+ X_F | S | X_I \nu_j\rangle$. Flavor states definition depends on the process.

Oscillation probability

$$P_{\sigma\to\rho}(L) \ = \ N \sum_{j,k} \mathcal{A}_{\sigma j}^P \mathcal{A}_{\rho j}^{D*} \mathcal{A}_{\sigma k}^{P*} \mathcal{A}_{\rho k}^D e^{-i\frac{\delta m_{kj}^2 L}{2E}}$$

with $\delta m_{kj}^2 \equiv m_k^2 - m_j^2$ and

$$N \ \equiv \ \Big(\sum_j |\mathcal{A}_{\sigma j}^P|^2\Big)^{-\frac{1}{2}} \Big(\sum_k |\mathcal{A}_{\rho k}^D|^2\Big)^{-\frac{1}{2}}$$

[†]C. Giunti and C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford Univ. Press, 2007)

# Quantum Field Theory of neutrino mixing and oscillations

# Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

can be written as*

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\,\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For $\nu_e$, we get $\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$ , $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

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*M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\;\ln\left(1-\sin^2\theta\,|V_{\mathbf{k}}|^2\right)^2} = 0$$

with

$$|V_{\mathbf{k}}|^2 \;\;\equiv\;\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^{s}_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

# Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite $\sharp$ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite $\sharp$ of degrees of freedom.

- $\infty$ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

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*F. Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[ (1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\, \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}||U_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\, |V_{\mathbf{k}}|^2\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} \Big] |0\rangle_{1,2}$$

- Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\, |V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

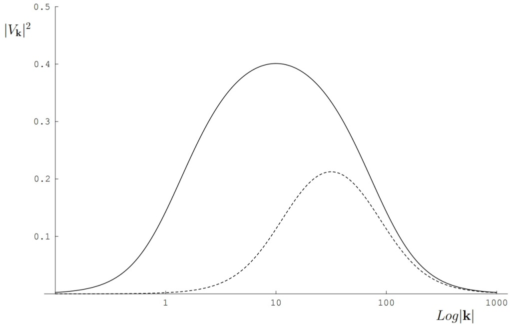
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensate: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

# Condensation density for mixed fermions




Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \quad = \quad u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

# Decomposition of mixing generator *

The mixing generator can be expressed in terms of a rotation and a Bogoliubov transformation. Define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\ \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1,2$$

Since $[B_1, B_2] = 0$ we put $B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$.

- We find:

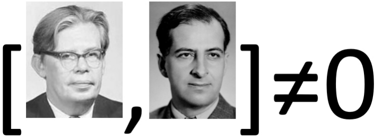
$$G_\theta \ =\ B(\Theta_1, \Theta_2)\ \ R(\theta)\ \ B^{-1}(\Theta_1, \Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k}\ \cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\ \sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})$$

*M. B., M.V. Gargiulo and G. Vitiello, Phys. Lett. B (2017)

- Bogoliubov and Pontecorvo do not commute!




As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \left[B(m_1, m_2)\,,\; R^{-1}(\theta)\right]\; |\widetilde{0}\rangle_{1,2}$$

with $|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2}$.

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\; \tilde{V}_\mathbf{k} \sum_r \epsilon^r \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2 - m_1)^2}{m_1 m_2}$.

# Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} \,=\, \bar{\nu}_m \left(i \not\!\partial - M_d\right) \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge $Q$= total charge.

  – Consider now the $SU(2)$ transformation:

$$\nu'_m \,=\, e^{i\alpha_j \tau_j}\, \nu_m \qquad ; \qquad j \,=\, 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and $\sigma_j$ being the Pauli matrices.

*M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} = i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m = -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} = \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] = i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for $\nu_1$ and $\nu_2$:

$$Q_1 = \frac{1}{2}Q + Q_{m,3} = \int d^3\mathbf{x}\,\nu_1^\dagger(x)\,\nu_1(x)$$

$$Q_2 = \frac{1}{2}Q - Q_{m,3} = \int d^3\mathbf{x}\,\nu_2^\dagger(x)\,\nu_2(x).$$

These are the flavor charges in the absence of mixing.

# The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left( i \not\partial - M \right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu'_f = e^{i\alpha_j \tau_j} \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and $\sigma_j$ being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x}\, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i\, \epsilon_{jkl}\, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2} Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between $\nu_e$ and $\nu_\mu$.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$Q_e(t) = \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) = \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x}\; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \sum_r \int d^3\mathbf{k}\, \left( \alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu.$$

Here :: ... :: denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv A \,-\, {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \,\equiv\, \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\, |0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle \,=\, |\nu^r_{\mathbf{k},\sigma}\rangle$$

# Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\begin{aligned}\mathcal{Q}_{\mathbf{k},\sigma}(t) &\equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle \\ &= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2\end{aligned}$$

with $Q_\sigma(t) \equiv \int d^3\mathbf{x}\, \nu^\dagger_\sigma(x)\, \nu_\sigma(x)$.

- Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0 \Rightarrow$ Pontecorvo formula is recovered.

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*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

# Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\, |\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\, |\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\, |\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\, |\nu^r_{\mathbf{k},2}\rangle \;,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}| :Q_e(0): |\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\, \sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

## Other results

- Rigorous mathematical treatment for any number of flavors [*]
- Three flavor fermion mixing: CP violation[†];
- QFT spacetime dependent neutrino oscillation formula[‡];
- Boson mixing[§]; Majorana neutrinos[¶];
- Flavor vacuum and cosmological constant[‖]
- Flavor vacuum induced by condensation of D-particles.[**]
- Geometric phase for mixed particles[††].

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[*] K. C. Hannabuss and D. C. Latimer, J. Phys. A (2000); J. Phys. A (2003);

[†] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

[‡] M.B., P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).

[§] M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D**(2001); M.Binger and C.R.Ji. Phys. Rev. **D**(1999); C.R.Ji and Y.Mishchenko, Phys. Rev. **D**(2001); Phys. Rev. **D**(2002).

[¶] M.B. and J.Palmer, Phys. Rev. **D** (2004)

[‖] M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);

[**] N.E.Mavromatos, S.Sarkar and W.Tarantino, Phys. Rev. D (2008); Phys. Rev. D (2011).

[††] M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999)

# Neutrino ontology: flavor or mass?

- In view of the unitary inequivalence of mass and flavor representations, we have the problem of the fundamental (ontological) nature of neutrino.

*Flavor or mass, that is the question...*

# Neutrino ontology: research directions

- How to verify the fundamental nature of neutrino states?

Two directions:

- Investigate the phenomenology of flavor neutrinos, with corrections expected in the non-relativistic regime: oscillations, beta decay endpoint, quantum correlations, ...
- Use the formal consistency of QFT, by comparing neutrino processes in two different frames (inertial and comoving) for accelerated particle: Unruh effect.*

*M. B., G. Lambiase, G. Luciano and L.Petruzziello, Phys. Rev. D (2018); G.Cozzella, S.Fulling, A.Landulfo, G.Matsas and D.Vanzella, Phys.Rev.(2018) M. B., G.Lambiase, G. Luciano and L.Petruzziello, Phys. Lett. B (2020)

- Dynamical generation of fermion mixing*.
- Flavor-energy uncertainty relations for mixed states†.
- Poincaré invariance for flavor neutrinos‡.
- Violation of equivalence principle§.

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*M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2019)
†M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)
‡M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2020) ; A. E. Lobanov, Ann. Phys. (2019)
§M.B., P.Jizba, G.Lambiase and L.Petruzziello, Phys. Lett. B (2020)

# Flavor neutrino as unstable particles

- Time-energy uncertainty relations (TEUR) in the Mandelstam–Tamm form, furnish lower-bounds on neutrino energy uncertainty compatible with flavor oscillations*.
- QFT formulation of neutrino oscillations suggests that these bounds can be read as flavor-energy uncertainty relations (FEUR)†. Energy uncertainty is connected with the intrinsic unstable nature of flavor neutrinos.

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*S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

†M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)

# Time-energy uncertainty relations

Mandelstam–Tamm TEUR is*:

$$\Delta E \, \Delta t \, \geq \frac{1}{2}$$

where

$$\Delta E \equiv \sigma_H \qquad \Delta t \equiv \sigma_O / \left| \frac{\mathrm{d}\langle O(t) \rangle}{\mathrm{d}t} \right|$$

Here $\langle \ldots \rangle \equiv \langle \psi | \ldots | \psi \rangle$ and $O(t)$ represents the "clock observable" whose dynamics quantifies temporal changes in a system.

– The above inequality is obtained by means of the Cauchy-Schwarz inequality and using the fact that $[\hat{O}, \hat{H}] \neq 0$.

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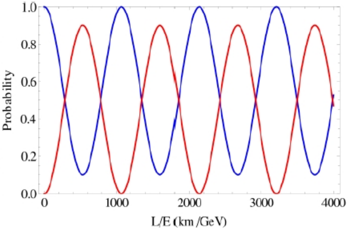
*L. Mandelstam and I.G. Tamm, J. Phys. USSR (1945)

# Clock observables



# Flavor-energy uncertainty relations

Choose flavor charges as clock observables. Then $[Q_{\nu_\sigma}(t)\,,\,H] \neq 0 \Rightarrow$ flavor-energy uncertainty relation[†]:

$$\langle \Delta H \rangle \, \langle \Delta Q_{\nu_\sigma}(t) \rangle \;\geq\; \frac{1}{2}\left|\frac{\mathrm{d}\langle Q_{\nu_\sigma}(t)\rangle}{\mathrm{d}t}\right|$$

Taking the state $|\psi\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$ we have $\langle Q_{\nu_\sigma}(t)\rangle = \mathcal{Q}_{\sigma\to\sigma}(t)$ and

$$\langle \Delta Q_{\nu_\sigma}(t) \rangle \;=\; \sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)(1-\mathcal{Q}_{\sigma\to\sigma}(t))} \leq \frac{1}{2}.$$

Integrating over time from 0 to $T$, and using the triangular inequality, we obtain:

$$\Delta E\, T \;\geq\; \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

[†]M. B., P. Jizba and L.Smaldone, Phys. Rev. D (2019)

# Neutrino oscillation condition

When $m_i/|\mathbf{k}| \to 0$:

$$\Delta E \;\geq\; \frac{2\,\sin^2 2\theta}{L_{osc}}$$

This relation is usually interpreted as neutrino oscillation condition[‡].

The situation is similar to that of unstable particles:

$$\Delta E \;\approx\; \frac{1}{2\tau}$$

where the $\tau$ is the particle life-time.

– As for unstable particles only energy distribution are meaningful.
The width of the distribution is related to the oscillation length.

[‡]S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

# Mixing in the interaction picture: QM toy model, boson field, neutrinos

# Motivations

- Both Pontecorvo neutrino states and weak process states are not eigenstates of flavor charge;
- Exact flavor (lepton) charge eigenstates require the introduction of the flavor vacuum which breaks Poincaré invariance*.

Consider another approach: treat the mixing term of the Lagrangian as a perturbation and compute oscillation formula from QFT at finite time†.

