Spatially oscillating correlations in strongly-interacting four-fermion model with generalized PT-symmetry

Marc Winstel

[MW, Physical Review D **110**, 034008 (2024), arXiv: 2403.07430]

Applications of Field Theory to Hermitian and Non-Hermitian Systems, King's College London

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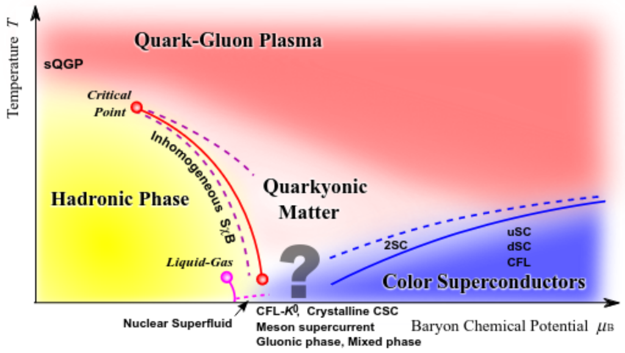
CRC-TR 211
Strong-interaction matter under extreme conditions
GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Introduction

- QCD Phase diagram in $T - \mu_B$ plane: A lot of open questions
  - Moat regimes, inhomogeneous chiral phases, quantum pion liquid: Spatial modulations of the order parameter?

The phase diagram of dense QCD



[Fukushima, Hatsuda, Rept. Prog. Phy. **74** (2011), arXiv: 1005.4814.]

Generalized $\mathcal{PT}$-symmetry in QCD at $\mu \neq 0$

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- Charge conjugation in Euclidean spacetime $A_\mu \to -A_\mu^T$; Wilson loop $W \Rightarrow W^\dagger$
- At $\mu = 0$: Fermion determinant can be expanded in Wilson loops $\mathrm{tr}_F W$ where $\mathrm{tr}_F W$ and $\mathrm{tr}_F W^\dagger$ appear with similar coefficients $\Rightarrow \ln\det[A_\mu] \in \mathbb{R}$

see also: talk by M. Ogilvie on Wednesday

[Nishimura, Ogilvie, Pangeni, PRD **90**, 045039 (2014) & PRD **91**, 054004 (2015)]

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- Charge conjugation $\mathcal{C}$ exchanges $\mathrm{tr}_F W$ and $\mathrm{tr}_F W^\dagger$
- At $\mu \neq 0$: Wilson loops $\mathrm{tr}_F W / W^\dagger$ with non-trivial winding number n are weighted by $\mathrm{e}^{\pm\beta\mu}$ $\Rightarrow$ Broken $\mathcal{C}$ symmetry

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- But: Invariance under $\mathcal{CK}$ operation: $\mathrm{tr}_F W \to \mathrm{tr}_F W^T = \mathrm{tr}_F W$ ($\mathcal{K}$ is complex conjugation). This is a $\mathcal{PT}$-type symmetry!

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Generalized $\mathcal{PT}$-symmetry: Implications on the Hessian matrix

- Consider a model with homogeneous (thermal) vev's $\langle\vec{\phi}\rangle$ and propagators G_{ϕ_j}
- Low momentum representation of (scalar) propagators with Mass / Hessian matrix $\mathcal{M}$

$$G^{-1}(q^2) = q^2 + \mathcal{M}$$

[Schindler, Schindler, Medina, Ogilvie, PRD **102**, 114510 (2020)]

[Schindler, Schindler, Ogilvie, J. Phys. Conf. Ser. 2038 (2021)]

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$$\mathcal{M} = \Sigma\mathcal{M}^*\Sigma \Rightarrow \text{Compl. conj. EV pairs}$$

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What are the implications?

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$$G^{-1}(q^2) = q^2 + \mathcal{M}$$

$\mathcal{PT}$-type symmetry: What are the implications?

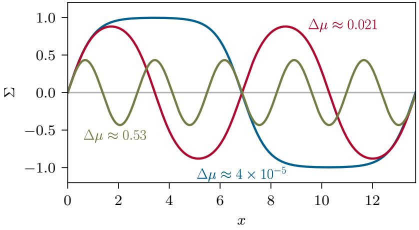
Eigenvalues E_i of $\mathcal{M}$	**Position-space propagator behavior**	**Region**
All positive	Exponential decay	Normal
Odd number of $E_i < 0$	Exponential growth	Unstable
Some $E_i = E_j^*$	Sinusoidally-modulated exponential	$\mathcal{PT}$ broken
Even number of $E_i < 0$	Homogeneous solution unstable at some $p \neq 0$	Patterned vacuum

[Schindler, Schindler, Ogilvie, PoS LATTICE2021 (2022)]

Exotic phases in finite density QCD: Spatial modulations?

Inhomogeneous phase (IP)

- "Patterned"
- Long-range order & Translational SSB!
- $\langle \phi_j \rangle \sim \langle \bar{\psi} \Gamma_j \psi \rangle = f_{\text{os}}(\mathbf{x})$
- $\langle \phi(x)\phi(0) \rangle \sim C_{\text{osc}}(x)$



[Buballa, Carignano, PPNP **81**, 39-96 (2015)]

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- Disordering through Goldstone modes of chiral SSB???
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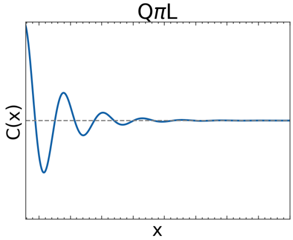

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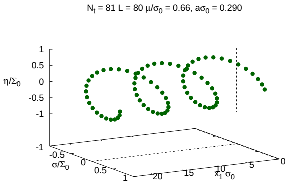

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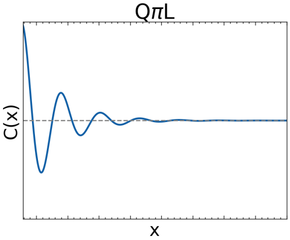

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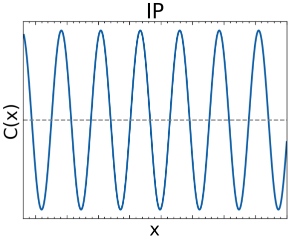

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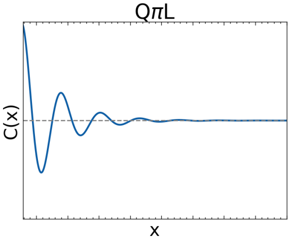

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Exotic phases at $\mu \neq 0$ from functional methods: FRG

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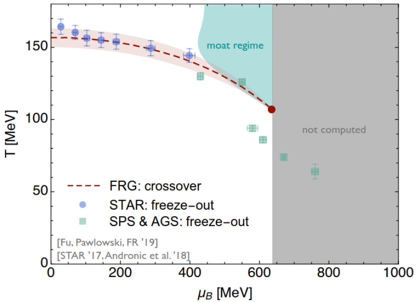
- Moat regime: $E^2 = Wp^4 + Zp^2 + m^2 + \mathcal{O}\left(p^6\right)$ with $Z < 0$

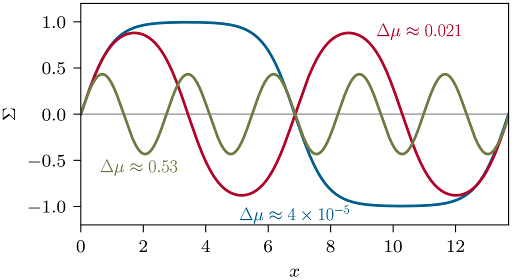
Recent FRG study

[Fu, Pawlowski, Rennecke, PRD **101**, 054032 (2020)]



$1+1$-dimensional Four-Fermion model

[Thies, Urlichs, PRD **67**, 125015 (2003)]



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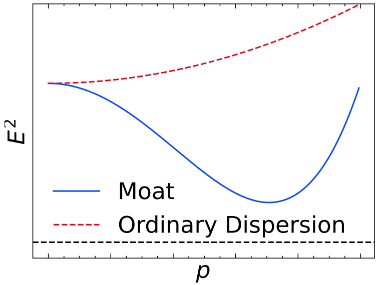

