Integrability in Non-Hermitian Physics and Field Theories

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In collaboration with : M. Cárdenas, J. Cen, A. Fring, V. Jakubský, K. Lara, M. Pino, M. Plyushchay and T. Taira.

Applications of Field Theory to Hermitian and Non-Hermitian Systems

September 10-13, 2024

Integrable systems & Gravity

Inverse scattering method

Einstein-Maxwell equations

Gravity-Electromagnetism

-Cosmological models

-Collision of exact gravity waves

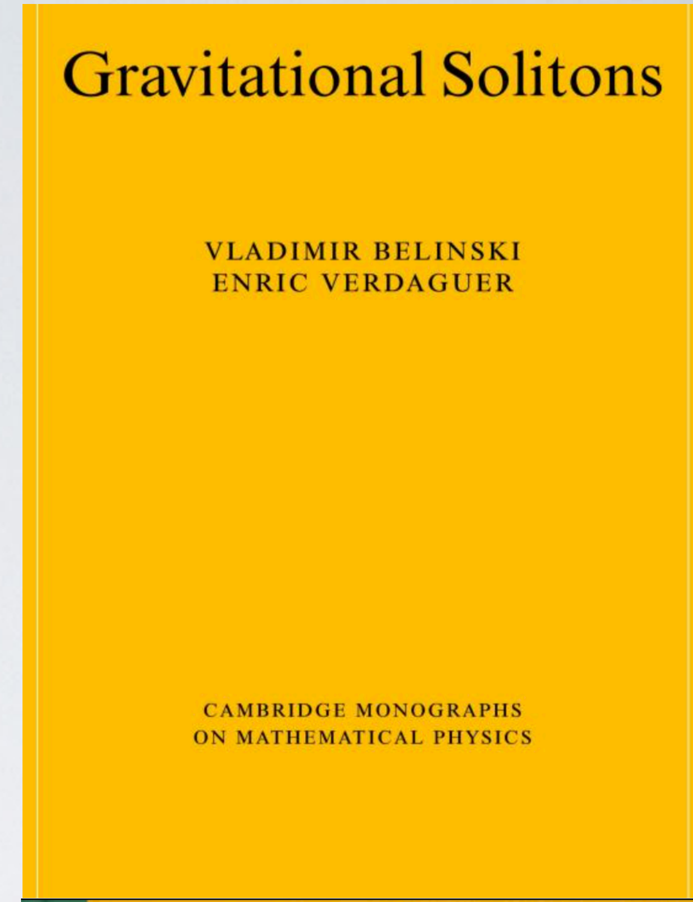
-Schwarzschild black hole

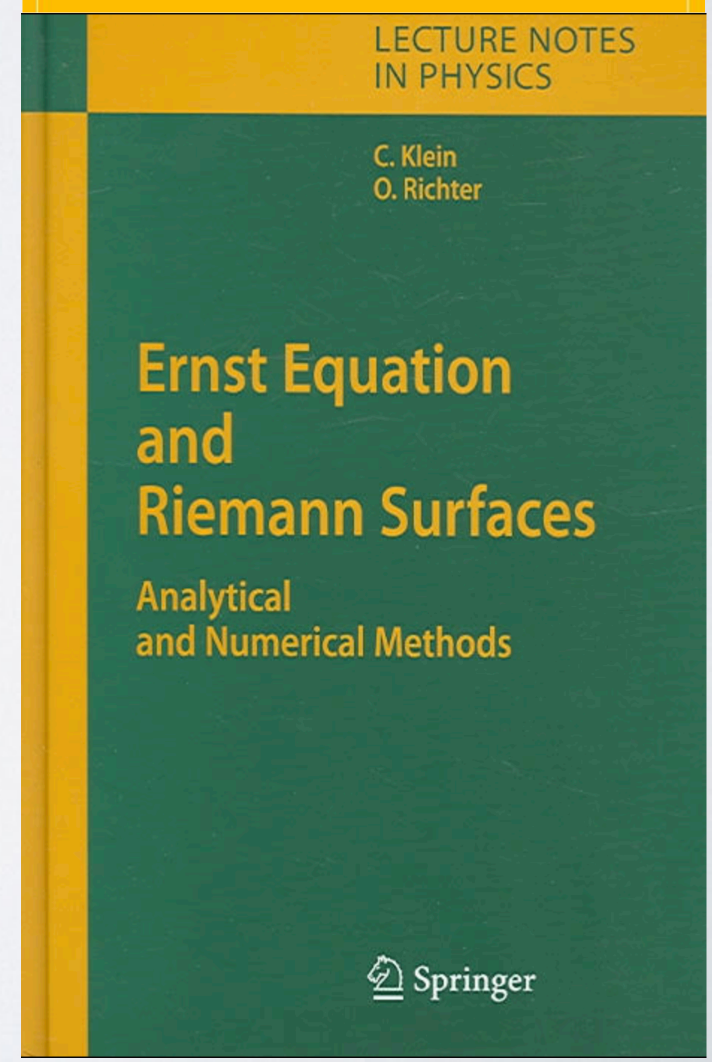
-Kerr black hole

-Taub-NUT

And more...

Universality in Binary Black Hole Dynamics: An Integrability Conjecture
J.L. Jaramillo, B. Krishnan, C. F. Sopuerta, **arXiv:2305.08554**





Integrable systems & Field Theories

Quantum Chromodynamics & self-interacting fermions

$$\mathcal{L}_{\mathrm{GN}} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi\right)^2$$

Gross-Neveu model

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi} i \gamma_5 \psi\right)^2\right]$$

Nambu-Jona-Lasinio model

Integrable systems & Field Theories

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Nambu-Jona-Lasinio model

Relativistic Hartree-Fock problem $H\psi = E\psi$ $\quad H = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix}$