*M. B., P. Jizba, N.E. Mavromatos and L. Smaldone, Phys. Rev. D (2020);
†M. B., F. Giacosa, L. Smaldone and G.Torrieri, EPJC (2023)

# Neutrino mixing and time-evolution operator

• Decompose neutrino Lagrangian as ($g = 0$)

$$\mathcal{L} \;=\; \mathcal{L}_0 + \mathcal{L}_{int}$$

with

$$\begin{aligned}\mathcal{L}_0 &= \sum_{\sigma=e,\mu} \overline{\nu}_\sigma \left(i\partial\!\!\!/ - m_\sigma\right) \nu_\sigma \\ \mathcal{L}_{int} &= -m_{e\mu} \left(\overline{\nu}_e \nu_\mu + \overline{\nu}_\mu \nu_e\right)\end{aligned}$$

Time-evolution operator

$$U(t_i, t_f) \;=\; \mathcal{T} \exp\left[-i \int_{t_i}^{t_f} \mathrm{d}^4x \; : \mathcal{H}_{int}(x) : \right]$$

$\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x)$.

• Why finite time? Analogy with unstable particles[‡].

Flavor-energy uncertainty relation[§]

$$\Delta E\, T \,\geq\, \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma\neq\rho$$

It follows (at $T_h$ oscillation probability$=\frac{1}{2}$)

$$\Delta E\, T_h \geq \frac{1}{2}$$

For unstable particles:

$$\Delta E \;\approx\; \frac{1}{2\tau}$$

where the $\tau$ is the particle life-time.

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[‡] C.Bernardini, L.Maiani and M.Testa, Phys. Rev. Lett. (1993).
P. Facchi and S. Pascazio, La regola d'oro di Fermi, (Bibliopolis, 1999).
[§] M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)

# A toy model

$0+1D$ field theory (QM)

$$L = \frac{1}{2}\left(\frac{dx_A}{dt}\right)^2 - \frac{\omega_A^2}{2}x_A^2 + \frac{1}{2}\left(\frac{dx_B}{dt}\right)^2 - \frac{\omega_B^2}{2}y^2 - \omega_{AB}^2 x_A x_B$$

In the interaction picture

$$x_\sigma(t) \;=\; \frac{1}{\sqrt{2\omega_A}}\left(a_\sigma e^{-i\omega_\sigma t} + a_\sigma^\dagger e^{i\omega_\sigma t}\right)\,, \qquad \sigma = A, B$$

"Flavor" states

$$|A\rangle = a_A^\dagger|0\rangle\,, \qquad |B\rangle = a_B^\dagger|0\rangle$$

Interaction term

$$H_{int} = \omega_{AB}^2 x_A(t) x_B(t) = \frac{\omega_{AB}^2}{2\sqrt{\omega_A\omega_B}}\left[a_B^\dagger a_A e^{i(\omega_B-\omega_A)t} + a_B^\dagger a_A^\dagger e^{i(\omega_A+\omega_B)t} + h.c.\right]$$

# Decay probability

- Consider the process


$$|A\rangle \;\to\; |B\rangle$$

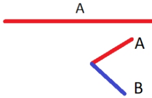
Amplitude at first order in $\omega_{AB}^2$

$$\langle B|U(t_f,t_i)|A\rangle \;=\; \frac{\omega_{AB}^2}{\sqrt{2\omega_A}\sqrt{2\omega_B}}\,\frac{e^{-i(\omega_A-\omega_B)t_f}-e^{-i(\omega_A-\omega_B)t_i}}{(\omega_A-\omega_B)}$$

Transition probability

$$\mathcal{P}_{A\to B}(\Delta t)=\frac{\omega_{AB}^4}{\omega_A\omega_B}\,\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2}\qquad \Delta t=t_f-t_i$$

# Decay probability




- The other non-trivial process

$$|A\rangle \;\to\; |A\rangle|A\rangle|B\rangle$$

Normalized amplitude (first order in $\omega_{AB}^2$)

$$\frac{1}{\sqrt{2}}\langle 0|a_B a_A^2\, U(t_f,t_i)\, a_A^\dagger|0\rangle \;=\; \frac{\sqrt{2}\omega_{AB}^2}{\sqrt{2\omega_A}\sqrt{2\omega_B}}\,\frac{e^{-i(\omega_A+\omega_B)t_f} - e^{-i(\omega_A+\omega_B)t_i}}{(\omega_A+\omega_B)}$$

Hence

$$\mathcal{P}_{A\to AAB}(\Delta t) = \frac{2\omega_{AB}^4}{\omega_A\omega_B}\,\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}$$

# Decay probability

Flavor transition (decay) probability $\mathcal{P}_{A\to B}(\Delta t) + \mathcal{P}_{A\to AAB}(\Delta t)$

$$\mathcal{P}^A_D(\Delta t) \; = \; \frac{\omega^4_{AB}}{\omega_A\omega_B}\left[\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2}+2\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}\right]$$

# Survival probability

Survival amplitude

$$\langle A|U(t_f,t_i)|A\rangle = 1 - i\,\mathcal{T}\langle 0|a_A \int_{t_i}^{t_f} dt_1 H_{int}(t_1) \int_{t_i}^{t_1} dt_2 H_{int}(t_2) a_A^\dagger |0\rangle$$

one gets the survival probability of the state $|A\rangle$ as:

$$\begin{aligned}\mathcal{P}_{A\to A}(\Delta t) &= \Bigg|1 - \frac{\omega_{AB}^4}{4\omega_A\omega_B}\Bigg[2\frac{t}{i(\omega_A+\omega_B)} - 2\frac{e^{-i(\omega_A+\omega_B)\Delta t}-1}{(\omega_A+\omega_B)^2} \\ &\qquad + \frac{t}{-i(\omega_A-\omega_B)} - \frac{e^{i(\omega_A-\omega_B)\Delta t}-1}{(\omega_A-\omega_B)^2}\Bigg]\Bigg|^2 \\ &= |1 - R - iI|^2 = 1 - 2R + \ldots\,,\end{aligned}$$

with

$$R = \frac{\omega_{AB}^4}{2\omega_A\omega_B}\left(\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2} + 2\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}\right),$$

# Survival probability

$$\mathcal{P}_S^A(\Delta t) \;=\; 1 - \frac{\omega_{AB}^4}{\omega_A\omega_B}\left(\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2} + 2\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}\right)$$

Unitarity

$$\mathcal{P}_S^A(\Delta t) + \mathcal{P}_D^A(\Delta t) \;=\; 1$$

## Diagonalization

Of course, the problem can be also solved by introducing the rotation

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with

$$\begin{aligned} \tan 2\theta &= \frac{2\omega_{AB}^2}{\omega_B^2 - \omega_A^2}\,, \\ \omega_A^2 &= \omega_1^2 \cos^2\theta + \omega_2^2 \sin^2\theta \;, \\ \omega_B^2 &= \omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta \;. \end{aligned}$$

Denoting $|\Omega\rangle$ the vacuum of the full Hamiltonian ($a_i\,|\Omega\rangle = 0$, $i = 1, 2$), one may also consider the state

$$|a\rangle = \cos\theta a_1^\dagger\,|\Omega\rangle + \sin\theta a_2^\dagger\,|\Omega\rangle \;,$$

yet it is clear that $|a\rangle \neq |A\rangle = a_A^\dagger\,|a\rangle$,

# Diagonalization

In terms of $|a\rangle$, the survival probability takes the form:

$$\mathcal{P}_S^a(\Delta t) \; = \; 1 - \sin^2 2\theta \sin^2 \left[\frac{(\omega_1 - \omega_2)\Delta t}{2}\right] .$$

In the limit of small $\theta$, the previous expression is approximated by:

$$\begin{aligned} \mathcal{P}_S^a(\Delta t) \quad &\simeq \quad 1 - \frac{4\omega_{AB}^4}{\left(\omega_B^2 - \omega_A^2\right)^2} \sin^2 \left[\frac{(\omega_A - \omega_B)\Delta t}{2}\right] \\ &= \quad 1 - \frac{4\omega_{AB}^4}{\left(\omega_B + \omega_A\right)^2} \frac{\sin^2\left[\frac{(\omega_A - \omega_B)\Delta t}{2}\right]}{\left(\omega_B - \omega_A\right)^2} \quad . \end{aligned}$$

which is different from the perturbative formula both for the absence of the fast oscillating term and for the amplitude of the standard oscillating term.

# Scalar field mixing

Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\alpha \phi_A\right)^2 - \frac{m_A^2}{2}\phi_A^2 + \frac{1}{2}\left(\partial_\alpha \phi_B\right)^2 - \frac{m_B^2}{2}\phi_B^2 - m_{AB}^2 \phi_A \phi_B$$

In the interaction picture (in a volume $V$ box)

$$\phi_A(x) \; = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}=2\pi\mathbf{n}/L} \frac{1}{\sqrt{2\omega_{\mathbf{k},A}}} \left( a_{\mathbf{k},A} e^{-ikx} + a^\dagger_{\mathbf{k},A} e^{ikx} \right)$$

One-particle state

$$|A, \mathbf{p}\rangle = a^\dagger_{\mathbf{p},A} \, |0\rangle$$

The same for $\phi_B$.

# Decay probability

- Process $|A, \mathbf{p}\rangle \to |B, \mathbf{k}\rangle$

$$\mathcal{A}_{A\to B}\left(\mathbf{p},\mathbf{k};t_i,t_f\right) = \frac{m_{AB}^2}{\sqrt{2\omega_{\mathbf{p},A}}\sqrt{2\omega_{\mathbf{k},B}}}\delta_{\mathbf{k},\mathbf{p}}\frac{e^{-i\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{k},B}\right)t_f}-e^{-i\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{k},B}\right)t_i}}{\omega_{\mathbf{p},A}-\omega_{\mathbf{k},B}}$$

Probability

$$\mathcal{P}_{A\to B}(\mathbf{p};\Delta t) = \sum_{\mathbf{k}}\left|\mathcal{A}_{A\to B}\left(\mathbf{p},\mathbf{k};t_i,t_f\right)\right|^2$$

Then

$$\mathcal{P}_{A\to B}(\mathbf{p};\Delta t) = \frac{m_{AB}^4}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}}\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B}\right)^2}$$

# Decay probability

- The other non-trivial process

$$|A,\mathbf{p}\rangle \to |A,\mathbf{k}_1\rangle\,|A,\mathbf{k}_2\rangle\,|B,\mathbf{k}_3\rangle$$

– When $\mathbf{k}_1 \neq \mathbf{k}_2$

$$\mathcal{P}^{\mathbf{k}_1\neq\mathbf{k}_2}_{A\to AAB}(\mathbf{p};\Delta t) \;=\; \sum_{\mathbf{k}_3}\frac{m^4_{AB}}{\omega_{\mathbf{k}_3,A}\,\omega_{\mathbf{k}_3,B}}\,\frac{\sin^2\left[\frac{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)^2} - \frac{m^4_{AB}}{\omega_{\mathbf{p},A}\,\omega_{\mathbf{p},B}}\,\frac{\sin^2\left[\frac{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}$$