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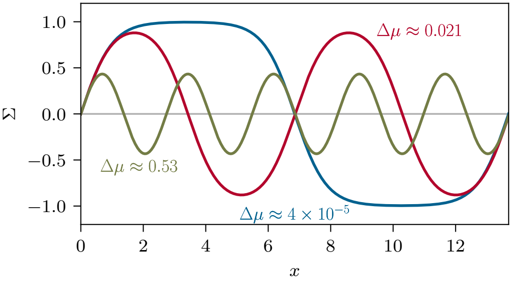

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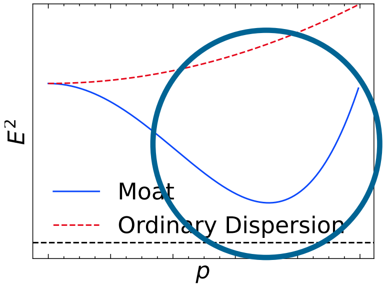

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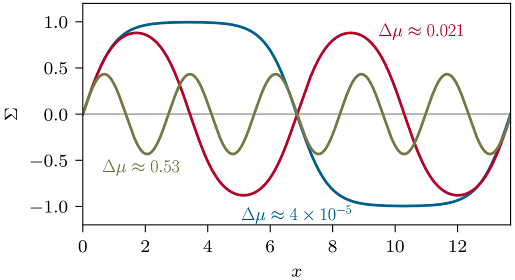

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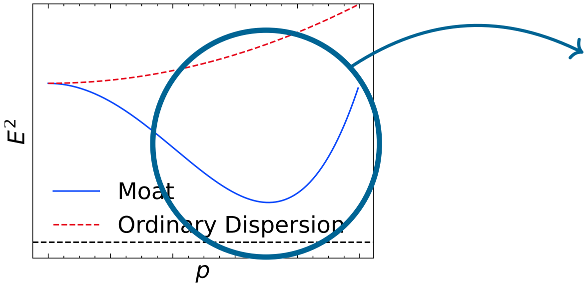

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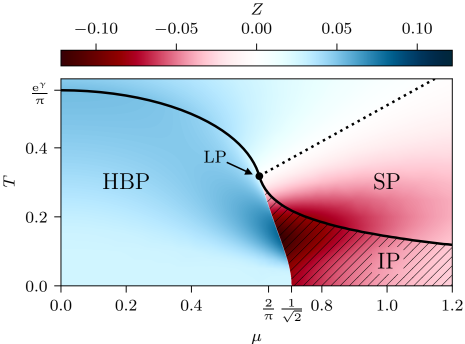

- Four-quark interactions / quark-meson as low-energy approximation for QCD from $\Lambda \gtrsim 1\text{GeV}$
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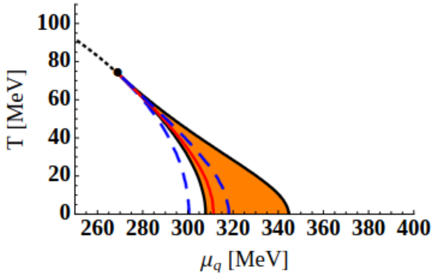

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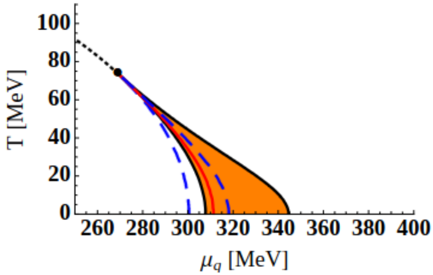

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BUT: Evidence that IPs are cutoff artifacts from multiple studies!

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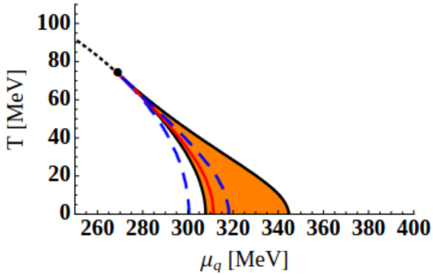

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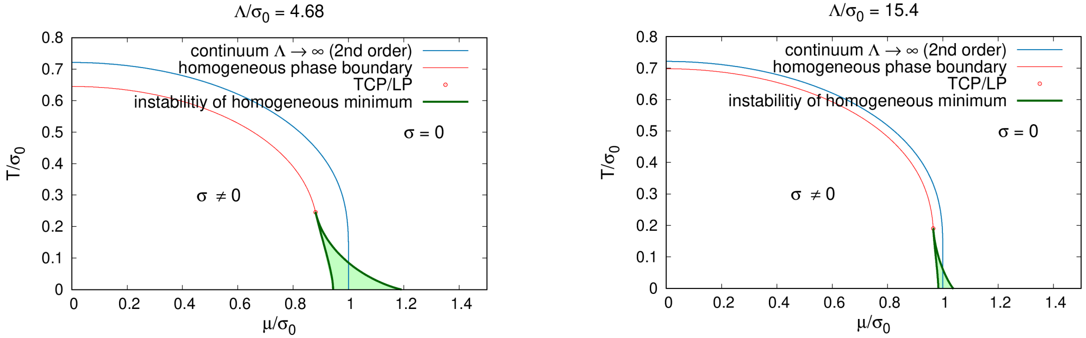
[Narayanan, PRD (2021)], [Buballa et al., PRD (2021)], [Pannullo, MW, PRD (2023)], [Pannullo, PRD (2023)], [Koenigstein, Pannullo, PRD (2023)], [Pannullo, MW, Wagner, PRD (2024)]

Regulator dependence of IPs: 2+1-dimensional GN model



- Strong regulator dependence of results & IP vanishes when renormalizing the theory !

[Buballa, Kurth, Wagner, MW, PRD **103**, 034503 (2020), arXiv: 2012.09588]

- No instability towards IP in general four-fermion model with scalar $(S = 0)$ channels

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

- Strong regularization scheme dependence in non-renorm. NJL model

[Pannullo, MW, Wagner, PRD (2024)]

Generic four-fermion model setup

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- Study of $2+1$-dimensional models

$$\mathcal{S}_{\mathrm{FF}}[\bar\psi,\psi] = \int \mathrm{d}^3x \left\{ \bar\psi \left(\partial\!\!\!/ + \gamma_0\mu\right)\psi - \sum_j \left(\frac{\lambda_j}{2N} \left(\bar\psi\, c_j\, \psi\right)^2 \right) \right\}$$

- 4×4 Dirac basis for chiral symmetry, 2 flavors ($\gamma_{45} = \mathrm{i}\gamma_4\gamma_5$)

$$(c_j) \in (1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec\tau, \mathrm{i}\vec\tau\gamma_4, \mathrm{i}\vec\tau\gamma_5, \vec\tau\gamma_{45}) \times (1, \mathrm{i}\gamma_\mu)$$

- Bosonized version

$$S[\bar\psi,\psi,\vec\phi] = \int \mathrm{d}^3x \left\{ N \sum_j \frac{\phi_j^2}{2\lambda_j} + \bar\psi Q \psi \right\}, \quad Q = \partial\!\!\!/ + \gamma_0\mu + \sum_j c_j\,\phi_j,$$

- Integrate out $\bar\psi,\psi \Rightarrow S_{\mathrm{eff}}[\vec\phi] \sim \ln \mathrm{Det} Q$

How to detect the IP

Large-N limit / Mean-field approximation: No integration about $\vec{\phi}$

$$\Rightarrow \Omega \sim \min_{\vec{\phi}} S_{\mathrm{eff}}[\vec{\phi}]$$

- Global minimization of $S_{\mathrm{eff}}[\vec{\phi}(\mathbf{x})]$ using ansatzes / lattice field theory

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  - $(-)$ Less info about ground state: Only momentum $\mathbf{q}$ with most negative curvature

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Stability analysis

$$\phi_j(x) = \bar{\phi}_j + \delta\phi_j(\mathbf{x})$$

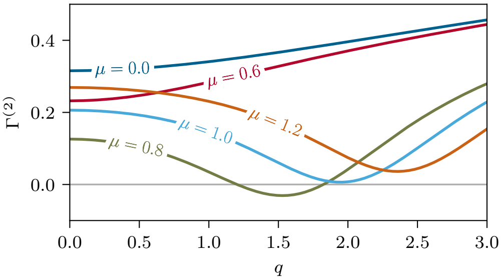
- First non-vanishing correction expressed by Hessian matrix (analog of mass matrix $\mathcal{M}$)