Consistency condition $$\langle \bar{\psi}\psi \rangle - i \langle \bar{\psi} i \gamma^5 \psi \rangle = -\Delta / g^2$$

ility

G. Dunne, G. Basar (2008)

The nonlinear Schrödinger equation

Integrability & non-Hermitian physics

PT-SYMMETRY

Classical and quantum many particle models

Solitons and non-linear integrable equations

Quantum spin chains

Integrability & non-Hermitian physics

PT-SYMMETRY

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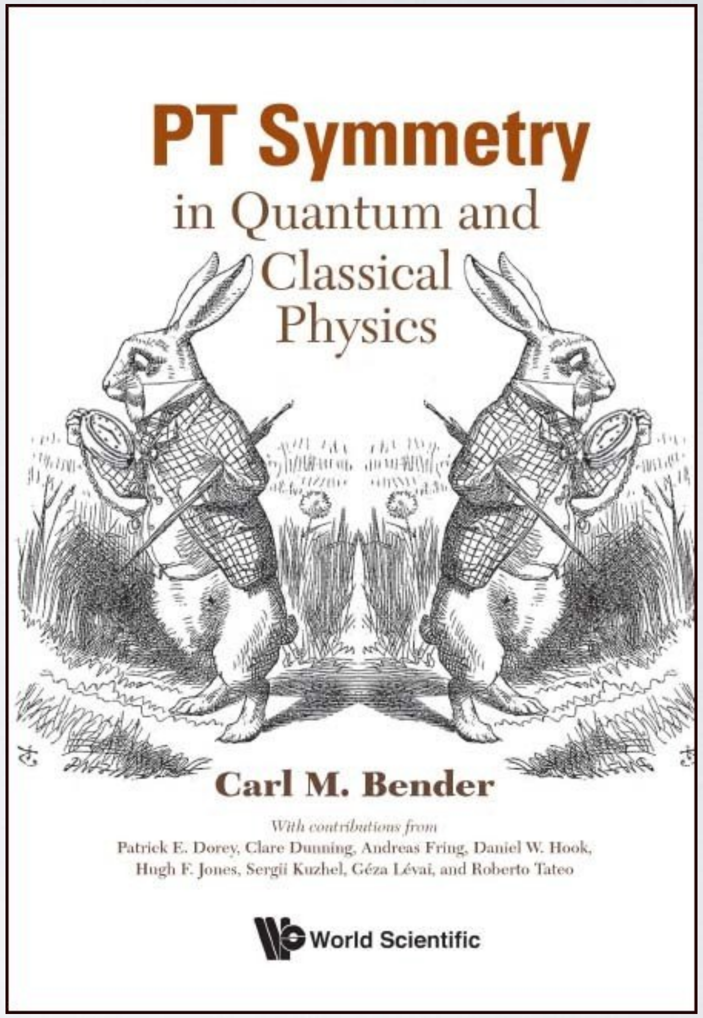
A. Fring, Phil. Trans. R. Soc. A (2012)



Integrability & non-Hermitian physics

PRL **100,** 030402 (2008) PHYSICAL REVIEW LETTERS week ending 25 JANUARY 2008

Optical Solitons in $\mathcal{PT}$ Periodic Potentials

Z. H. Musslimani
Department of Mathematics, Florida State University, Tallahassee, Florida 32306-4510, USA

K. G. Makris, R. El-Ganainy, and D. N. Christodoulides
College of Optics & Photonics-CREOL, University of Central Florida, Orlando, Florida 32816, USA
(Received 1 September 2007; revised manuscript received 24 October 2007; published 23 January 2008)

We investigate the effect of nonlinearity on beam dynamics in parity-time ($\mathcal{PT}$) symmetric potentials. We show that a novel class of one- and two-dimensional nonlinear self-trapped modes can exist in optical $\mathcal{PT}$ synthetic lattices. These solitons are shown to be stable over a wide range of potential parameters. The transverse power flow within these complex solitons is also examined.

$$i\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + [V(x) + iW(x)]\psi + |\psi|^2\psi = 0$$

Integrability & non-Hermitian physics

PRL **110,** 064105 (2013) PHYSICAL REVIEW LETTERS week ending 8 FEBRUARY 2013

Integrable Nonlocal Nonlinear Schrödinger Equation

Mark J. Ablowitz[1] and Ziad H. Musslimani[2]

[1] *Department of Applied Mathematics, University of Colorado, Campus Box 526, Boulder, Colorado 80309-0526*

[2] *Department of Mathematics, Florida State University, Tallahassee, Florida 32306-4510*

(Received 22 August 2012; published 7 February 2013)

A new integrable nonlocal nonlinear Schrödinger equation is introduced. It possesses a Lax pair and an infinite number of conservation laws and is *PT* symmetric. The inverse scattering transform and scattering data with suitable symmetries are discussed. A method to find pure soliton solutions is given. An explicit breathing one soliton solution is found. Key properties are discussed and contrasted with the classical nonlinear Schrödinger equation.

Outline

1. Integrability and non-Hermitian physics via a simple example
2. Features of integrable non-Hermitian field theories
3. Applications to black hole physics
4. Discussion