Large $V$

$$\mathcal{P}^{\mathbf{k}_1\neq\mathbf{k}_2}_{A\to AAB}(\mathbf{p};\Delta t) = V\int\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,\frac{m^4_{AB}}{\omega_{\mathbf{k}_3,A}\,\omega_{\mathbf{k}_3,B}}\,\frac{\sin^2\left[\frac{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)^2} - \frac{m^4_{AB}}{\omega_{\mathbf{p},A}\,\omega_{\mathbf{p},B}}\,\frac{\sin^2\left[\frac{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}$$

First piece on the r.h.s. IR divergent vacuum contribution

# Total decay probability

– When $\mathbf{k}_1 = \mathbf{k}_2$

$$\mathcal{P}^{\mathbf{k}_1=\mathbf{k}_2}_{A\to AAB}(\mathbf{p},\Delta t) = 2\frac{m^4_{AB}}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}}\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2} \, .$$

- Total decay probability

$$\mathcal{P}^{A}_{D}(\mathbf{p};\Delta t) \;=\; \mathcal{P}_{A\to B}(\mathbf{p};\Delta t) + \mathcal{P}^{\mathbf{k}_1\neq\mathbf{k}_2}_{A\to AAB}(\mathbf{p};\Delta t) + \mathcal{P}^{\mathbf{k}_1=\mathbf{k}_2}_{A\to AAB}(\mathbf{p};\Delta t)$$

Subtracting the divergent term

$$\mathcal{P}^{A}_{D}(\mathbf{p};\Delta t) = \frac{m^4_{AB}}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}}\left(\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B}\right)^2} + \frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}\right)$$

# Survival probability

- Survival process

$$|A, \mathbf{p}\rangle \rightarrow |A, \mathbf{k}\rangle$$

Amplitude

$$\mathcal{A}_{A\rightarrow A}(\mathbf{p},\mathbf{k};t_i,t_f) = \delta_{\mathbf{k},\mathbf{p}} + (-i)^2 \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_1} dt_2 \langle 0| a_{\mathbf{k},A} H_{int}(t_1) H_{int}(t_2) a^{\dagger}_{\mathbf{p},A} |0\rangle$$

Unitarity

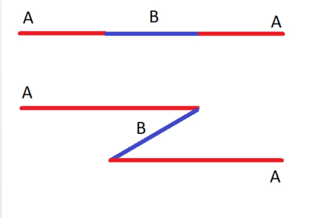
$$\mathcal{P}^{A}_{D}(\mathbf{p};\Delta t) + \mathcal{P}^{A}_{S}(\mathbf{p};\Delta t) \; = \; 1$$

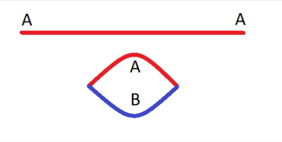
- Survival probability

$$\mathcal{P}_S^A(\mathbf{p};\Delta t) = \sum_{\mathbf{k}} |\mathcal{A}_{A\to A}(\mathbf{p},\mathbf{k};t_i,t_f)|^2$$

$$= 1 - \frac{m_{AB}^4}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}} \frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B})\Delta t}{2}\right]}{(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B})^2} - \frac{m_{AB}^4}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}} \frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B})\Delta t}{2}\right]}{(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B})^2}$$

$$-V\int \frac{d^3\mathbf{q}_1}{(2\pi)^3} \frac{m_{AB}^4}{\omega_{\mathbf{q}_1,A}2\omega_{\mathbf{q}_1,B}} \frac{\sin^2\left[\frac{(\omega_{\mathbf{q}_1,A}+\omega_{\mathbf{q}_1,B})\Delta t}{2}\right]}{(\omega_{\mathbf{q}_1,A}+\omega_{\mathbf{q}_1,B})^2},$$



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# Neutrino oscillations in the interaction picture

Neutrino fields

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[ u^r_{\mathbf{k},\sigma}(t)\, \alpha^r_{\mathbf{k},\sigma} + v^r_{-\mathbf{k},\sigma}(t)\, \beta^{r\dagger}_{-\mathbf{k},\sigma} \right] e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $\sigma = e, \mu$. Spinor normalization

$$u^{r\dagger}_{\mathbf{k},\rho} u^s_{\mathbf{k},\rho} = v^{r\dagger}_{\mathbf{k},\rho} v^s_{\mathbf{k},\rho} \ = \ \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},\rho} v^s_{-\mathbf{k},\rho} = 0$$

Neutrino flavor state

$$|\nu^r_{\mathbf{p},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{p},\sigma} |0\rangle$$

Interaction Hamiltonian

$$H_{int}(t) = m_{e\mu} \sum_{s,s'=1,2} \sum_{\mathbf{p}} \Big[ \beta^{s}_{\mathbf{p},\mu} \beta^{s\dagger}_{\mathbf{p},e} \delta_{ss'} W^{*}_{\mathbf{p}}(t) + \alpha^{r\dagger}_{\mathbf{p},\mu} \alpha^{r}_{\mathbf{p},e} \delta_{ss'} W_{\mathbf{p}}(t)$$

$$+ \quad \beta^{s}_{-\mathbf{p},\mu} \alpha^{s'}_{e,\mathbf{p}} \left( Y^{ss'}_{\mathbf{p}}(t) \right)^{*} + \alpha^{s\dagger}_{\mathbf{p},\mu} \beta^{s'\dagger}_{-\mathbf{p},e} Y^{ss'}_{\mathbf{p}}(t) \, + \, e \leftrightarrow \mu \Big]$$

where

$$W_{\mathbf{p}}(t) \;=\; \overline{u}^{s}_{\mathbf{p},\mu} u^{s}_{\mathbf{p},e} e^{i(\omega_{\mathbf{k},\mu} - \omega_{\mathbf{k},e})t} \;=\; W_{\mathbf{p}}\, e^{i(\omega_{\mathbf{p},\mu} - \omega_{\mathbf{p},e})t}$$

$$Y^{ss'}_{\mathbf{p}}(t) \;=\; \overline{u}^{s}_{\mathbf{p},\mu} v^{s'}_{-\mathbf{p},e} e^{i(\omega_{\mathbf{k},\mu} + \omega_{\mathbf{k},e})t} \;=\; Y^{ss'}_{\mathbf{p}}\, e^{i(\omega_{\mathbf{p},\mu} + \omega_{\mathbf{p},e})t}$$

Explicit form of coefficients:

$$W_{\mathbf{p}} \;=\; \sqrt{\frac{(\omega_{\mathbf{p},e}+m_e)\,(\omega_{\mathbf{p},\mu}+m_\mu)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(1-\frac{|\mathbf{p}|^2}{(\omega_{\mathbf{p},e}+m_e)(\omega_{\mathbf{p},\mu}+m_\mu)}\right)$$

$$Y^{22}_{\mathbf{p}} \;=\; -Y^{11}_{\mathbf{p}} \;=\; \frac{p_3}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right)$$

$$Y^{12}_{\mathbf{p}} \;=\; \left(Y^{21}_{\mathbf{p}}\right)^* \;=\; -\frac{p_1-ip_2}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right)$$

# Decay probability

- Amplitude of the $|\nu^r_{\mathbf{p},e}\rangle \;\rightarrow\; |\nu^s_{\mathbf{k},\mu}\rangle$ process

$$\mathcal{A}^{rs}_{e\to\mu}(\mathbf{p},\mathbf{k},;t_i,t_f) \;=\; \delta_{rs}\delta_{\mathbf{k},\mathbf{p}}\frac{m_{e\mu}\,W_{\mathbf{p}}}{\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}}\,\left(e^{i(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})t_f}-e^{i(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})t_i}\right)$$

Probability

$$\mathcal{P}_{e\to\mu}(\mathbf{p};\Delta t) \;=\; \sum_{\mathbf{k},s}|\mathcal{A}^{rs}_{e\to\mu}(\mathbf{p},\mathbf{k};t_i,t_f)|^2$$

Explicitly

$$\mathcal{P}_{e\to\mu}(\mathbf{p};\Delta t) \;=\; W^2_{\mathbf{p}}\,\frac{4m^2_{e\mu}}{\left(\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left[\frac{(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})\,\Delta t}{2}\right]$$

# Decay probability

- Consider the process

$$|\nu^r_{\mathbf{p},e}\rangle \;\to\; |\nu^{s_1}_{\mathbf{k}_1,e}\rangle|\nu^{s_2}_{\mathbf{k}_2,e}\rangle|\overline{\nu}^{s_3}_{\mathbf{k}_3,\mu}\rangle\,, \qquad \mathbf{k}_1 \neq \mathbf{k}_2 \;\vee\; s_1 \neq s_2\,.$$

Probability (After non-trivial subtractions!)

$$\mathcal{P}_{e\to ee\overline{\mu}}(\mathbf{p};\Delta t) \;=\; \frac{4m^2_{e\mu}Y^2_{\mathbf{p}}}{\left(\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e})\,\Delta t}{2}\right)$$

where

$$Y^2_{\mathbf{p}} \;=\; \sum_{\mathbf{s}}\left(Y^{rs}_{\mathbf{p}}\right)^* Y^{rs}_{\mathbf{p}}$$

# Neutrino oscillation formula

Total flavor transition probability

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) \;=\; 4m^2_{e\mu}\left[\frac{W^2_{\mathbf{p}}}{\left(\omega^-_{\mathbf{p}}\right)^2}\sin^2\left(\frac{\omega^-_{\mathbf{p}}\Delta t}{2}\right)+\frac{Y^2_{\mathbf{p}}}{\left(\omega^+_{\mathbf{p}}\right)^2}\sin^2\left(\frac{\omega^+_{\mathbf{p}}\Delta t}{2}\right)\right]$$

with $\omega^\pm_{\mathbf{p}} \equiv \omega_{\mathbf{p},e} \pm \omega_{\mathbf{p},\mu}$. Note that

$$|U_{\mathbf{p}}| \;=\; W_{\mathbf{p}}\frac{m_\mu - m_e}{\omega^-_{\mathbf{p}}}\,, \qquad |V_{\mathbf{p}}| \;=\; Y_{\mathbf{p}}\frac{m_\mu - m_e}{\omega^+_{\mathbf{p}}}$$

when $m_1 \approx m_e$, $m_2 \approx m_\mu$. Then

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) \;=\; \sin^2(2\theta)\left[|U_{\mathbf{p}}|^2\sin^2\left(\frac{\omega^-_{\mathbf{p}}\Delta t}{2}\right)+|V_{\mathbf{p}}|^2\sin^2\left(\frac{\omega^+_{\mathbf{p}}\Delta t}{2}\right)\right]$$

with $\theta = m_{e\mu}/(m_\mu - m_e) \approx \sin\theta$. Oscillation formula of the flavor Fock-space approach!!