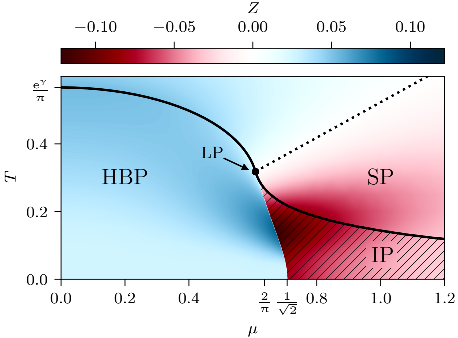
$$\frac{S_{\text{eff}}^{(2)}}{N} = \frac{\beta}{2}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, \delta\vec{\phi}^{\,T}(-\mathbf{q}) H(|\mathbf{q}|) \delta\vec{\phi}(\mathbf{q}), \quad H_{\phi_j\phi_k}(q) = \left(\frac{\delta_{j,k}}{\lambda_j}\right) + \frac{1}{\beta}\sum\!\!\!\!\!\!\int_p \operatorname{tr}\left[S(p+(0,\vec{q}))\, c_j\, S(p)\, c_k\right]$$

- Free fermion propagator S with mass $M^2(\bar{\phi}_j)$ at fixed μ and T
- Eigenvalues of Hessian $\Rightarrow$ Bosonic two-point functions $\Gamma^{(2)}_{\varphi_j}(q) = \left(\langle \varphi_j \varphi_j \rangle_c\right)^{-1}$
- Inhomogeneous phase: $\Gamma^{(2)}(q) < 0$ for $q \neq 0$ Moat: $Z = \frac{\mathrm{d}^2 \Gamma^{(2)}}{\mathrm{d}q^2}(q=0) < 0$
- Quantum pion liquid: $\Gamma^{(2)}(q=0) \in \mathbb{C}$ but appear in complex-conjugate pairs!

$$S[\bar\psi,\psi] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \left\{ \bar\psi \left(\partial\!\!\!/ + \gamma_0\mu\right)\psi - \tfrac{\lambda}{2N}\left(\bar\psi\,\psi\right)^2 \right\} \xrightarrow{\text{Bosonize}} S[\bar\psi,\psi,\sigma] = \int \mathrm{d}^2x \left\{ \frac{\sigma^2}{2\lambda} + \bar\psi\left(\partial\!\!\!/ + \gamma_0\mu + \sigma\right)\psi \right\}$$

$T/\Sigma_0 = 0.15$





[Koenigstein, Pannullo, Rechenberger, MW, Steil, (2022)]

- $\mu = 1.2$ corresponds to a Moat regime
- $\mu = 0.8$ corresponds to the IP

Four-fermion model with $\mathcal{PT}$-type symmetry in $D = 2+1$

$$S_{\text{mix}}[\bar\psi, \psi] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2x \left\{ \bar\psi \left(\partial\!\!\!/ + \gamma_3 \mu \right) \psi - \left[\frac{\lambda_S}{2N} \left(\bar\psi\, \psi \right)^2 + \frac{\lambda_V}{2N} \left(\bar\psi\, \mathrm{i}\gamma_\nu\, \psi \right)^2 \right] \right\}$$

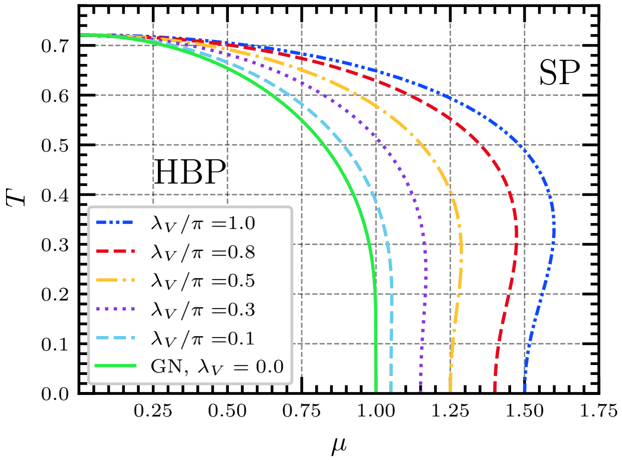
- Bosonization similar to before

$$S[\bar\psi, \psi, \sigma, \omega_\nu] = \int \mathrm{d}^3x \left[\bar\psi \left(\partial\!\!\!/ + \mathrm{i}\gamma_\nu \omega_\nu + \gamma_0 \mu + \sigma \right) \psi + \frac{\omega_\nu \omega_\nu}{2\lambda_V} + \frac{\sigma^2}{2\lambda_S} \right]$$

- Homogeneous condensation:
  - $\bar\omega_j = 0$ & $\omega_0 \sim \mathrm{i}\langle \psi^\dagger \psi \rangle / N$ purely imaginary; shift in chemical potential $\bar\mu = \mu + \mathrm{i}\bar\omega_0$
- Charge symmetry breaking at $\mu \neq 0$: $\mathcal{C}\omega_0 = -\omega_0$, but $\mathcal{CK}$ invariance – as in QCD!
- Complex saddle points $(\sigma, \omega_\nu) = \left(\bar\sigma, \mathrm{i}\bar n \delta_{0,\nu}\right)$!
- The following results are generic for all models with local $\left(\bar\psi \Gamma \psi\right)^2$ in $D = 2+1$

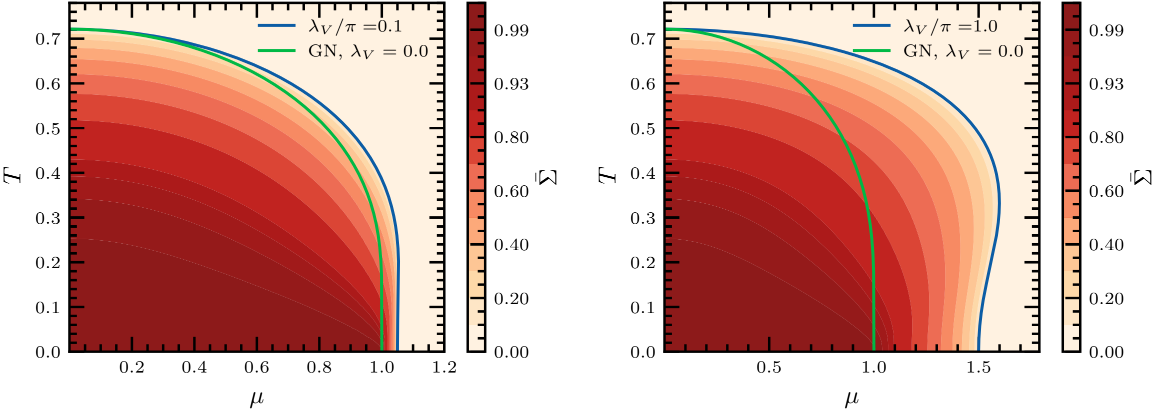
[MW, PRD **110**, 034008 (2024)]

$D = 2 + 1$ four-fermion model: Homogeneous phase boundary



In general: Broken phase enlarged by vector coupling

Homogeneous phase diagram: Chiral condensate



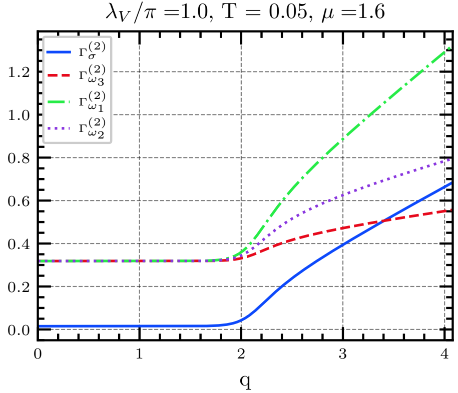
Now to the interesting object $H_{\phi_j \phi_k}$!

Mixing effect $H_{\sigma\omega_0} \sim \bar{\Sigma} \Rightarrow$ New physics in broken phase!