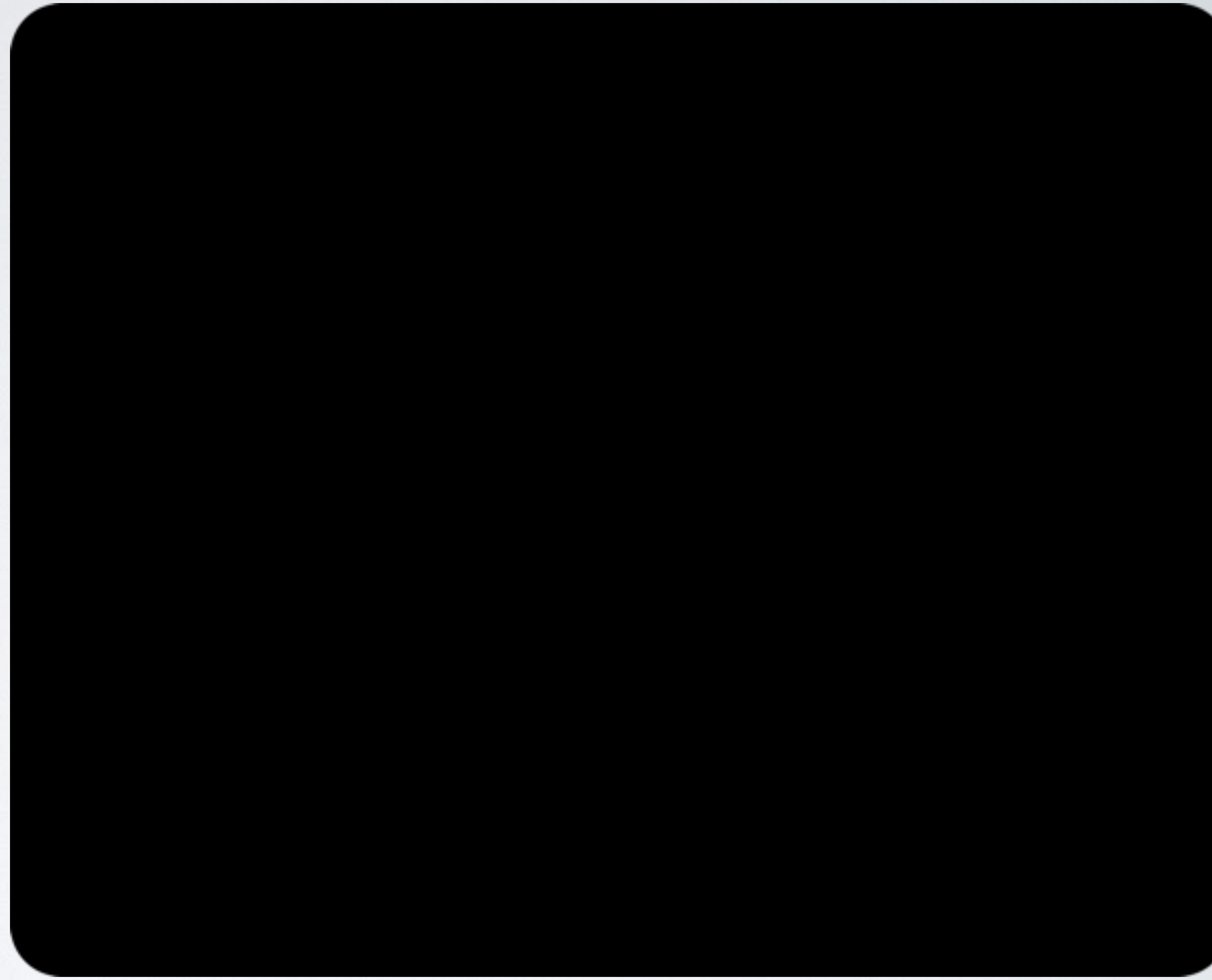
Outline

1. **Integrability and non-Hermitian physics via a simple example**

Solitons and integrable equations

$$u_t + 6uu_x + u_{xxx} = 0$$

Korteweg-de Vries equation



https://youtu.be/wEbYELtGZwI

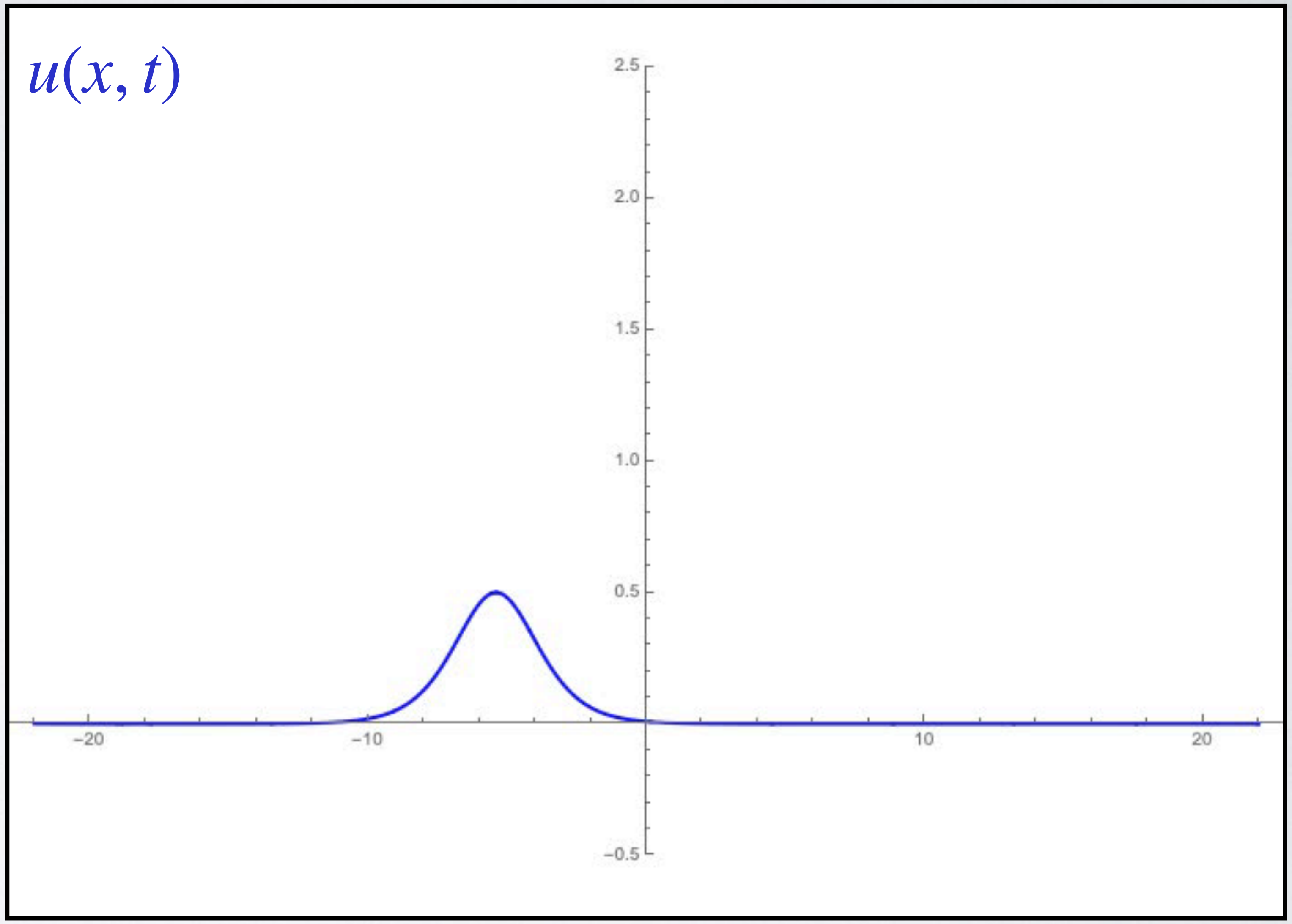
Laboratoire Interdisciplinaire CARNOT de Bourgogne, Équipe Solitons, Laser et Communications optiques