# Survival probability

- The amplitude of $|\nu^r_{\mathbf{p},e}\rangle \rightarrow |\nu^s_{\mathbf{k},e}\rangle$ is decomposed as

$$\mathcal{A}^{rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k};t_i,t_f) = \delta_{\mathbf{k},\mathbf{p}}\delta_{rs} + \frac{1}{2}\mathcal{A}^{(2)rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k};t_i,t_f)$$

Probability

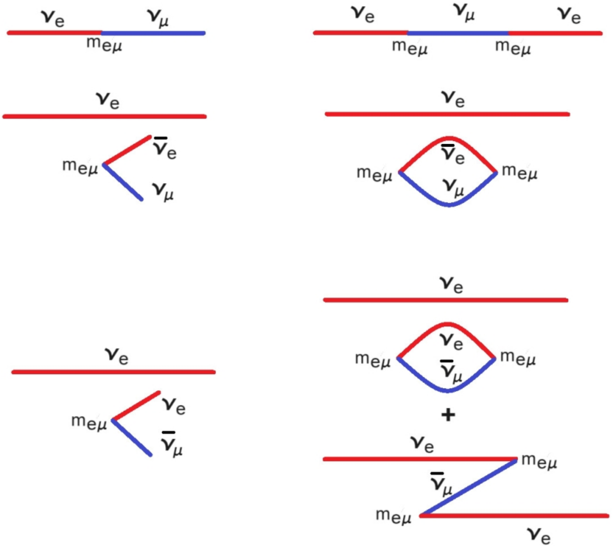
$$\mathcal{P}^e_S(\mathbf{p};\Delta t) = \sum_{\mathbf{k},s}\mathcal{A}^{rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k},t_i,t_f) \approx 1 + 2\,\Re e\left(\tilde{\mathcal{A}}^{(2)}_{e\rightarrow e}(\mathbf{p};t_i,t_f)\right)$$

with

$$\tilde{\mathcal{A}}^{(2)}_{e\rightarrow e}(\mathbf{p};t_i,t_f) \;\equiv\; \sum_{\mathbf{k},s} A^{(2)rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k};t_i,t_f)$$

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) + \mathcal{P}^e_S(\mathbf{p};\Delta t) \;=\; 1.$$

# Diagrams for neutrino oscillations



# Conclusions and perspectives

• The interaction picture approach* matches results of the flavor Fock space approach, at the lowest order in $m_{e\mu}$

• It should be possible to sum up the perturbative series and recover the flavor space (nonperturbative) result.

• Chiral oscillations should be also accommodated in this scheme.

*M.B., F.Giacosa, L.Smaldone and G.Torrieri, EPJC (2023)

# Chiral oscillations

# Chiral oscillations

- Taking into account (bi)spinorial nature of neutrinos and chiral nature of weak interaction, one naturally gets chiral oscillations *
- Interplay with flavor oscillations in the non-relativistic region[†]
- For C$\nu$B, chiral oscillations reduce detection by a factor of 2.[‡]
- Application: lepton-antineutrino entanglement and chiral oscillations in pion decay.[§]

[*] A. Bernardini and S. De Leo, Phys. Rev. D (2005)

[†] V.A.Bittencourt, A.Bernardini and M.B.,Eur.Phys.J.C(2021);EPL Persp.(2022); M. W. Li, Z. L. Huang and X. G. He, Phys. Lett. B (2024); K. Kimura and A. Takamura, arXiv:2101.03555 [hep-ph].

[‡] S.-F. Ge and P.Pasquini, Phys. Lett. B (2020)

[§] V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)

# Chiral oscillations

Chiral representation of the Dirac matrices

$$\alpha_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix},$$

and $\gamma_5 = (I_2, -I_2)$. Any bispinor $|\xi\rangle$ can be written in this representation as

$$|\xi\rangle = \begin{bmatrix} |\xi_R\rangle \\ |\xi_L\rangle \end{bmatrix},$$

The Dirac equation $H_D\,|\xi\rangle = i|\dot{\xi}\rangle$ can then be written as

$$i\partial_t\,|\xi_R\rangle - \mathbf{p}\cdot\sigma\,|\xi_R\rangle = m\,|\xi_L\rangle\,,$$
$$i\partial_t\,|\xi_L\rangle + \mathbf{p}\cdot\sigma\,|\xi_L\rangle = m\,|\xi_R\rangle\,,$$

• Evolution under the free Dirac Hamiltonian $\hat{H}_D$ induces left-right chiral oscillations.

Take initial state $|\psi(0)\rangle = [0,\ 0,\ 0,\ 1]^T$ which has negative helicity and negative chirality: $\hat{\gamma}_5\,|\psi(0)\rangle = -\,|\psi(0)\rangle$.

The time evolved state $|\psi_m(t)\rangle = e^{-i\hat{H}_D t}\,|\psi(0)\rangle$ is given by

$$
\begin{aligned}
|\psi_m(t)\rangle \quad &= \quad \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}\left[\left(1+\frac{p}{E_{p,m}+m}\right)e^{-iE_{p,m}t}\,|u_-(p,m)\rangle\right. \\
&\qquad\qquad \left. -\left(1-\frac{p}{E_{p,m}+m}\right)e^{iE_{p,m}t}\,|v_-(-p,m)\rangle\right],
\end{aligned}
$$

with (for one-dimensional propagation along the $\mathbf{e}_z$ direction)

$$
|u_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}\begin{bmatrix}\left(1\pm\frac{p}{E_{p,m}+m}\right)|\pm\rangle \\ \left(1\mp\frac{p}{E_{p,m}+m}\right)|\pm\rangle\end{bmatrix},
$$

$$
|v_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}}\begin{bmatrix}\left(1\pm\frac{p}{E_{p,m}+m}\right)|\pm\rangle \\ -\left(1\mp\frac{p}{E_{p,m}+m}\right)|\pm\rangle\end{bmatrix},
$$

• Survival probability of initial left-handed state

$$\mathcal{P}(t) = |\langle\psi_m(0)|\psi_m(t)\rangle|^2 = 1 - \frac{m^2}{E_{p,m}^2}\sin^2(E_{p,m}t)\,,$$

Average value of the chiral operator $\langle\hat{\gamma}_5\rangle(t)$

$$\langle\hat{\gamma}_5\rangle(t) = \langle\psi_m(t)|\hat{\gamma}_5\,|\psi_m(t)\rangle = -1 + \frac{2m^2}{E_{p,m}^2}\sin^2(E_{p,m}t)\,.$$

– Chiral oscillation period: $T_{ch} = \frac{2\pi}{E_{p,m}}$

– Chiral oscillation length: $L_{ch} = v\frac{2\pi}{E_{p,m}} = \frac{2\pi p}{E_{p,m}^2}$

# Chiral and flavor oscillations

- State of a neutrino of flavor $\alpha$ at a given $t$:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha,i} \, |\psi_{m_i}(t)\rangle \otimes |\nu_i\rangle \,,$$

where $|\psi_{m_i}(t)\rangle$ are bispinors.

- The state at $t = 0$ reads

$$|\nu_\alpha(0)\rangle = |\psi(0)\rangle \otimes \sum_i U_{\alpha,i} \, |\nu_i\rangle = |\psi(0)\rangle \otimes |\nu_\alpha\rangle \,,$$

where $|\psi(0)\rangle$ is a left handed bispinor.

- Survival probability:

$$\mathcal{P}_{\alpha\to\alpha} = |\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle|^2 = \left|\sum_i |U_{\alpha,i}|^2 \, \langle\psi(0)|\psi_{m_i}(t)\rangle\right|^2 .$$

Two flavor mixing:

$$
\begin{aligned}
|\nu_e(t)\rangle &= \left[\cos^2\theta\, |\psi_{m_1}(t)\rangle + \sin^2\theta\, |\psi_{m_2}(t)\rangle\right] \otimes |\nu_e\rangle \\
&\quad + \sin\theta\cos\theta\, \left[|\psi_{m_1}(t)\rangle - |\psi_{m_2}(t)\rangle\right] \otimes |\nu_\mu\rangle\, ,
\end{aligned}
$$

- The survival probability can be decomposed as

$$
\mathcal{P}_{e\to e}(t) = \mathcal{P}^S_{e\to e}(t) + \mathcal{A}_e(t) + \mathcal{B}_e(t).
$$

$\mathcal{P}^S_{e\to e}(t)$ is the standard flavor oscillation formula

$$
\mathcal{P}^S_{e\to e}(t) = 1 - \sin^2 2\theta \sin^2\left(\frac{E_{p,m_2} - E_{p,m_1}}{2}t\right)
$$

and

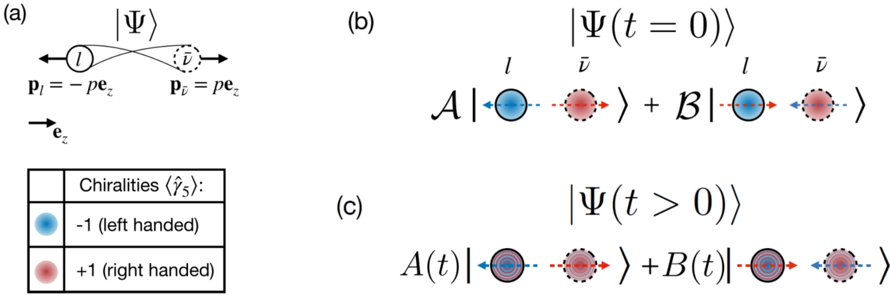
$$
\begin{aligned}
\mathcal{A}_e(t) &= -\left[\frac{m_1}{E_{p,m_1}}\cos^2\theta \sin\left(E_{p,m_1}t\right) + \frac{m_2}{E_{p,m_2}}\sin^2\theta\sin\left(E_{p,m_2}t\right)\right]^2, \\
\mathcal{B}_e(t) &= \frac{1}{2}\sin^2 2\theta \sin(E_{p,m_1}t)\sin(E_{p,m_2}t)\left(\frac{p^2 + m_1 m_2}{E_{p,m_1}E_{p,m_2}} - 1\right),
\end{aligned}
$$

are correction terms due to the bispinorial structure.

- Agreement with the QFT formula.

# Lepton-antineutrino entanglement and chiral oscillations*

• As an application of chiral oscillations, we consider induced spin correlations in pion decay products ($\pi \to l + \bar{\nu}$)




*V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)
V.A.Bittencourt, M.B. and G.Zanfardino, arXiv:2308.14574

# Quantum field theory of chiral oscillations[†]

• Dirac Lagrangian density

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

Invariance under global phase transformations ⇒ conserved charge

$$Q = \int d^3\mathbf{x}\,\psi^\dagger(x)\psi(x)$$

Dirac field $\psi$ can be split as $\psi = \psi_L + \psi_R$ where

$$\psi_L \equiv P_L \psi(x) = \frac{1-\gamma^5}{2}\psi(x), \qquad \psi_R \equiv P_R \psi(x) = \frac{1+\gamma^5}{2}\psi(x)$$

and hence Dirac Lagrangian can be written as

$$\mathcal{L} = \overline{\psi_L}\, i\gamma^\mu\, \partial_\mu\, \psi_L + \overline{\psi_R}\, i\gamma^\mu \partial_\mu \psi_R - m\left(\overline{\psi_L}\,\psi_R + \overline{\psi_R}\,\psi_L\right)$$

---

[†] V.Bittencourt, M.B. and G.Zanfardino, arXiv:2408.16742[hep-ph]

• Chiral symmetry is explicitly broken by the Dirac mass term.