[MW, PRD **110**, 034008 (2024)]

Mean-field stability analysis: Diagonalize Hessian $H_{\phi_j\phi_k}(q)$

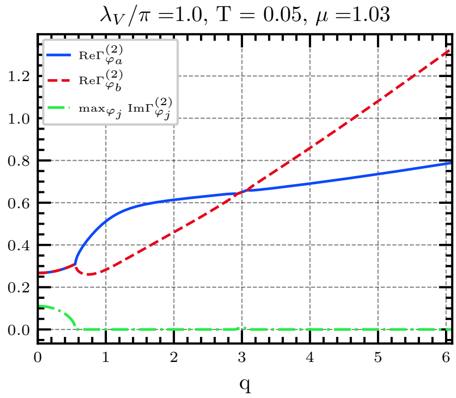
- Symmetric phase: No IP & no moat regime
- Exponentially decaying propagators, "normal" symmetric phase



Mean-field stability analysis: Diagonalize Hessian $H_{\phi_j \phi_k}(q)$

- Broken phase: $\Gamma^{(2)}_{\phi_j} \in \mathbb{C}$ with $\Gamma^{(2)*}_{\phi_j} = \Gamma^{(2)}_{\phi_k}$!
- Low momentum expansion of inverse propagators
- Poles are $q^2 = -\Gamma^{(2)}_{\phi}(q=0) \left(\frac{\mathrm{d}^2\Gamma^{(2)}_{\phi}}{\mathrm{d}q^2}\big|_{q=0} \right)^{-1} \in \mathbb{C}$

$$\Rightarrow \langle \phi(x)\phi(0) \rangle \sim \mathrm{e}^{-mx} \sin(px)$$



Only σ and ω_0 studied

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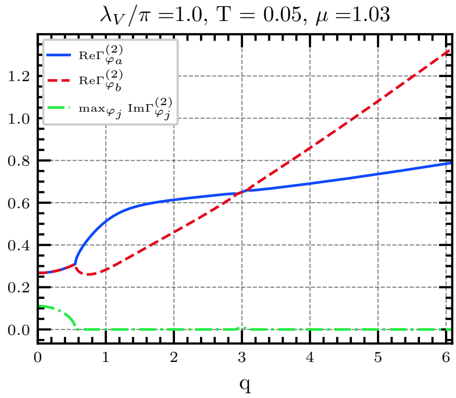
$$\Rightarrow \langle\phi(x)\phi(0)\rangle \sim \mathrm{e}^{-mx}\sin(px)$$

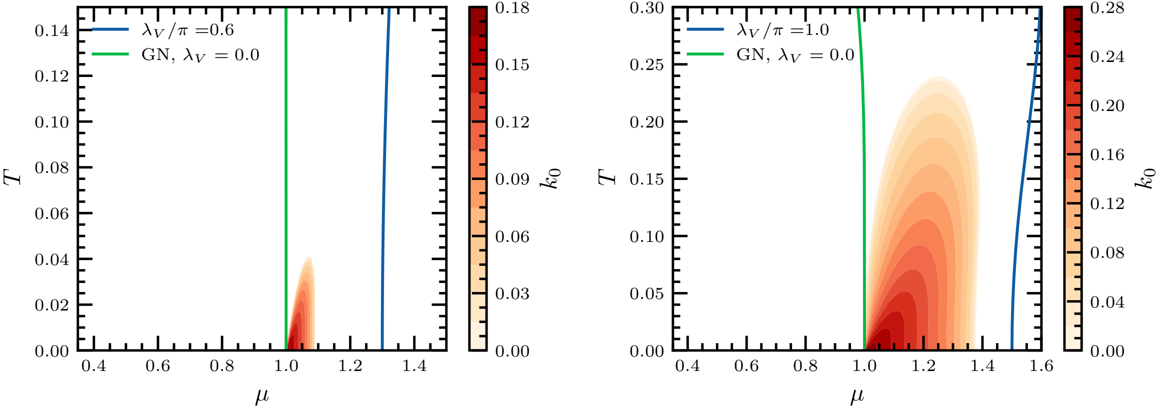
Quantum pion liquid, $\mathcal{PT}$ broken!



Only σ and ω_0 studied

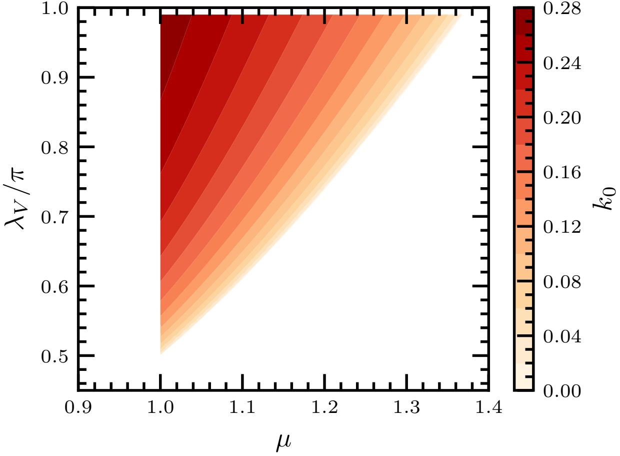
Static Hessian: Complex-conjugate eigenvalues

$k_0 = \max_j \mathrm{Im}\Gamma^{(2)}_{\phi_j}(0)$ – scale for oscillation

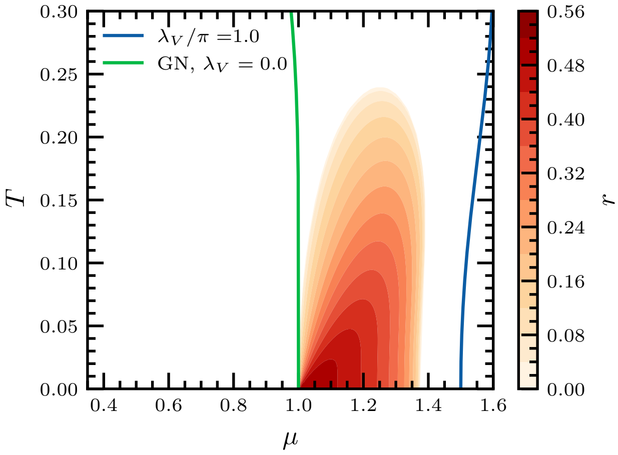


Static Hessian at zero temperature

Smooth onset of k_0 when increasing λ_V at fixed μ



$r \approx p/m$ with $C(x) \sim \mathrm{e}^{-mx}\cos(px)$



Frequency roughly of same order as m in large parts of the $\mathrm{Q}\pi\mathrm{L}$

Conclusions

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Observation of quantum pion liquid ($\mathrm{Q}\pi\mathrm{L}$)

- Spatially oscillating correlators $C(x) \sim \mathrm{e}^{-mx} \sin(kx)$ in NJL-type models ($D = 2 + 1$)
  [MW, PRD **110**, 034008 (2024)]
  - Related to $\mathcal{CK}$ invariance of FF model
  - Mixing between scalar and vector mesons; Competing attractive and repulsive interactions
  - Similar effects reported in Polyakov-Loop quark-meson model with ω
    [Haensch, Rennecke, von Smekal, PRD **110**, 036018(2024) arXiv:2308.16244]
- Regimes with spatially modulations very relevant in QCD at $\mu \neq 0$ ($\mathcal{PT}$ symmetry!)
  - $\mathcal{CK}$ symmetry as generalized $\mathcal{PT}$ symmetry – Results relevant in other contexts?
  - All mechanisms from above also apply to QCD at $\mu \neq 0$

Outlook

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- $\mathrm{Q}\pi\mathrm{L}$ seems stable against quantum fluctuations compared to inhomogeneous phase

  [MW, Valgushev, arXiv:2403.18640 (2024) & in preparation]

- $\mathrm{Q}\pi\mathrm{L}$: How to get realistic estimates for the ratio between decay rates and frequency?