Solitons and integrable equations

$$u_t + 6uu_x + u_{xxx} = 0$$

Korteweg-de Vries equation

Two soliton



Solitons and integrable equations

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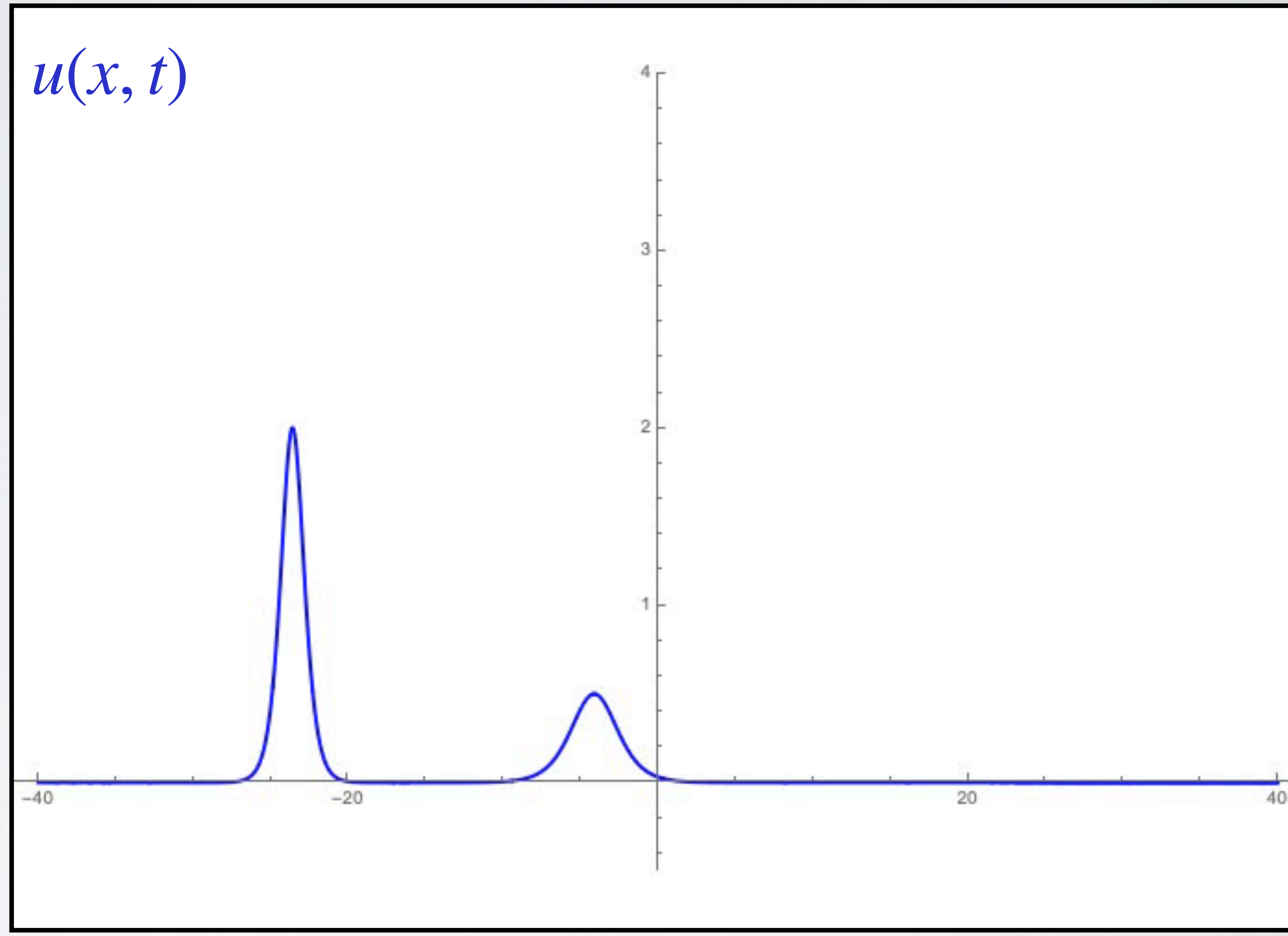
Three soliton



- **Different velocities**
- **Conserved quantities are functions of the velocities**
- **Total energy is the sum of individual solitons with different energies**
- **There is no a concept of a two-soliton with twice one soliton energy….**

$$m = \int_{-\infty}^{\infty} u dx \qquad p = \int_{-\infty}^{\infty} u^2 dx \qquad H = E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 - u^3 \right)$$

Mass **Momentum** **Energy**

Solitons and integrable equations

$$u_t + 6uu_x + u_{xxx} = 0$$

Korteweg-de Vries equation

KdV equation remains invariant under

$$\mathcal{PT}: \; x \to -x, \; t \to -t, \; i \to -i, \; u \to u$$

Complex Solitons and integrable equations

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$$u = p + iq$$

p even function

q odd function

A. Khare, A. Saxena PLA (2016)

J. Cen, A. Fring , JPhA (2016)

F.C, A. Fring, JHEP (2016)