• Separate global phase transformations for $\psi_L$ and $\psi_R$ lead to the non-conserved chiral charges

$$Q_L(t) = \int d^3\mathbf{x}\, \psi_L^\dagger(x)\psi_L(x), \qquad Q_R(t) = \int d^3\mathbf{x}\, \psi_R^\dagger(x)\psi_R(x).$$

• The total (conserved) charge is equal to the sum of the (time dependent) chiral charges

$$Q \,=\, Q_L(t) \,+\, Q_R(t).$$

Quantization of Dirac field

$$\{\psi_\alpha(\mathbf{x},t), \psi^\dagger_\beta(\mathbf{y},t)\} = \delta_{\alpha,\beta}\delta^3(\mathbf{x}-\mathbf{y}),$$
$$\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} = \{\psi^\dagger_\alpha(\mathbf{x},t), \psi^\dagger_\beta(\mathbf{y},t)\} = 0$$

The expansion of the field in terms of creation and annihilation is

$$\psi(x) = \sum_{r=1,2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[u^r_{\mathbf{k}}\,\alpha^r_{\mathbf{k}}\,e^{-i\omega_k t} \,+\, v^r_{-\mathbf{k}}\,\beta^{r\dagger}_{-\mathbf{k}}\,e^{i\omega_k t}\right]e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $\omega_k = \sqrt{k^2+m^2}$ and

$$\left\{\alpha^r_{\mathbf{k}}, \alpha^{s\dagger}_{\mathbf{p}}\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}\,; \qquad \left\{\beta^r_{\mathbf{k}}, \beta^{s\dagger}_{\mathbf{p}}\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}$$

and other anticommutators vanishing. We denote by $|0\rangle$ the vacuum state annihilated by the above operators for the Dirac field:

$$\alpha^r_{\mathbf{k}}\,|0\rangle \,=\, \beta^r_{\mathbf{k}}\,|0\rangle \,=\, 0.$$

In terms of ladder operators, we have

$$Q = \sum_r \int d^3\mathbf{k} \left( \alpha^{r\dagger}_{\mathbf{k}} \alpha^r_{\mathbf{k}} - \beta^{r\dagger}_{-\mathbf{k}} \beta^r_{-\mathbf{k}} \right)$$

If we now consider the chiral charges, we find:

$$Q_{L/R}(t) \;=\; \frac{1}{2}\left( Q \mp \int d^3\mathbf{x}\, \psi^\dagger(x) \gamma^5 \psi(x) \right)$$

Time dependence comes from the second piece in the above equation, which is indeed non-diagonal in the ladder operators:

$$\begin{aligned}\int d^3\mathbf{x}\, \psi^\dagger(x)\gamma^5\psi(x) \;=\; & \sum_{r=1,2} \int d^3\mathbf{k} \Big[ (u^{r\dagger}_{\mathbf{k}}\, \gamma^5\, u^s_{\mathbf{k}})\, \alpha^{r\dagger}_{\mathbf{k}} \alpha^r_{\mathbf{k}} + (v^{r\dagger}_{-\mathbf{k}}\, \gamma^5\, v^s_{-\mathbf{k}})\, \beta^r_{-\mathbf{k}} \beta^{r\dagger}_{-\mathbf{k}} \\ & + \; e^{-2i\omega_k t}\, (v^{r\dagger}_{-\mathbf{k}}\, \gamma^5\, u^s_{\mathbf{k}})\, \beta^r_{-\mathbf{k}} \alpha^r_{\mathbf{k}} + e^{2i\omega_k t}\, (u^{r\dagger}_{k}\, \gamma^5\, v^s_{-\mathbf{k}})\, \alpha^{r\dagger}_{\mathbf{k}} \beta^{r\dagger}_{-\mathbf{k}} \Big].\end{aligned}$$

Explicit form for $Q_L$ (similar expression for $Q_R$):

$$Q_L = \frac{1}{2}\sum_{r=1,2}\int d^3\mathbf{k}\left[\left(1+\epsilon^r\frac{|\mathbf{k}|}{\omega_k}\right)\alpha^{r\dagger}_{\mathbf{k}}\alpha^r_{\mathbf{k}} - \left(1-\epsilon^r\frac{|\mathbf{k}|}{\omega_k}\right)\beta^{r\dagger}_{-\mathbf{k}}\beta^r_{-\mathbf{k}} - \frac{m}{\omega_k}\left(e^{-2i\omega_k t}\,\beta^r_{-\mathbf{k}}\alpha^r_{\mathbf{k}} + e^{2i\omega_k t}\,\alpha^{r\dagger}_{\mathbf{k}}\beta^{r\dagger}_{-\mathbf{k}}\right)\right]$$

with $\epsilon^r = (-1)^r$.

In the relativistic limit $\omega_k \gg m$:

$$Q_L(t)|_{m=0} = \int d^3\mathbf{k}\left(\alpha^{2\dagger}_{\mathbf{k}}\alpha^2_{\mathbf{k}} - \beta^{1\dagger}_{-\mathbf{k}}\beta^1_{-\mathbf{k}}\right)$$

$$Q_R(t)|_{m=0} = \int d^3\mathbf{k}\left(\alpha^{1\dagger}_{\mathbf{k}}\alpha^1_{\mathbf{k}} - \beta^{2\dagger}_{-\mathbf{k}}\beta^2_{-\mathbf{k}}\right)$$

conserved (Noether) charges for the Weyl fields $\psi_L$ and $\psi_R$.

# Diagonalization of chiral charges

- Introduce the following canonical (Bogoliubov) transformation:

$$\begin{aligned}
\alpha_{\mathbf{k},L} &= \cos\theta_k\, \alpha^2_{\mathbf{k}} - \, e^{i\phi_k}\, \sin\theta_k\, \beta^{2\dagger}_{-\mathbf{k}} \\
\beta^{\dagger}_{-\mathbf{k},L} &= \cos\theta_k\, \beta^{1\dagger}_{-\mathbf{k}} - \, e^{-i\phi_k}\, \sin\theta_k\, \alpha^{1}_{\mathbf{k}} \\
\alpha_{\mathbf{k},R} &= \cos\theta_k\, \alpha^1_{\mathbf{k}} + \, e^{i\phi_k}\, \sin\theta_k\, \beta^{1\dagger}_{-\mathbf{k}} \\
\beta^{\dagger}_{-\mathbf{k},R} &= \cos\theta_k\, \beta^{2\dagger}_{-\mathbf{k}} + \, e^{-i\phi_k}\, \sin\theta_k\, \alpha^{1}_{\mathbf{k}}
\end{aligned}$$

- Condition for diagonalization

$$\begin{aligned}
\cos^2\theta_k &= \frac{1}{2}\left(1+\frac{|\mathbf{k}|}{\omega_k}\right), \quad \sin^2\theta_k = \frac{1}{2}\left(1-\frac{|\mathbf{k}|}{\omega_k}\right), \\
\cos 2\theta_k &= \frac{|\mathbf{k}|}{\omega_k}, \quad \sin 2\theta_k = -\frac{m}{\omega_k}, \quad \phi_k = 2\omega_k t.
\end{aligned}$$

Thus the above defined chiral ladder operators are time-dependent and satisfy (equal time) canonical anticommutation relations (CAR):

$$\left\{\alpha^r_{\mathbf{k},L}(t), \alpha^{s\dagger}_{\mathbf{k},L}(t)\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}\,, \qquad \left\{\beta^r_{\mathbf{k},L}(t), \beta^{s\dagger}_{\mathbf{k},L}(t)\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}$$

- Chiral charges are diagonal in the new operators

$$\begin{aligned} Q_L(t) &= \int d^3\mathbf{k}\, \left(\alpha^\dagger_{\mathbf{k},L}(t)\alpha_{\mathbf{k},L}(t) - \beta^\dagger_{-\mathbf{k},L}(t)\beta_{-\mathbf{k},L}(t)\right), \\ Q_R(t) &= \int d^3\mathbf{k}\, \left(\alpha^\dagger_{\mathbf{k},R}(t)\alpha_{\mathbf{k},R}(t) - \beta^\dagger_{-\mathbf{k},R}(t)\beta_{-\mathbf{k},R}(t)\right). \end{aligned}$$

Dirac field expansion:

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Big[ u^1_{\mathbf{k}} \left( \cos\theta_k \, \alpha_{\mathbf{k},R} - \, e^{i\phi_k} \, \sin\theta_k \, \beta^\dagger_{-\mathbf{k},L} \right) e^{-i\omega_k t}$$
$$+ u^2_{\mathbf{k}} \left( \cos\theta_k \, \alpha_{\mathbf{k},L} + \, e^{i\phi_k} \, \sin\theta_k \, \beta^\dagger_{-\mathbf{k},R} \right) e^{-i\omega_k t}$$
$$+ v^1_{-\mathbf{k}} \left( \cos\theta_k \, \beta^\dagger_{-\mathbf{k},L} + \, e^{-i\phi_k} \, \sin\theta_k \, \alpha_{\mathbf{k},R} \right) e^{i\omega_k t}$$
$$+ v^2_{-\mathbf{k}} \left( \cos\theta_k \, \beta^\dagger_{-\mathbf{k},R} - \, e^{-i\phi_k} \, \sin\theta_k \, \alpha_{\mathbf{k},L} \right) e^{i\omega_k t} \Big]$$

can be rearranged in the following form (using $\phi_k = 2\omega_k t$)

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[ u_{\mathbf{k},L} \, \alpha_{\mathbf{k},L}(t) \, e^{-i\omega_k t} + \, v_{-\mathbf{k},L} \, \beta^\dagger_{-\mathbf{k},L}(t) \, e^{i\omega_k t} \Big] e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$+ \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[ u_{\mathbf{k},R} \, \alpha_{\mathbf{k},R}(t) \, e^{-i\omega_k t} + \, v_{-\mathbf{k},R} \, \beta^\dagger_{-\mathbf{k},R}(t) \, e^{i\omega_k t} \Big] e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$= \psi_L(x) + \psi_R(x)$$

with

$$
\begin{aligned}
u_{\mathbf{k},L} &\equiv \cos\theta_k\, u^2_{\mathbf{k}} - \sin\theta_k\, v^2_{-\mathbf{k}}, \quad u_{\mathbf{k},R} \equiv \cos\theta_k\, u^1_{\mathbf{k}} + \sin\theta_k\, v^1_{-\mathbf{k}} \\
v_{-\mathbf{k},L} &\equiv \cos\theta_k\, v^1_{-\mathbf{k}} - \sin\theta_k\, u^1_{\mathbf{k}}, \quad v_{-\mathbf{k},R} \equiv \cos\theta_k\, v^2_{-\mathbf{k}} + \sin\theta_k\, u^2_{\mathbf{k}}
\end{aligned}
$$

$$
\begin{aligned}
u^\dagger_{\mathbf{k},L} u_{\mathbf{k},L} = u^\dagger_{\mathbf{k},R} u_{\mathbf{k},R} = 1\,, \quad v^\dagger_{-\mathbf{k},L} v_{-\mathbf{k},L} = v^\dagger_{-\mathbf{k},R} v_{-\mathbf{k},R} = 1 \\
u^\dagger_{\mathbf{k},L} u_{\mathbf{k},R} = v^\dagger_{-\mathbf{k},L} v_{-\mathbf{k},R} = 0\,, \quad u^\dagger_{\mathbf{k},L} v_{-\mathbf{k},L} = u^\dagger_{\mathbf{k},R} v_{-\mathbf{k},L} = 0
\end{aligned}
$$

and the completeness relation:

$$
u_{\mathbf{k},R}\, u^\dagger_{\mathbf{k},R} + u_{\mathbf{k},L}\, u^\dagger_{\mathbf{k},L} + v_{-\mathbf{k},R}\, v^\dagger_{-\mathbf{k},R} + v_{-\mathbf{k},L}\, v^\dagger_{-\mathbf{k},L} = 1\!\!1
$$