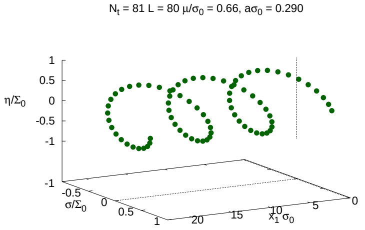
- Dilepton production rate as a experimental observable, $\pi^+ + \pi^- \to \gamma \to l^+ + l^-$: Spike at threshold given by non-trivial minimum of the dispersion

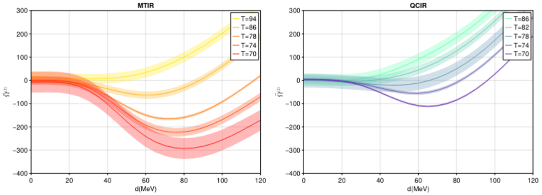
  [Hayashi, Tsue, arXiv: 2407.08523], [Nussinov, Ogilvie, Pannullo, Pisarski, Rennecke, Schindler, Winstel, Valgushev, in preparation]

- Think of more characteristic heavy ion collision observables for the moat regime & other spatially modulated regimes !

Appendix

Exotic phases at $\mu \neq 0$ from functional methods: DSE





- QCD Dyson-Schwinger setup with gluon propagator fitted to quenched lattice data
- Perturbative quark-loop effects

$\Rightarrow$ Chiral density wave is self-consistent solution of DSE

[Müller, Buballa, Wambach, PL B **727**, 240 (2013)]

- Stability analysis of 2PI effective action in rainbow ladder approximation
- Below certain temperature: $\langle\bar{\psi}\psi\rangle = 0$ is unstable with respect to IP
- Analysis can only be trusted on left spinodal where $\langle\bar{\psi}\psi\rangle = \text{const.} \neq 0$

[Motta et al., arXiv:2406.00205 (2024)]

$D = 2 + 1$ four-fermion model: General stability analysis

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$$S[\bar{\psi}, \psi, \vec{\phi}] = \int \mathrm{d}^3x \left\{ N \sum_j \frac{\phi_j^2}{2\lambda} + \bar{\psi} Q \psi \right\}, \quad Q = \not{\partial} + \gamma_0 \mu + \sum_j c_j \, \phi_j ,$$

- Curvature is diagonalizable

$$\Gamma^{(2)}_{\phi_j}\left(M^2, \mu, T, q^2\right) = \frac{1}{\lambda} - \ell_1\left(M^2, \mu, T\right) + L_{2,\phi_j}(M^2, \mu, T, q^2)$$

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CRC-TR 211

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- Momentum dependence fully contained in $L_{2,\phi_j}(M^2, \mu, T, q^2)$

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- For each combination of the 16 interaction channels we find:

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- For each combination of the 16 interaction channels we find:
  - $L_{2,\phi_j} = L_{2,+} = -\left(4M^2 + \mathbf{q}^2\right)\ell_2(\mathbf{q}^2)$ – known from GN model – no instabilities

$D = 2 + 1$ four-fermion model: General stability analysis

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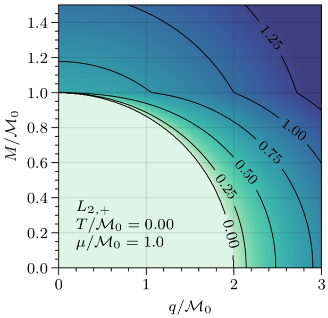
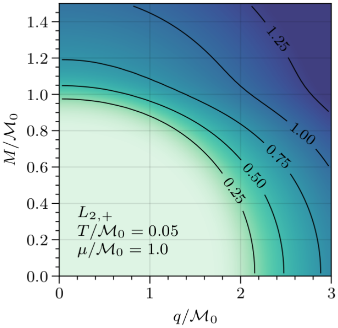
  - $L_{2,\phi_j} = L_{2,-} = -\mathbf{q}^2 \ell_2(\mathbf{q}^2)$ – also monotonically increasing – no instabilities
- Remember: $\Gamma^{(2)}(q \neq 0) < 0$ necessary for IP

Momentum dependence of two point function $L_{2,+}/L_{2,-}$

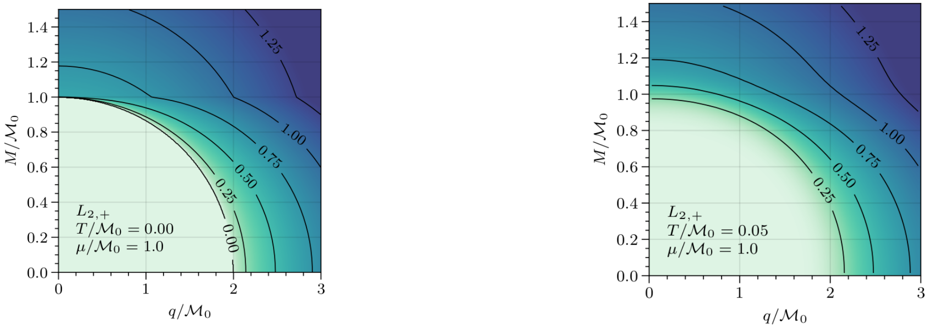




- $L_{2,+}/L_{2,-}$ are monotonically increasing functions of q for all M

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

Momentum dependence of two point function $L_{2,+}/L_{2,-}$



- $L_{2,+}/L_{2,-}$ are monotonically increasing functions of q for all M

No IP and no moat regime in all models containing (some of) these 16 interaction channels

$$\bar{\psi} c_j \phi_j \psi$$

$$(c_j)_{j=1,\ldots,16} = (1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \vec{\tau}\gamma_{45})$$

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

Examples (I): FF and Yukawa models

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Model	Used channels c_j	Field basis $\vec{\varphi}_j$ diagonalizing $\mathcal{S}^{(2)}_{\text{eff}}$	Momentum dependence of $\Gamma^{(2)}_{\varphi_j}$		Symmetry groups
			$L_{2,+}$	$L_{2,-}$	
GN	1	σ	σ		$\mathrm{U}_{\mathbb{1}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathbb{Z}_{\gamma_5}(2) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
NJL	$1, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5$	$\sigma, \vec{\pi}_4, \vec{\pi}_5$	σ	$\vec{\pi}_4, \vec{\pi}_5$	$\mathrm{U}_{\mathbb{1}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathrm{SU}_{A,\gamma_4}(2N) \times \mathrm{SU}_{A,\gamma_5}(2N) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
χHGN$_P$	$1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}$	$\sigma, \eta_4, \eta_5, \eta_{45}$ (for $\eta_{45} = 0$)	σ, η_{45}	η_4, η_5	$\mathrm{U}_{\gamma}(2N) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
PSFF	$1, \mathrm{i}\gamma_4, \mathrm{i}\gamma_5, \gamma_{45}, \vec{\tau}, \mathrm{i}\vec{\tau}\gamma_4, \mathrm{i}\vec{\tau}\gamma_5, \mathrm{i}\vec{\tau}\gamma_{45}$	$\sigma, \eta_4, \eta_5, \eta_{45}, \vec{a}_0, \vec{\pi}_4, \vec{\pi}_5, \vec{\pi}_{45}$ (for $\eta_{45} = \vec{\pi}_{45} = 0$)	$\sigma, \eta_{45}, \varsigma, \vec{\pi}_{45}$	$\eta_4, \eta_5, \vec{\pi}_4, \vec{\pi}_5$	$\mathrm{U}_{\gamma}(2N) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$