Complex Solitons and integrable equations

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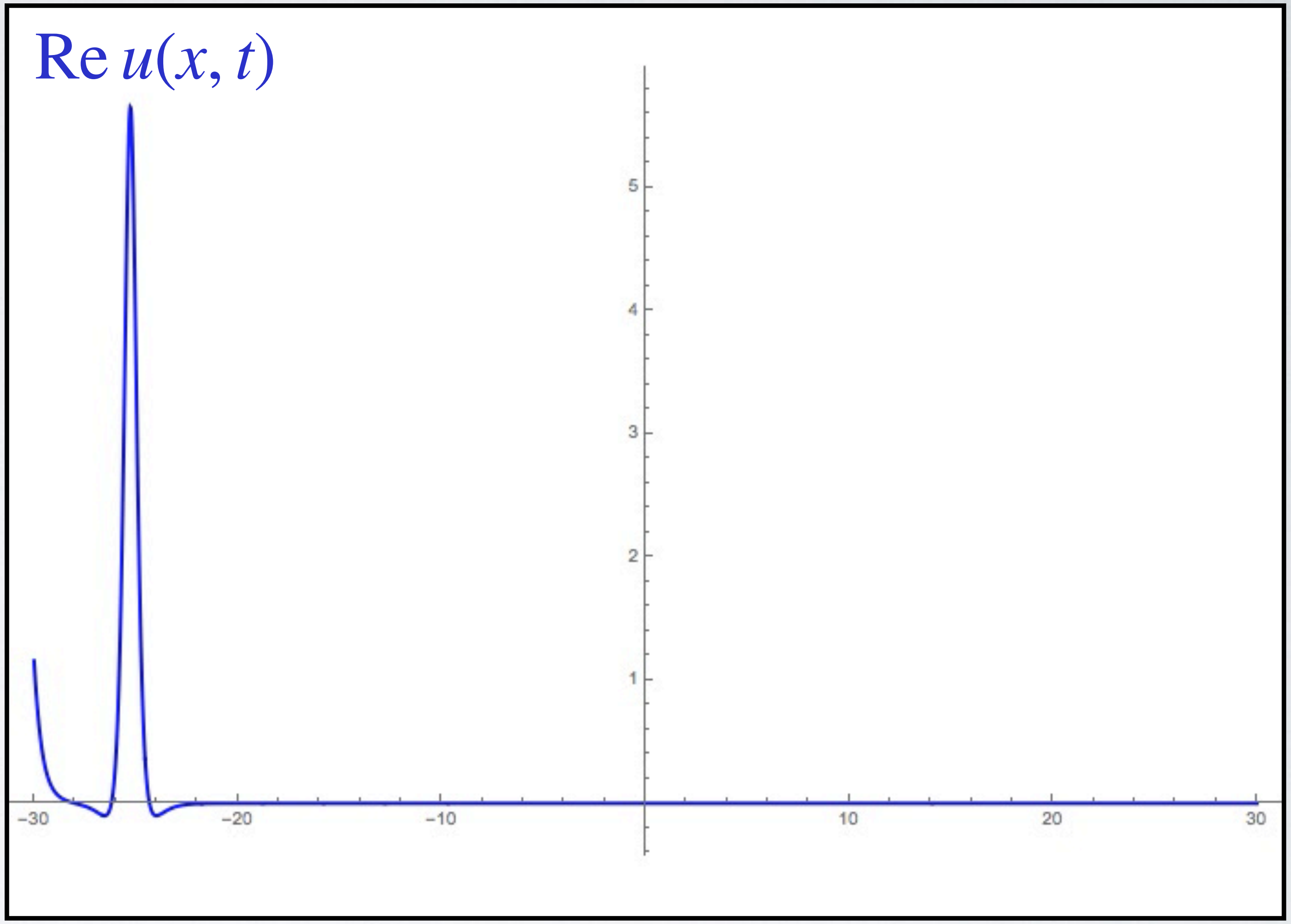
Two degenerate soliton



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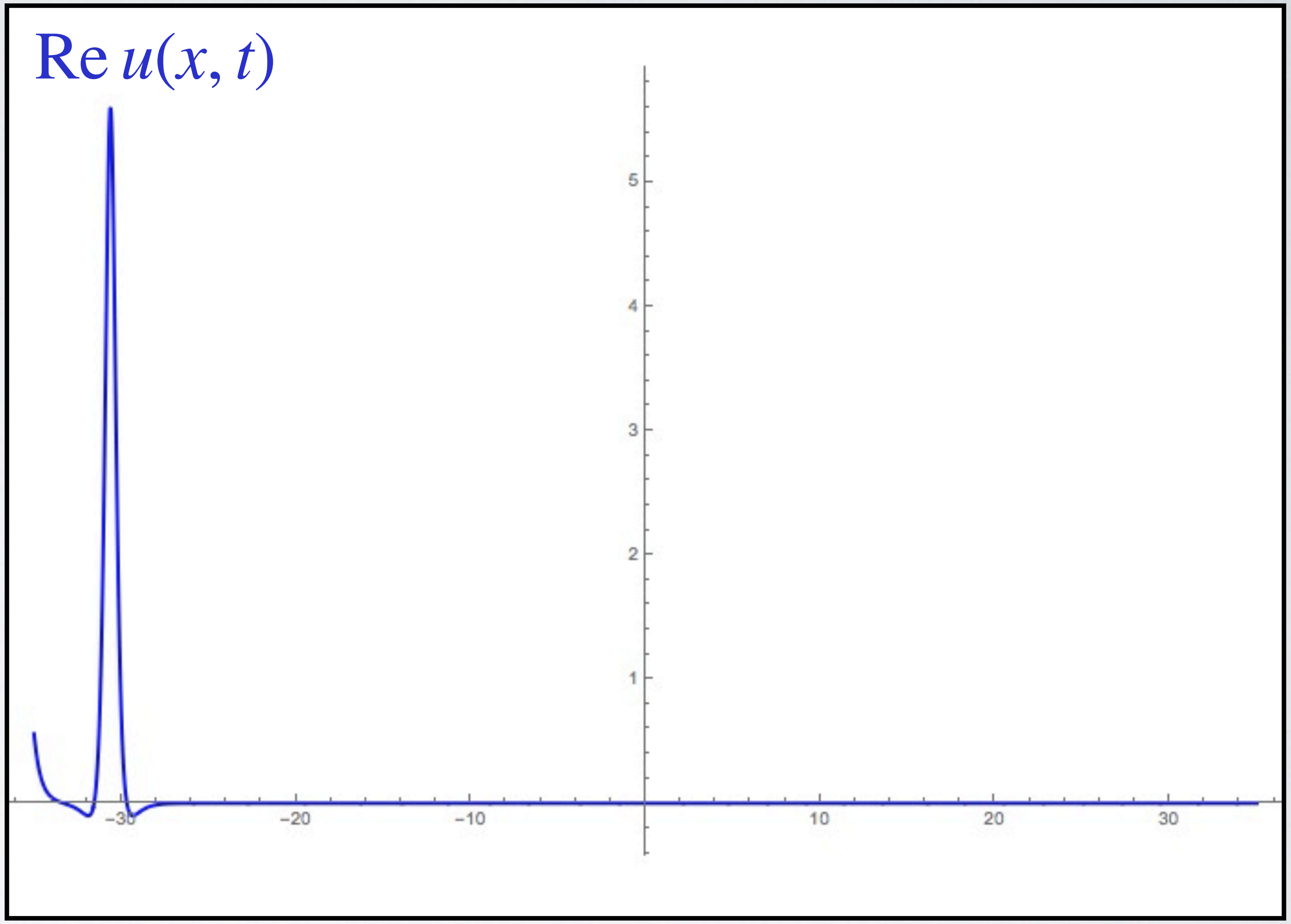