Consistency relations:

$$
\begin{aligned}
P_L\, u_{\mathbf{k},L} &= u_{\mathbf{k},L}\,, \qquad P_L\, v_{-\mathbf{k},L} = v_{-\mathbf{k},L} \\
P_R\, u_{\mathbf{k},R} &= u_{\mathbf{k},R}\,, \qquad P_R\, v_{-\mathbf{k},R} = v_{-\mathbf{k},R} \\
P_R\, u_{\mathbf{k},L} &= P_R\, v_{-\mathbf{k},L} = P_L\, u_{\mathbf{k},R} = P_L\, v_{-\mathbf{k},R} = 0
\end{aligned}
$$

The Bogoliubov is written as

$$\alpha_{\mathbf{k},L} = G_t^{-1}\alpha_{\mathbf{k}}^2 G_t \quad , \quad \beta_{\mathbf{k},L} = G_t^{-1}\beta_{\mathbf{k}}^1 G_t$$

$$\alpha_{\mathbf{k},R} = G_t^{-1}\,\alpha_{\mathbf{k}}^1\,G_t \quad , \quad \beta_{\mathbf{k},R} = G_t^{-1}\,\beta_{\mathbf{k}}^2\,G_t$$

with generator

$$G_t(\theta,\phi) = \exp\left[\sum_r \int d^3\mathbf{k}\,\theta_k \epsilon^r \left(e^{-i\phi_k}\alpha_{\mathbf{k}}^r \beta_{-\mathbf{k}}^r - e^{i\phi_k}\beta_{-\mathbf{k}}^{r\dagger}\alpha_{\mathbf{k}}^{r\dagger}\right)\right]$$

• Explicit form for the *chiral vacuum*:

$$|\tilde{0}(t)\rangle_{LR} = \prod_{\mathbf{k},r}\left[\cos\theta_k + \epsilon^r e^{i\phi_k}\sin\theta_k \alpha_{\mathbf{k}}^{r\dagger}\beta_{-\mathbf{k}}^{r\dagger}\right]|0\rangle$$

• The chiral vacuum $|\tilde{0}(t)\rangle_{LR}$ and the Dirac vacuum $|0\rangle$ are orthogonal in the infinite volume limit:

$$\lim_{V\to\infty}\langle 0|\tilde{0}(t)\rangle_{LR} = 0,$$

generating *unitarily inequivalent representations* of the field algebra.

# Chiral oscillation formula

Define the state $|\alpha_L\rangle \equiv \alpha^{\dagger}_{\mathbf{k},L}|\tilde{0}\rangle_{LR}$, with $|\tilde{0}\rangle_{LR} \equiv |\tilde{0}(0)\rangle_{LR}$.

Left chiral operator at time $t$

$$\alpha_{\mathbf{k},L}(t) = \cos\theta_k \, e^{-i\omega_k t} \, \alpha^2_{\mathbf{k}} \, - \, \sin\theta_k \, e^{i\omega_k t} \, \beta^{2\dagger}_{-\mathbf{k}}$$

- Chiral oscillation formula

$$\langle \alpha_{\mathbf{k},L}|Q_L(t)|\alpha_{\mathbf{k},L}\rangle = |\{\alpha_{\mathbf{k},L}(t), \alpha^{\dagger}_{\mathbf{k},L}(0)\}|^2$$

with

$$\{\alpha_{\mathbf{k},L}(t), \alpha^{\dagger}_{\mathbf{k},L}(0)\} = \cos^2\theta_k e^{-i\omega_k t} + \sin^2\theta_k e^{i\omega_k t}$$

We obtain

$$\langle \alpha_{\mathbf{k},L}|Q_L(t)|\alpha_{\mathbf{k},L}\rangle = 1 - \sin^2(2\theta_k)\sin^2(\omega_k t) = 1 - \frac{m^2}{\omega_k^2}\sin^2(\omega_k t)$$

# Conclusions and perspectives

- Consistent treatment of oscillating particles in QFT
- Unified approach for flavor and chiral oscillations
- Weak interactions appear to be non trivial at representation (particle) level.

# Other aspects

# Three-flavor fermion mixing[‡]

Mixing relations:

$$\Psi_f(x) \;=\; \mathbf{M}\,\Psi_m(x)$$

where $\Psi_f^T = (\nu_e, \nu_\mu, \nu_\tau)$ , $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and

$$\mathbf{M} \,=\, \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$

[‡] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

We have:

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\,\nu_i^\alpha(x)\,G_\theta(t),$$

where $(\sigma, i) = (e,1), (\mu,2), (\tau,3),$ and

$$G_\theta(t) \;=\; G_{2_3}(t)G_{1_3}(t)G_{12}(t)$$

$$G_{12}(t) \;=\; exp\left[\theta_{12}\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x))\right],$$

$$G_{13}(t) \;=\; exp\left[\theta_{13}\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_3(x)e^{-i\delta} - \nu_3^\dagger(x)\nu_1(x)e^{i\delta})\right],$$

$$G_{23}(t) \;=\; exp\left[\theta_{23}\int d^3\mathbf{x}(\nu_2^\dagger(x)\nu_3(x) - \nu_3^\dagger(x)\nu_2(x))\right],$$

Flavor vacuum:

$$|0\rangle_f \;=\; G_\theta^{-1}(t)\;|0\rangle_m$$

# Flavor annihilation operators:

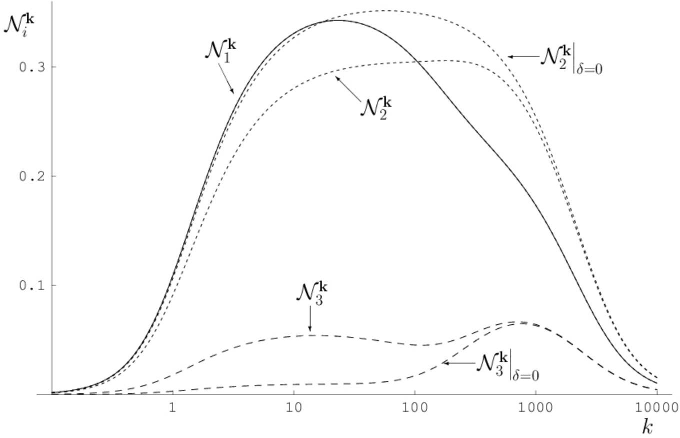
$$\alpha^r_{\mathbf{k},e} = c_{12}c_{13}\ \alpha^r_{\mathbf{k},1} + s_{12}c_{13}\left(U^{\mathbf{k}*}_{12}\ \alpha^r_{\mathbf{k},2} + \epsilon^r V^{\mathbf{k}}_{12}\ \beta^{r\dagger}_{-\mathbf{k},2}\right) + e^{-i\delta}\ s_{13}\left(U^{\mathbf{k}*}_{13}\ \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{13}\ \beta^{r\dagger}_{-\mathbf{k},3}\right)\ ,$$

$$\alpha^r_{\mathbf{k},\mu} = \left(c_{12}c_{23} - e^{i\delta}\ s_{12}s_{23}s_{13}\right)\ \alpha^r_{\mathbf{k},2} - \left(s_{12}c_{23} + e^{i\delta}\ c_{12}s_{23}s_{13}\right)\left(U^{\mathbf{k}}_{12}\ \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{12}\ \beta^{r\dagger}_{-\mathbf{k},1}\right)$$
$$+ s_{23}c_{13}\left(U^{\mathbf{k}*}_{23}\ \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{23}\ \beta^{r\dagger}_{-\mathbf{k},3}\right)\ ,$$

$$\alpha^r_{\mathbf{k},\tau} = c_{23}c_{13}\ \alpha^r_{\mathbf{k},3} - \left(c_{12}s_{23} + e^{i\delta}\ s_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{23}\ \alpha^r_{\mathbf{k},2} - \epsilon^r V^{\mathbf{k}}_{23}\ \beta^{r\dagger}_{-\mathbf{k},2}\right)$$
$$+ \left(s_{12}s_{23} - e^{i\delta}\ c_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{13}\ \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{13}\ \beta^{r\dagger}_{-\mathbf{k},1}\right)$$

and similar ones for antiparticles ($\delta \to -\delta$).

# Condensation densities




Condensation densities $\mathcal{N}_i^{\mathbf{k}}$ for sample values of masses and mixings

# Parameterizations of mixing matrix

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\, \nu_i^\alpha(x)\, G_\theta(t),$$

Define the more general generators:

$$G_{12} \equiv \exp\left[\theta_{12} \int d^3x \left(\nu_1^\dagger \nu_2 e^{-i\delta_2} - \nu_2^\dagger \nu_1 e^{i\delta_2}\right)\right]$$

$$G_{1_3} \equiv \exp\left[\theta_{1_3} \int d^3x \left(\nu_1^\dagger \nu_3 e^{-i\delta_5} - \nu_3^\dagger \nu_1 e^{i\delta_5}\right)\right]$$

$$G_{2_3} \equiv \exp\left[\theta_{2_3} \int d^3x \left(\nu_2^\dagger \nu_3 e^{-i\delta_7} - \nu_3^\dagger \nu_2 e^{i\delta_7}\right)\right]$$

There are six different matrices obtained by permutations of the above generators.

We can obtain all possible parameterizations of the matrix by setting to zero two of the phases and permuting rows/columns.

# Currents and charges for 3-flavor fermion mixing

Lagrangian for three free Dirac fields with different masses

$$\mathcal{L}(x) \,=\, \bar{\Psi}_m(x)\,(i\,\not{\partial} - M_d)\,\Psi_m(x)$$

where $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and $M_d = diag(m_1, m_2, m_3)$.

The $SU(3)$ transformations:

$$\Psi'_m(x) \,=\, e^{i\alpha_j \lambda_j/2}\,\Psi_m(x) \quad ; \qquad j = 1,..,8$$

with $\alpha_j$ real constants, and $\lambda_j$ the Gell-Mann matrices, give the currents:

$$J^\mu_{m,j}(x) \,=\, \frac{1}{2}\bar{\Psi}_m(x)\,\gamma^\mu\,\lambda_j\,\Psi_m(x)$$

The combinations:

$$Q_1 \equiv \frac{1}{3}Q + Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8},$$

$$Q_2 \equiv \frac{1}{3}Q - Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8}$$

$$Q_3 \equiv \frac{1}{3}Q - \frac{2}{\sqrt{3}}Q_{m,8}$$

$$Q_i = \sum_r \int d^3\mathbf{k} \left( \alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},i} \right), \;\; i = 1, 2, 3.$$

are the Noether charges for the fields $\nu_i$ with $\sum_i Q_i = Q$.