IPs and moat regime absent also for Yukawa versions of these models

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

Examples (II): Multiple chemical potentials

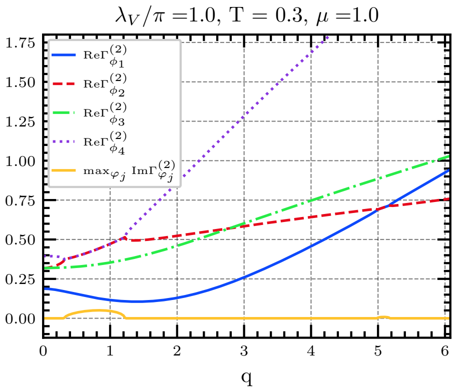
Used channels c_j	Bosonic auxiliary fields ϕ_j	Non-zero chemical potentials	Field basis $\vec{\varphi}_j$ diagonalizing $\mathcal{S}_{\mathrm{eff}}^{(2)}$	Momentum dependence of $\Gamma^{(2)}_{\varphi_j}$ $f(M^2,\mu) = L_{2,+}(M^2,\mu,T,\mathbf{q}^2)$	Underlying symmetry group
$1, \gamma_{45}$	σ, η_{45}	$\mu_L = \mu + \mu_{45}$ $\mu_R = \mu - \mu_{45}$	$\phi_L = (\sigma + \eta_{45})$ $\phi_R = (\sigma - \eta_{45})$	$f(\bar{\phi}_L^2, \mu_L)$ $f(\bar{\phi}_R^2, \mu_R)$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathbb{Z}_{\gamma_5}(2) \times \mathrm{SU}_{\vec{\tau}}(2) \times P_4 \times P_5$
$1, \tau_3$	$\sigma, a_{0,3}$	$\mu_\uparrow = \mu + \mu_I$ $\mu_\downarrow = \mu - \mu_I$	$\phi_\uparrow = (\sigma + \varsigma_3)$ $\phi_\downarrow = (\sigma - \varsigma_3)$	$f(\bar{\phi}_\uparrow^2, \mu_\uparrow)$ $f(\bar{\phi}_\downarrow^2, \mu_\downarrow)$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathbb{Z}_{\gamma_5}(2) \times \mathrm{U}_{\tau_3}(1) \times P_4 \times P_5$
$1, \tau_3\gamma_{45}$	$\sigma, \pi_{45,3}$	$\mu_{L,\uparrow} = \mu_L + \mu_I$ $\mu_{L,\downarrow} = \mu_L - \mu_I$ $\mu_{R,\uparrow} = \mu_R + \mu_I$ $\mu_{R,\downarrow} = \mu_R - \mu_I$	$\varphi_+ = (\sigma + \pi_{45,3})$ $\varphi_- = (\sigma - \pi_{45,3})$	$f(\bar{\varphi}_+^2, \mu_{L,\uparrow}) + f(\bar{\varphi}_+^2, \mu_{R,\downarrow})$ $f(\bar{\varphi}_-^2, \mu_{L,\downarrow}) + f(\bar{\varphi}_-^2, \mu_{R,\uparrow})$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathbb{Z}_{\gamma_5}(2) \times \mathrm{U}_{\tau_3}(1) \times P_4 \times P_5$
$1, \tau_3, \gamma_{45}$	$\sigma, a_{0,3}, \eta_{45}$	$\mu_{L,\uparrow}, \mu_{L,\downarrow}, \mu_{R,\uparrow}, \mu_{R,\downarrow}$	$\phi_L = (\sigma + \eta_{45})$ $\phi_R = (\sigma - \eta_{45})$ $a_{0,3}$	$f((\bar{\phi}_L + \bar{a}_{0,3})^2, \mu_{L,\uparrow}) + f(\bar{\phi}_L - \bar{a}_{0,3})^2, \mu_{L,\downarrow})$ $f((\bar{\phi}_R + \bar{a}_{0,3})^2, \mu_{R,\uparrow}) + f(\bar{\phi}_R - \bar{a}_{0,3})^2, \mu_{R,\downarrow})$ $\Gamma^{(2)}_{\phi_L} + \Gamma^{(2)}_{\phi_R}$	$\mathrm{U}_{\mathbb{I}_4}(N) \times \mathrm{U}_{\gamma_{45}}(N) \times \mathbb{Z}_{\gamma_5}(2) \times \mathrm{U}_{\tau_3}(1) \times P_4 \times P_5$

With multiple chemical potentials the analysis is straightforward only for a limited number of interaction channels

[Pannullo, MW, PRD **108**, 036011 (2023) arXiv:2305.09444]

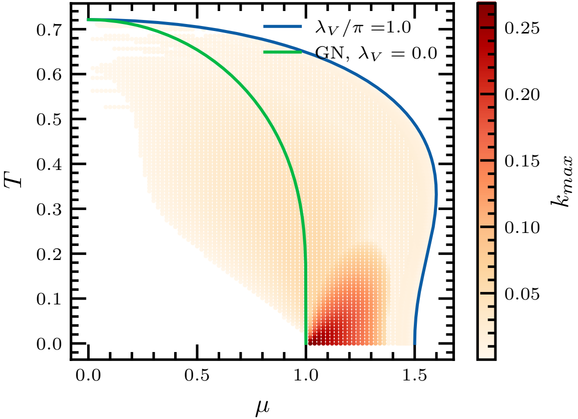
Non-vanishing imaginary part at $q \neq 0$

- At larger T: $H(0)$ has real eigenvalues, but $H(q \neq 0)$ develops compl. conj. eigenvalue pairs
- Caused by mixing of spatial vector components ω_j
- Interpretation of this phenomenon unclear so far

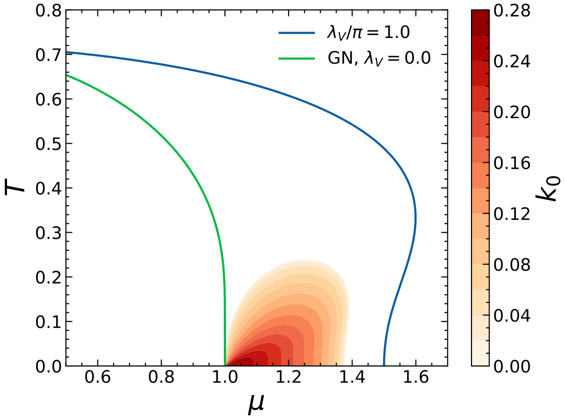


Complex-conjugate eigenvalues for $q \neq 0$

$$k_{\max} = \max_{j,q}\left(\mathrm{Im}\Gamma^{(2)}_{\phi_j}(q)\right)$$



Static case – $k_0 = \max_j \mathrm{Im}\Gamma^{(2)}_{\phi_j}(0)$



Complex-conjugate eigenvalues for $q \neq 0$

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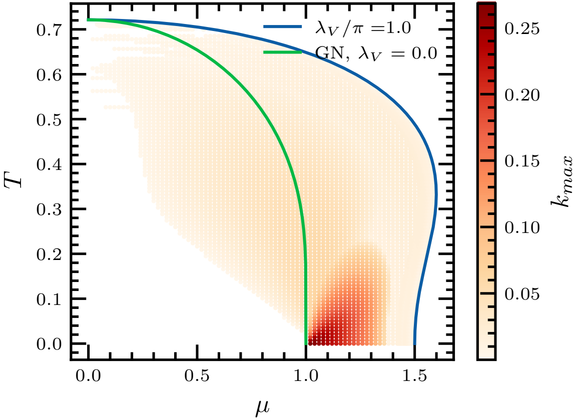
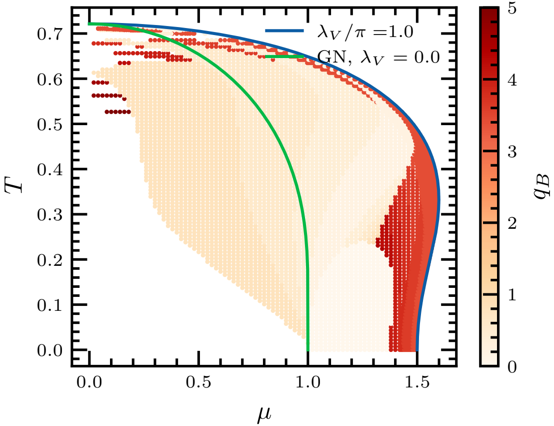
$$k_{\max} = \max_{j,q} \left(\mathrm{Im}\Gamma^{(2)}_{\phi_j}(q) \right)$$

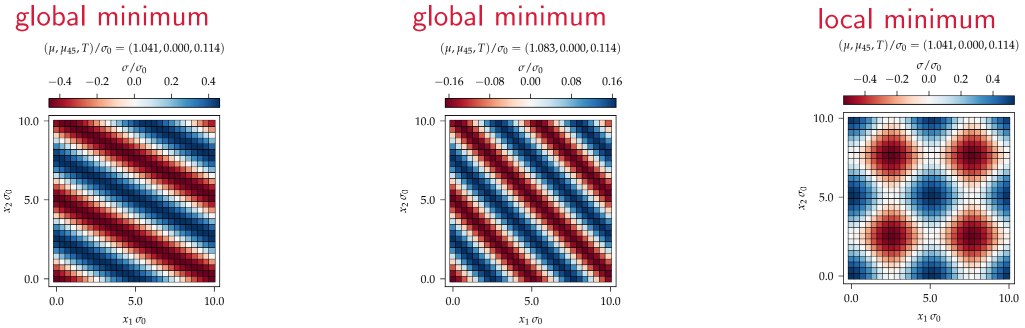
$$q_B = \min_{q \in C}, C = \{q \in [0, \infty) | \mathrm{Im}\Gamma^{(2)}_{\phi_j}(q) \neq 0\}$$