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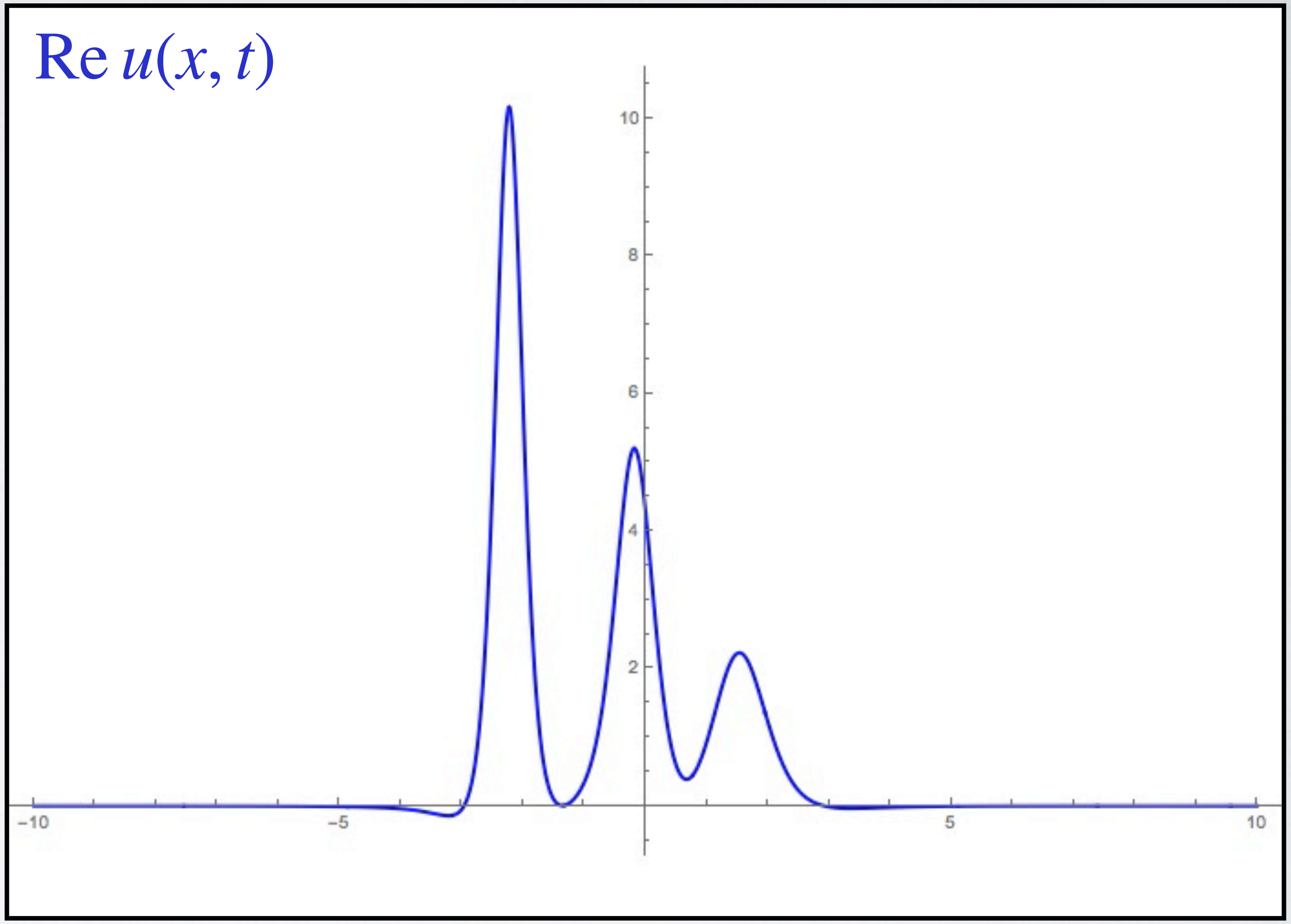