Flavor charges:

$$Q_\sigma(t) :\!:= G^{-1}_\theta(t) : Q_i : G_\theta(t) = \sum_r \int d^3\mathbf{k} \left( \alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right)$$

# CP violation and $SU(3)$

Modified Gell-Mann matrices:

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & e^{i\delta_2} & 0 \\ e^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\lambda}_2 = \begin{pmatrix} 0 & -ie^{i\delta_2} & 0 \\ ie^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & e^{-i\delta_5} \\ 0 & 0 & 0 \\ e^{i\delta_5} & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -ie^{-i\delta_5} \\ 0 & 0 & 0 \\ ie^{i\delta_5} & 0 & 0 \end{pmatrix}, \tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\delta_7} \\ 0 & e^{-i\delta_7} & 0 \end{pmatrix}, \tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\delta_7} \\ 0 & ie^{-i\delta_7} & 0 \end{pmatrix}$$

$$\tilde{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

# Neutrino mixing and accelerated proton decay

- Testing the consistency of QFT in curved background by comparing the decay rate of accelerated protons (*inverse $\beta$ decay*) in the inertial and comoving frames: a ‘theoretical check” of the Unruh effect*

- Clarifying some conceptual issues concerning the inverse $\beta$ decay in the context of neutrino mixing†

- Investigating the dichotomy between mass and flavor neutrinos as fundamental “objects” in QFT.

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*G. E. A. Matsas and D. A. T. Vanzella, Phys. Rev. D (1999)

†D. V. Ahluwalia, L. Labun and G. Torrieri, Eur. Phys. J. A (2016)

# The Unruh effect[‡]

*. . . the behavior of particle detectors under acceleration $a$ is investigated where it is shown that an accelerated detector even in flat spacetime will detect particles in the vacuum. . .*

*. . . This result is exactly what one would expect of a detector immersed in a thermal bath of temperature*

$$T_{\mathrm{U}} = a/2\pi$$

[‡]W.G.Unruh, Phys. Rev. D (1976)

# The Unruh effect

- Rindler coordinates

$$x^0 = \xi \sinh\eta, \quad x = \xi \cosh\eta$$

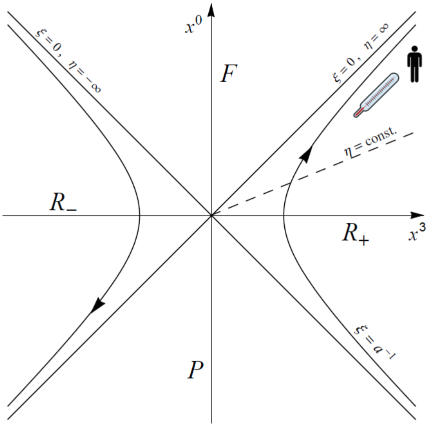
- Rindler vs Minkowski

$$ds^2_{\rm M} = \left(dx^0\right)^2 - (dx)^2$$

$$\Longrightarrow \; ds^2_{\rm R} = \xi^2 d\eta^2 - d\xi^2$$

- Rindler worldline

$$\eta = a\tau, \quad \xi = \text{const} \equiv a^{-1}$$




- Minkowski vacuum is a thermal bath for the Rindler observer

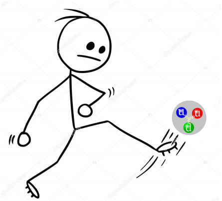
$$\langle 0_{\mathcal{M}} | \widehat{N}(\omega) | 0_{\mathcal{M}} \rangle = \frac{1}{e^{a\,\omega/T_{\rm U}} + 1}$$

# Decay of accelerated particles

Decay properties are not universal[§]

$$\tau_{proton} \gg \tau_{universe} \sim 10^{10}\,\mathrm{yr}$$

However, if we "kick" the proton…




$$p \to n + e^{+} + \nu_e$$

...the proton decay is kinematically allowed!

| acceleration | lifetime |
|---|---|
| $a_{LHC}$ | $\tau_p \sim 10^{3\times10^8}\,\mathrm{yr}$ |
| $a_{pulsar}$ | $\tau_p \sim 10^{-1}\,\mathrm{s}$ |

[§] R. Muller, Phys. Rev. D (1997)

# Accelerated proton decay and existence of Unruh effect*

Basic assumptions:

- Massless neutrino
- $|\mathbf{k}_e| \sim |\mathbf{k}_{\nu_e}| \ll M_{p,n}$
- Current-current Fermi theory

$$\widehat{S}_I = \int d^2x \sqrt{-g}\, \widehat{j}_\mu \left( \widehat{\overline{\Psi}}_\nu \gamma^\mu \widehat{\Psi}_e + \widehat{\overline{\Psi}}_e \gamma^\mu \widehat{\Psi}_\nu \right)$$

$$\widehat{j}^\mu = \widehat{q}(\tau)\, u^\mu\, \delta\big(u - a^{-1}\big)\,, \quad \widehat{q}(\tau) = e^{i\widehat{H}\tau}\, \widehat{q}_0\, e^{-i\widehat{H}\tau}$$

$$\widehat{H}\,|n\rangle = m_n\,|n\rangle\,, \quad \widehat{H}\,|p\rangle = m_p\,|p\rangle\,, \quad G_F = |\langle p|\, \hat{q}_0\, |n\rangle|$$

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*G.E.A. Matsas and D.A.T. Vanzella, Phys. Rev. D (1999); D.A.T. Vanzella and G.E.A. Matsas, Phys. Rev. D (2000); Phys. Rev. Lett. (2001).

• Laboratory frame

$$p \to n + e^+ + \nu_e$$

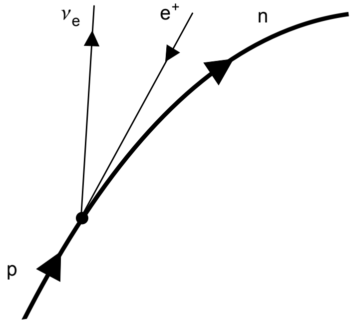


**Figure 1:** The decay occurs since the acceleration supplies the $p$-$n$ rest mass difference

- Comoving frame

$p + e \to n + \nu_e$ $\quad$ $p + \bar{\nu}_e \to n + e^+$ $\quad$ $p + e + \bar{\nu}_e \to n$

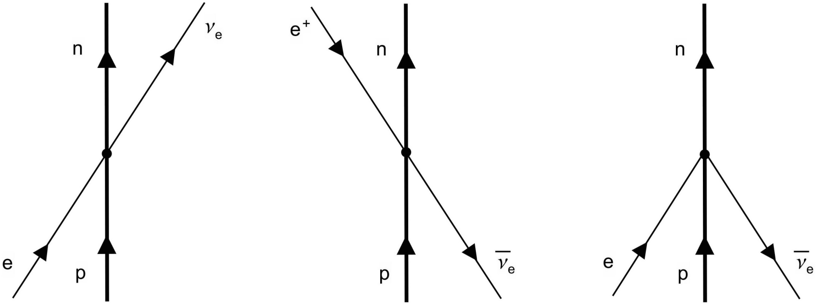


**Figure 2:** The decay occurs since $p$ interacts with the Unruh thermal bath of $e^-$ and $\nu_e$

- Tree-level transition amplitude

$$\mathcal{A}^{p\to n} = \langle n| \otimes \langle e^{+}_{k_e\,\sigma_e}, \nu_{k_\nu\,\sigma_\nu}| \widehat{S}_I \,|0\rangle \otimes |p\rangle$$

- Differential transition rate

$$\frac{d^2\mathcal{P}^{p\to n}_{in}}{dk_e dk_\nu} = \frac{1}{2}\sum_{\sigma_e=\pm}\sum_{\sigma_\nu=\pm} \left|\mathcal{A}^{p\to n}\right|^2$$

- Scalar decay rate (inertial frame)

$$\Gamma^{p\to n}_{in} \equiv \frac{\mathcal{P}^{p\to n}_{in}}{T} = \frac{4G_F^2 a}{\pi^2 e^{\pi\Delta m/a}} \int_0^\infty d\tilde{k}_e \int_0^\infty d\tilde{k}_\nu K_{2i\Delta m/a}\left[2\left(\tilde{\omega}_e + \tilde{\omega}_\nu\right)\right]$$

• Scalar decay rate (comoving frame)

$$\Gamma^{p\to n}_{com} = \Gamma^{p\to n}_{(i)} + \Gamma^{p\to n}_{(ii)} + \Gamma^{p\to n}_{(iii)}$$

$$= \frac{G_F^2\, m_e}{a\,\pi^2\, e^{\pi\Delta m/a}} \int_{-\infty}^{+\infty} d\omega\, \frac{K_{i\omega/a+1/2}(m_e/a) K_{i\omega/a-1/2}(m_e/a)}{\cosh\left[\pi\left(\omega-\Delta m\right)/a\right]}$$

• Result (tree level): $\Gamma^{p\to n}_{in} = \Gamma^{p\to n}_{com}$.

**Conclusion***

The Unruh effect is **mandatory** for the General Covariance of QFT

• Generalization to 4D with **massive neutrino**†: similar (analytical) results.

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*D. A. T. Vanzella and G. E. A. Matsas, Phys. Rev. Lett. (2001).

†H. Suzuki and K. Yamada, Phys. Rev. D (2003).

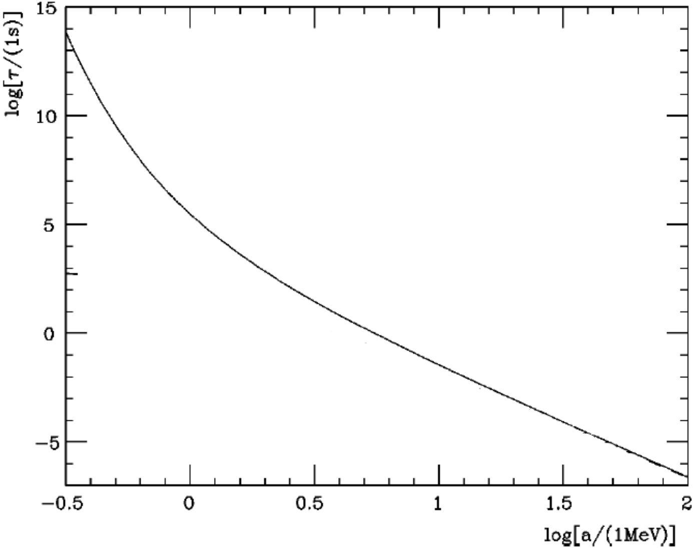


**Figure 3:** The mean proper lifetime $\tau$ of proton versus its proper acceleration $a$.

# Accelerated proton decay and neutrino mixing

– Neutrino mixing in the inverse $\beta$-decay *:

"*In the laboratory frame, the interaction is the electroweak vertex, hence neutrinos are in **flavor eigenstates**. In the comoving frame, the proton interacts with neutrinos in Rindler states, which display an effective thermal weight and are **mass eigenstates***".