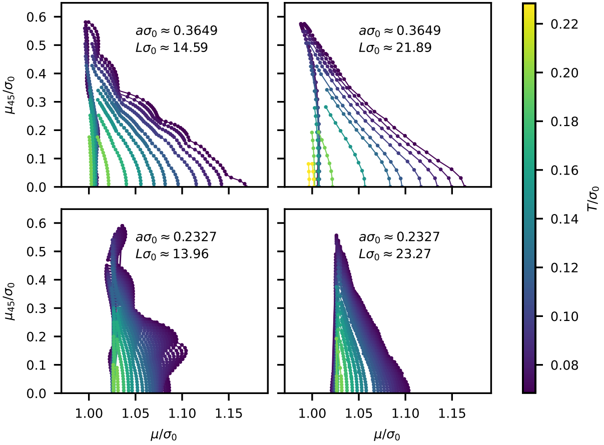
2+1-dimensional GN model : Search for inhomogeneities

- Lattice field theory: Stability analysis & brute for minimization
- At finite lattice spacing: inhomogeneous groundstates
- Thus: Do we also find an inhomogeneous phase in the continuum?



[Buballa, Kurth, Wagner, MW, PRD **103**, 034503 (2020), arXiv: 2012.09588], [Pannullo, Wagner, MW, Symmetry **14**, 265 (2022), arXiv:2112.11183]

Stability analysis on the lattice of GN model with μ_{45} or μ_I



[Pannullo, Wagner, MW, Symmetry **14**, 265 (2022), arXiv:2112.11183]

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$$L_{\text{eff}} = \frac{Z}{2}\left(\partial_j \vec{\phi}\right)^2 + \frac{1}{2M^2}\left(\sum_j \partial_j^2 \vec{\phi}\right)^2 + \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda N}{4}(\vec{\phi}^2)^2$$

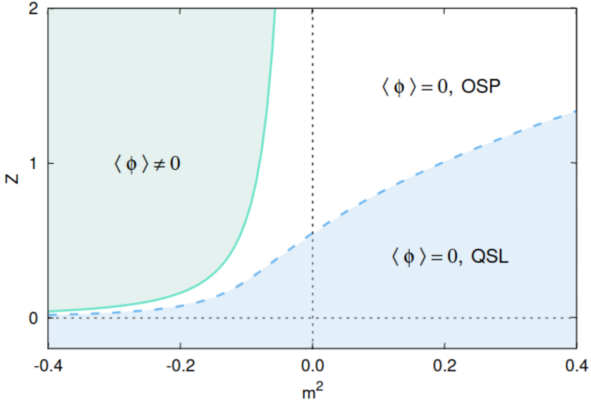
- Large-N limit easily solvable with constraint field approach, λN & M is fixed
  [Pisarski, Tsvelsik, Valgushev, PRD $\mathbf{102}, 016015\ (2020)$] [Moshe, Zinn-Justin, Phys. Rept. $\mathbf{385}, 69 - 228\ (2003)$]
- Vary only m^2 and Z
- Theory in $3+1$ dim. at nonzero T: Consider only static mode $E_n = 2\pi T n$ for $n = 0$
- Natural ansatz for ground state: The chiral spiral

$$\vec{\phi} = \phi_0 \left(\cos(k_0 z), \sin(k_0 z), \phi_\perp = \vec{0}\right)^T$$

- As expected: $k_0 \neq 0$ is solution for classical EoM

Solve theory in Large-N limit with chiral spiral ansatz

[Pisarski, Tsvelsik, Valgushev, PRD **102**, 016015 (2020)]



"Quantum spin liquid" (QSL)

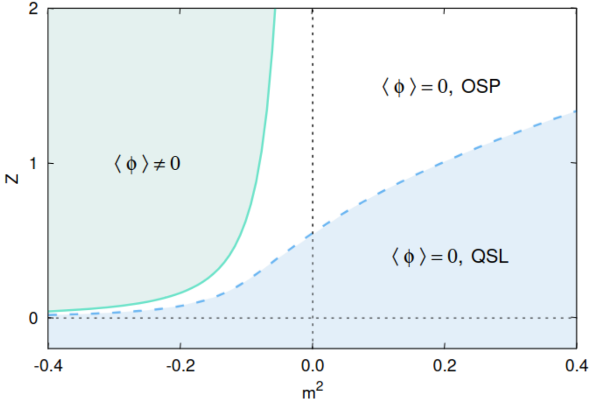
- Disordering of chiral spiral (IP) via fluctuation of transverse modes $\phi_\perp$
- Transverse modes $\equiv$ Goldstone modes of $O(N)$ symmetry breaking

$$\text{QSL} : \langle \phi^i(x)\phi^j(0)\rangle|_{x\to\infty} \sim \delta^{ij} \mathrm{e}^{-m_r} c_1 \cos(m_i x)!$$

Large-N phase diagram

Solve theory in Large-N limit with chiral spiral ansatz

[Pisarski, Tsvelsik, Valgushev, PRD **102**, 016015 (2020)]



"Quantum spin liquid" (QSL)

- Disordering of chiral spiral (IP) via fluctuation of transverse modes $\phi_\perp$
- Transverse modes $\equiv$ Goldstone modes of $O(N)$ symmetry breaking

QSL : $\langle \phi^i(x)\phi^j(0)\rangle|_{x\to\infty} \sim \delta^{ij} \mathrm{e}^{-m_r} c_1 \cos(m_i x)$!

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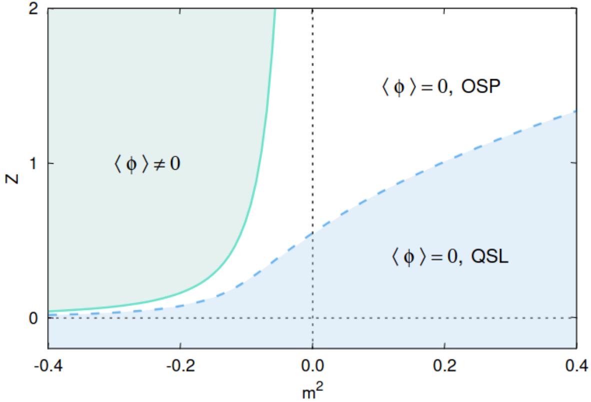

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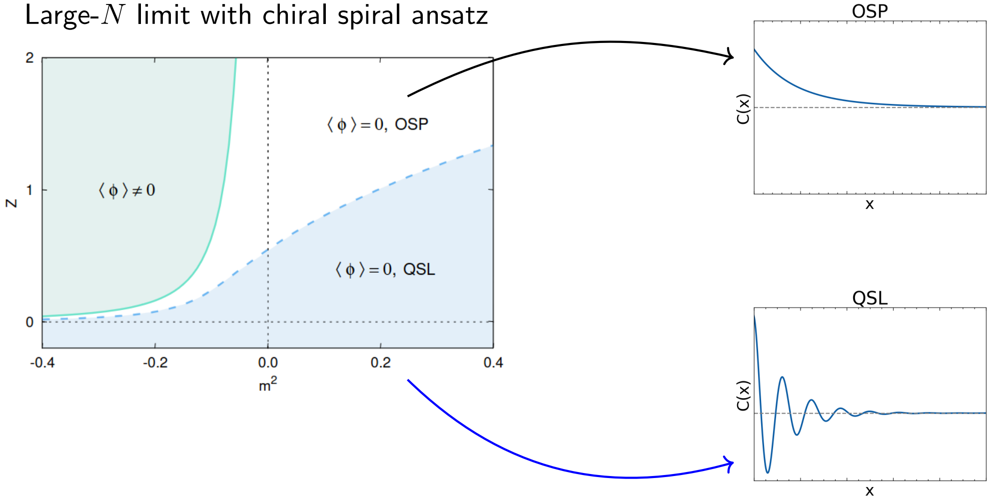
Large-N limit with chiral spiral ansatz