Rogue wave type

Complex Solitons and integrable equations

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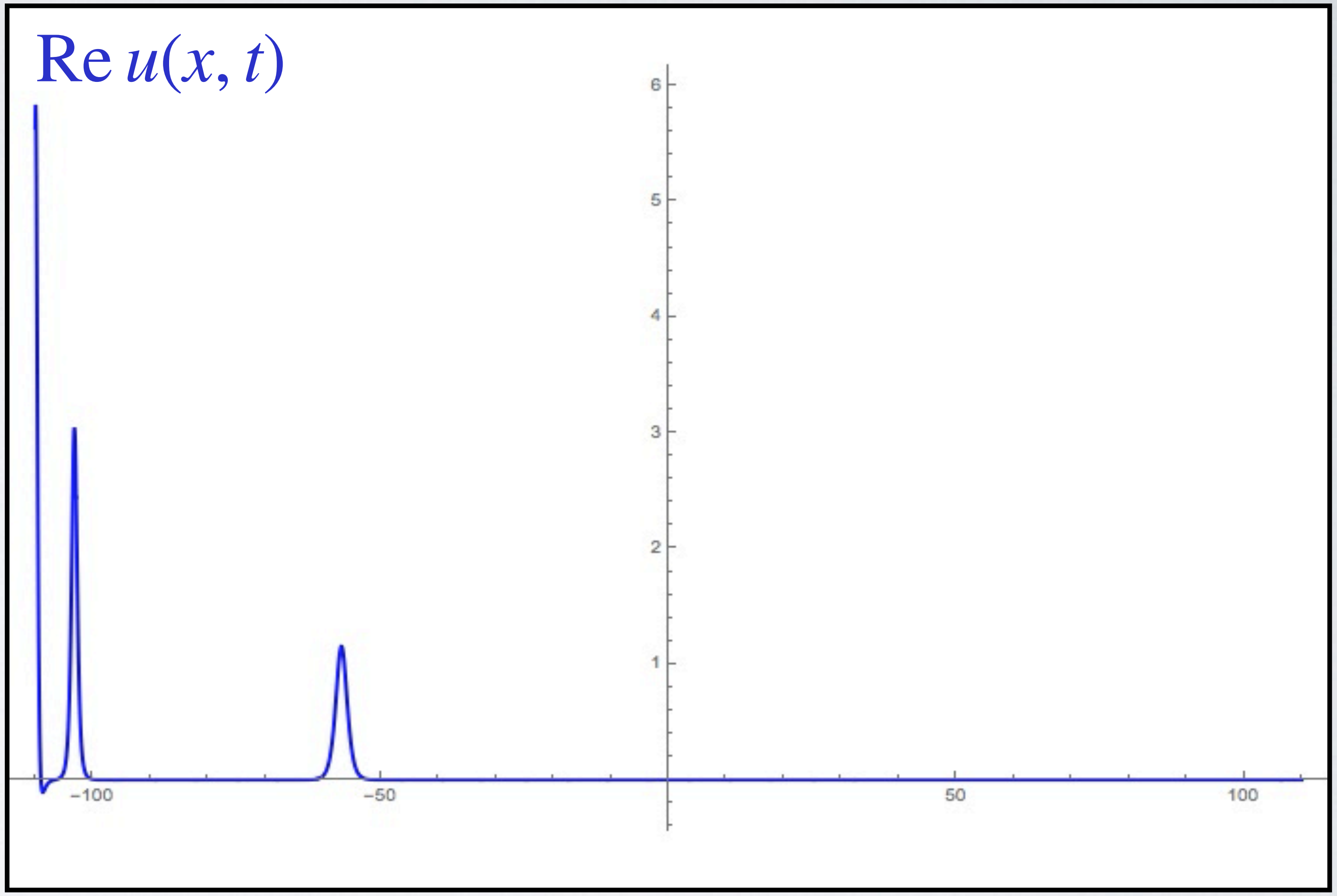
Korteweg-de Vries equation

$$u = p + iq$$

F.C, A. Fring, JHEP (2016)

two degenerate + one soliton



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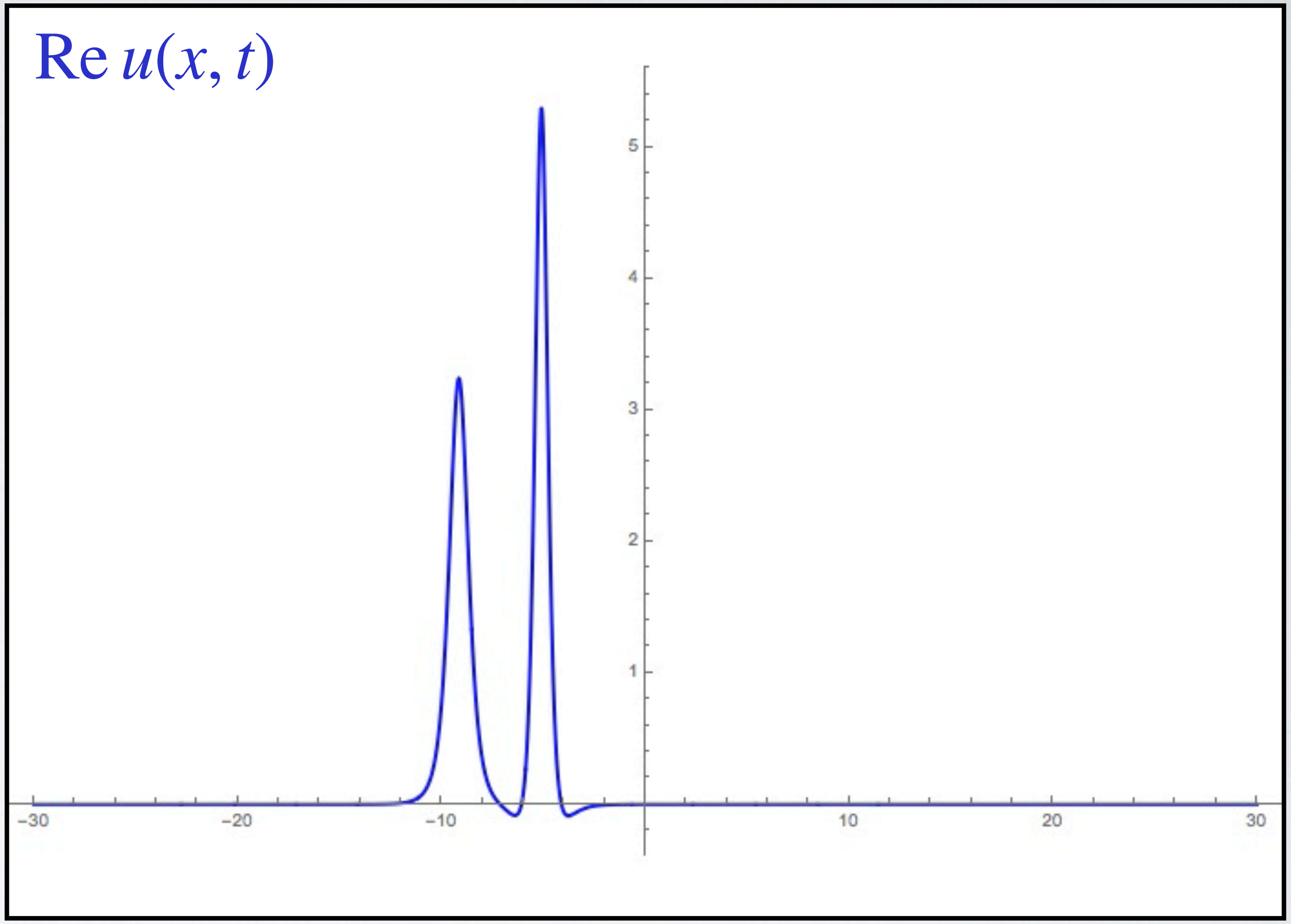
F.C, A. Fring, JHEP (2016)