"*...if charge eigenstates were the asymptotic states also in the accelerating frame, the **thermality** of the Unruh effect would be **violated**".*

"*...we conclude that the rates in the two frames **disagree** when taking into account neutrino mixings*".

* D. V. Ahluwalia, L. Labun and G. Torrieri, Eur. Phys. J. A (2016)

# Non-thermal Unruh effect for mixed neutrinos*

When flavor mixing for an accelerated observer is considered, the two Bogoliubov transformations involved:

$$\phi_{\mathcal{R}} \overset{\text{thermal Bogol. } (a)}{\longrightarrow} \phi_{\mathcal{M}} \Rightarrow \text{condensate in } |0_{\mathcal{M}}\rangle$$

$$\phi_1, \phi_2 \overset{\text{mixing Bogol. } (\theta)}{\longrightarrow} \phi_e, \phi_\mu \Rightarrow \text{condensate in } |0_{e,\mu}\rangle$$

combine with each other.

* M. B., G. Lambiase and G. Luciano, Phys. Rev. D (2017)

The Unruh spectrum for mixed neutrinos

$$ {}_{\rm M}\langle 0|\mathcal{N}(\theta,\omega)|0\rangle_{\rm M} \;=\; \underbrace{\frac{1}{e^{a\,\omega/T_{\rm U}}+1}}_{Thermal\ spectrum} \;+\; \underbrace{\sin^2\theta\left\{\mathcal{O}\left(\frac{\delta m}{m}\right)^2\right\}}_{Non-thermal\ corrections} $$

acquires non-thermal corrections.

# Inverse $\beta$-decay and neutrino mixing[†]

Working with **flavor neutrinos**, we compute

$$\mathcal{A}^{p\rightarrow n} = \langle n| \otimes \langle e^+_{k_e\,\sigma_e}, \nu_{k_\nu\,\sigma_\nu}|\widehat{S}_I\,|0\rangle \otimes |p\rangle$$

in the *laboratory frame...*

$$\Gamma^{p\rightarrow n}_{in} \;=\; \cos^4\theta\,\Gamma^{p\rightarrow n}_1 \,+\, \sin^4\theta\,\Gamma^{p\rightarrow n}_2 \,+\, \cos^2\theta\sin^2\theta\,\Gamma^{p\rightarrow n}_{12}$$

$$\Gamma^{p\rightarrow n}_i \;\equiv\; \frac{1}{T}\sum_{\sigma_\nu,\sigma_e} {G_F}^2 \int d^3k_\nu \int d^3k_e \left|\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_i},\omega_e)\right|^2, \quad i=1,2,$$

$$\Gamma^{p\rightarrow n}_{12} \;\equiv\; \frac{1}{T}\sum_{\sigma_\nu,\sigma_e} {G_F}^2 \int d^3k_\nu \int d^3k_e \left[\mathcal{I}_{\sigma_\nu\sigma_e}(\omega_{\nu_1},\omega_e)\,\mathcal{I}^*_{\sigma_\nu\sigma_e}(\omega_{\nu_2},\omega_e) \,+\, \text{c.c.}\right]$$

[†]M. B., G. Lambiase, G. Luciano and L.Petruzziello, Phys. Rev. D (2018)

...and in the *comoving frame*

$$\Gamma^{p\to n}_{com} = \cos^4\theta\,\widetilde{\Gamma}^{p\to n}_1 + \sin^4\theta\,\widetilde{\Gamma}^{p\to n}_2 + \cos^2\theta\sin^2\theta\,\widetilde{\Gamma}^{p\to n}_{12}$$

$$\begin{aligned}
\widetilde{\Gamma}^{p\to n}_{12} \;=\; & \frac{2\,G_F^2}{a^2\,\pi^7\,\sqrt{l_{\nu_1} l_{\nu_2}}\,e^{\pi\Delta m/a}}\int_{-\infty}^{+\infty} d\omega \left\{\int d^2k_e\, l_e \left|K_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)\right|^2\right.\\
& \times \int d^2k_\nu \left(\kappa_\nu^2 + m_{\nu_1}m_{\nu_2} + l_{\nu_1}l_{\nu_2}\right)\\
& \times \mathrm{Re}\left\{K_{i(\omega-\Delta m)/a+1/2}\left(\frac{l_{\nu_1}}{a}\right)K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}\\
& + m_e \int d^2k_e \int d^2k_\nu \left(l_{\nu_1}m_{\nu_2} + l_{\nu_2}m_{\nu_1}\right)\\
& \times \mathrm{Re}\left\{K^2_{i\omega/a+1/2}\left(\frac{l_e}{a}\right)K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_1}}{a}\right)\right.\\
& \left.\left.\times\; K_{i(\omega-\Delta m)/a-1/2}\left(\frac{l_{\nu_2}}{a}\right)\right\}\right\}, \quad \kappa_\nu \equiv (k_\nu^x, k_\nu^y)
\end{aligned}$$

# Comparing the rates

## Laboratory vs comoving decay rates

$$\Gamma_{in}^{p\to n} \;=\; \cos^4\theta\,\Gamma_1^{p\to n} \,+\, \sin^4\theta\,\Gamma_2^{p\to n} \,+\, \cos^2\theta\sin^2\theta\,\Gamma_{12}^{p\to n},$$

$$\Gamma_{com}^{p\to n} = \cos^4\theta\,\widetilde{\Gamma}_1^{p\to n} \,+\, \sin^4\theta\,\widetilde{\Gamma}_2^{p\to n} \,+\, \cos^2\theta\sin^2\theta\,\widetilde{\Gamma}_{12}^{p\to n}$$

$$\Gamma_i^{p\to n} = \widetilde{\Gamma}_i^{p\to n}, \qquad i = 1, 2$$

What about the "off-diagonal" terms?

$$\Gamma_{12}^{p\to n} \stackrel{?}{=} \widetilde{\Gamma}_{12}^{p\to n}$$

Non-trivial calculations…

…for $\frac{\delta m}{m} \ll 1$

$$\Gamma_{12}^{p\to n} = \widetilde{\Gamma}_{12}^{p\to n} \quad \text{up to } \mathcal{O}\left(\frac{\delta m}{m}\right)$$

**Result**

$$\Gamma_{in}^{p\to n} = \Gamma_{com}^{p\to n} \quad \text{up to } \mathcal{O}\left(\frac{\delta m}{m}\right)$$

# General Covariance and mass states?

Solving the problem with **mass eigenstates** [‡] ?

"[...] *a physical Fock space for flavor neutrinos cannot be constructed. Flavor states are only phenomenological since their definition depends on the specific considered process.*"

" *We should view the neutrino states with well defined mass as the fundamental ones.* [...] *The decay rates calculated in this way are perfectly in agreement*".

[‡]G.Cozzella, S.A.Fulling, A.G.S.Landulfo, G.E.A.Matsas and D.A.T.Vanzella, Phys.Rev. (2018)

## Why not mass states?

- A physical Fock space for flavor neutrinos can be rigorously defined
- The use of mass eigenstates wipes mixing out of calculations

$$\Gamma^{p\rightarrow n+\bar{\ell}_\alpha+\nu_i} = |U_{\alpha,i}|^2 \Gamma_i, \qquad i = 1, 2$$

- Inconsistency with the asymptotic occurrence of *flavor oscillations*[§]

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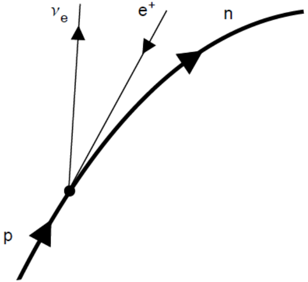
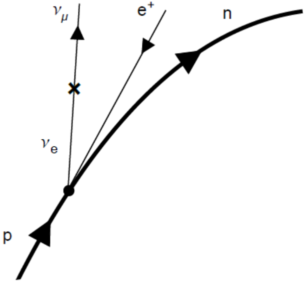
[§] M. B., G.Lambiase, G. Luciano and L.Petruzziello, Phys. Lett. B (2020)

# Neutrino oscillations (inertial frame)




a) Without oscillations

$$\Gamma_{in}^{(\nu_e)} = c_\theta^4\,\Gamma_1^{p\to n} + s_\theta^4\,\Gamma_2^{p\to n} + c_\theta^2\,s_\theta^2\,\Gamma_{12}^{p\to n}$$



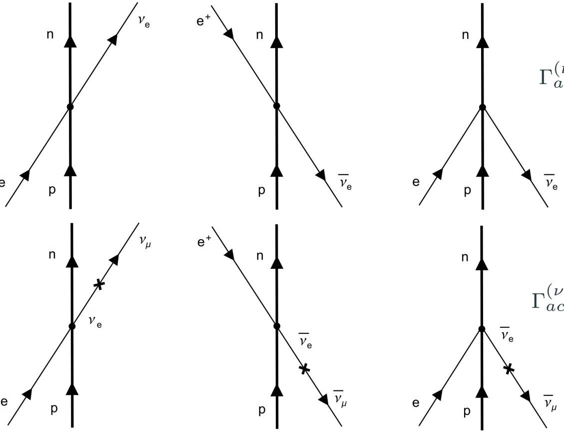

b) With oscillations

$$\Gamma_{in}^{(\nu_\mu)} = c_\theta^2\,s_\theta^2\left(\Gamma_1^{p\to n} + \Gamma_2^{p\to n} - \Gamma_{12}^{p\to n}\right)$$

## Total decay rate

$$\Gamma_{in}^{tot} \equiv \Gamma_{in}^{(\nu_e)} + \Gamma_{in}^{(\nu_\mu)} = \cos^2\theta\,\Gamma_1^{p\to n} + \sin^2\theta\,\Gamma_2^{p\to n}$$

# Neutrino oscillations (comoving frame)




a) Without oscillations

$$\Gamma_{acc}^{(\nu_e)} = c_\theta^4\,\widetilde{\Gamma}_1^{p\to n} + s_\theta^4\,\widetilde{\Gamma}_2^{p\to n} + c_\theta^2\,s_\theta^2\,\widetilde{\Gamma}_{12}^{p\to n}$$

b) With oscillations

$$\Gamma_{acc}^{(\nu_\mu)} = c_\theta^2\,s_\theta^2\left(\widetilde{\Gamma}_1^{p\to n} + \widetilde{\Gamma}_2^{p\to n} - \widetilde{\Gamma}_{12}^{p\to n}\right)$$

**Total decay rate**

$$\Gamma_{acc}^{tot} = \cos^2\theta\,\widetilde{\Gamma}_1^{p\to n} + \sin^2\theta\,\widetilde{\Gamma}_2^{p\to n} = \Gamma_{in}^{tot}$$

# Conclusion

- Asymptotic neutrinos must be in **flavor eigenstates** in order to preserve the General Covariance of QFT in curved background.

| | Ahluwalia's approach | Matsas's approach | Our approach |
|---|---|---|---|
| Asympt. neutrinos in the laboratory frame | Flavor | Mass | Flavor |
| Asympt. neutrinos in the comoving frame | Mass | Mass | Flavor |
| Agreement between the decay rates | ✗ | ✓ | ✓ |
| Consistency with neutrino oscillations | ✗ | ✗ | ✓ |