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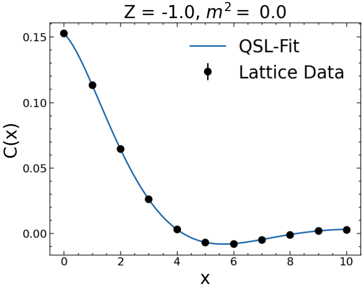
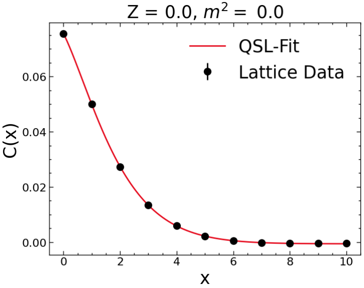
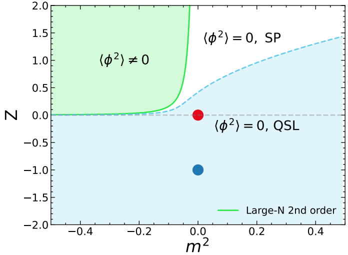
Correlators are (discrete) rotationally symmetric, study $C(\mathbf{x}) = C((x, 0, 0))$





Large-N



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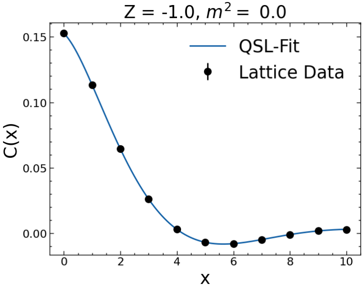

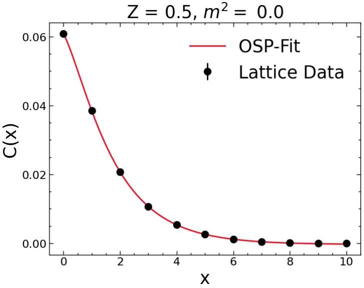

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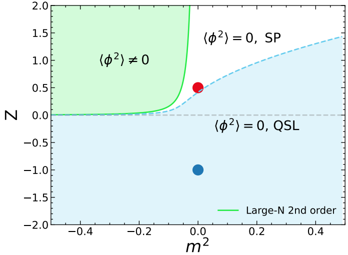

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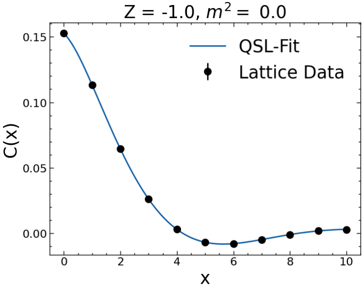

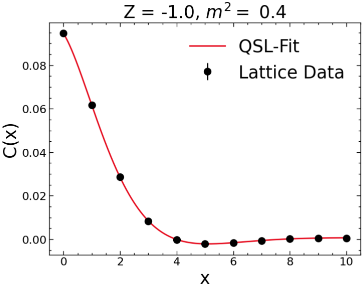

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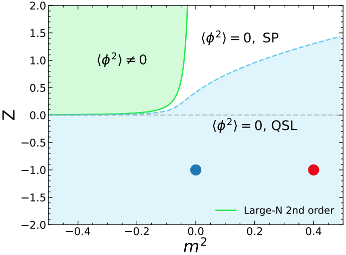

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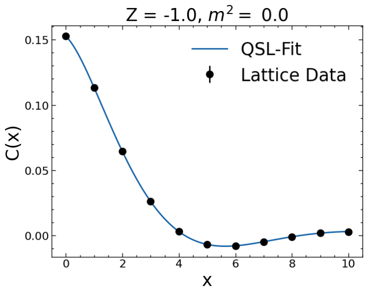

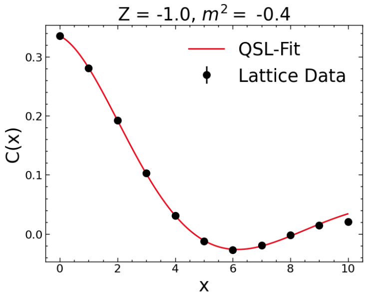

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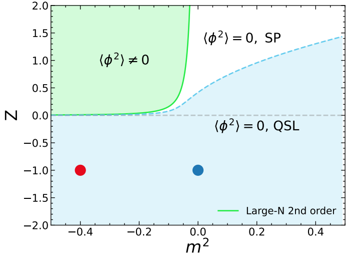

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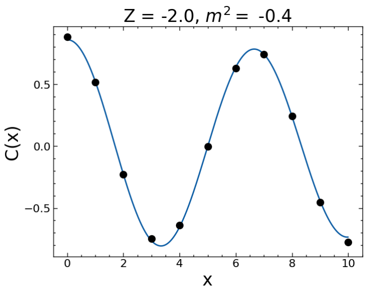

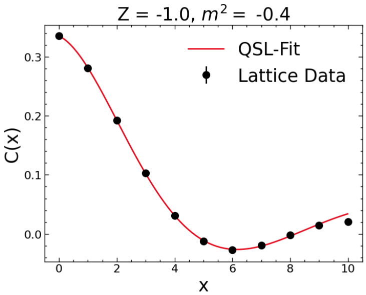

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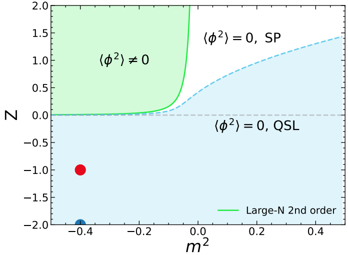
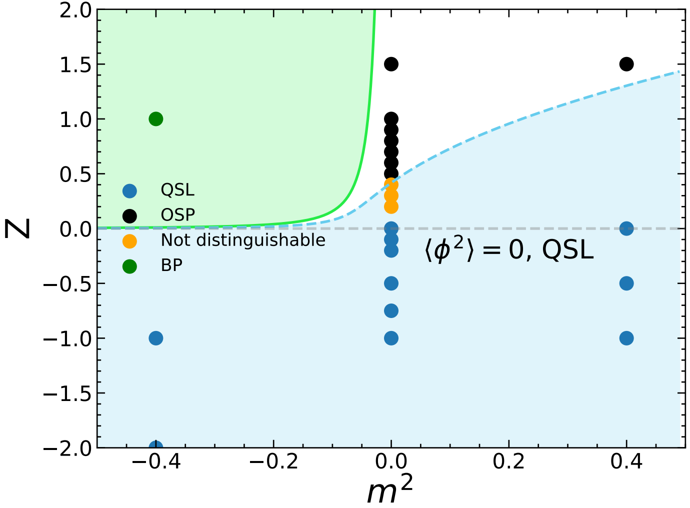
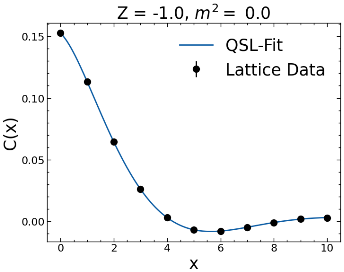


'Phase diagram' $N = 1$

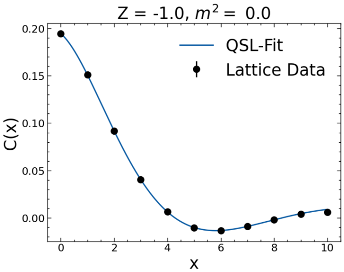


Independence of N

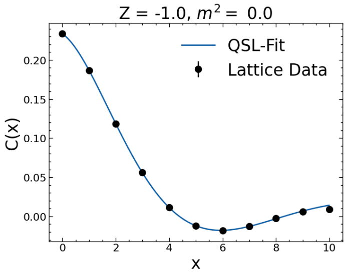
$N = 1$



$N = 2$



$N = 4$



- $m^2 = 0.0, Z \in [-1.0, 1.0]$ Resulting regimes (QSL /OSP) almost independent of $N (= 1, 2, 4, 8, 10)$
- Mechanism through disordering with transverse modes cannot be the full picture
- Lacking a (local) order parameter to fully map out regime in (Z, m^2) plane