Two degenerate soliton

- **Same velocities 😱**
- **Real conserved quantities (infinitely many) 😱 😱**
- **Energies could be n times one soliton energy 😱 😱 😱**

Complex Solitons and integrable equations

But if the KdV equation is so well known, how could this have gone unnoticed?

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V. Matveev, JMP (1994)

Singular solutions with infinite (divergent) energies

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PT-SYMMETRY

Complex Solitons and integrable equations

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Singular solutions with infinite (divergent) energies

PT-SYMMETRY

Non-physical solitons are regularized: removing singularities and making charges finite!

An additional symmetry ensures the reality of the integrals of motion

All conserved charges are real, even though solitons are complex

F.C, A. Fring, JHEP (2016)

J. Cen, F.C, A. Fring, Annals of Phys. (2017)

Why these solitons are relevant ???

Solitons + QM

Solitons + QM

M. Kac "Can one hear the shape of a drum ?"

Solitons + QM

M. Kac

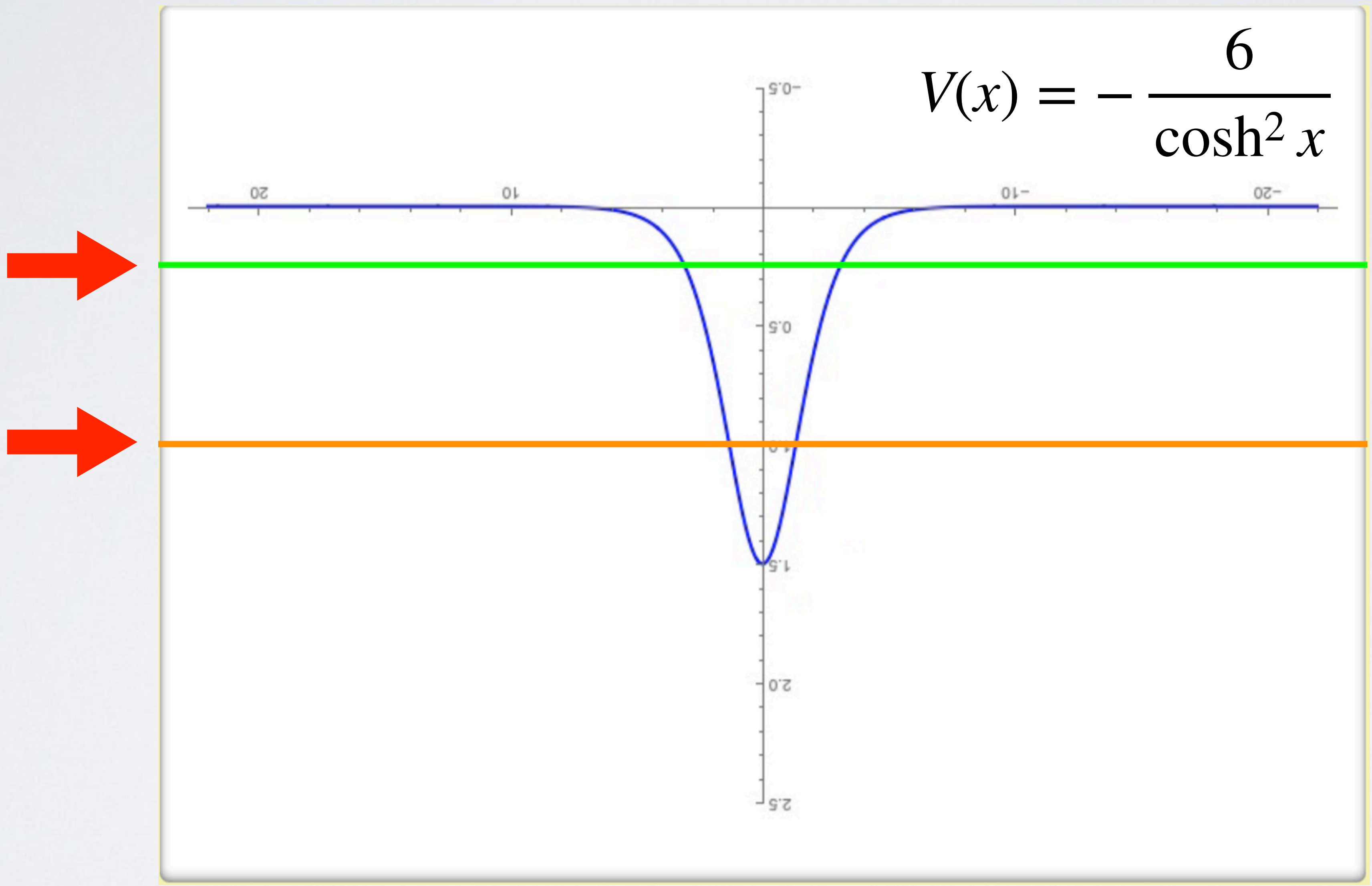
“Can one hear the eigenvalues of a potential ?”

Solitons + QM

M. Kac

“Can one hear the eigenvalues of a potential ?”

bound states

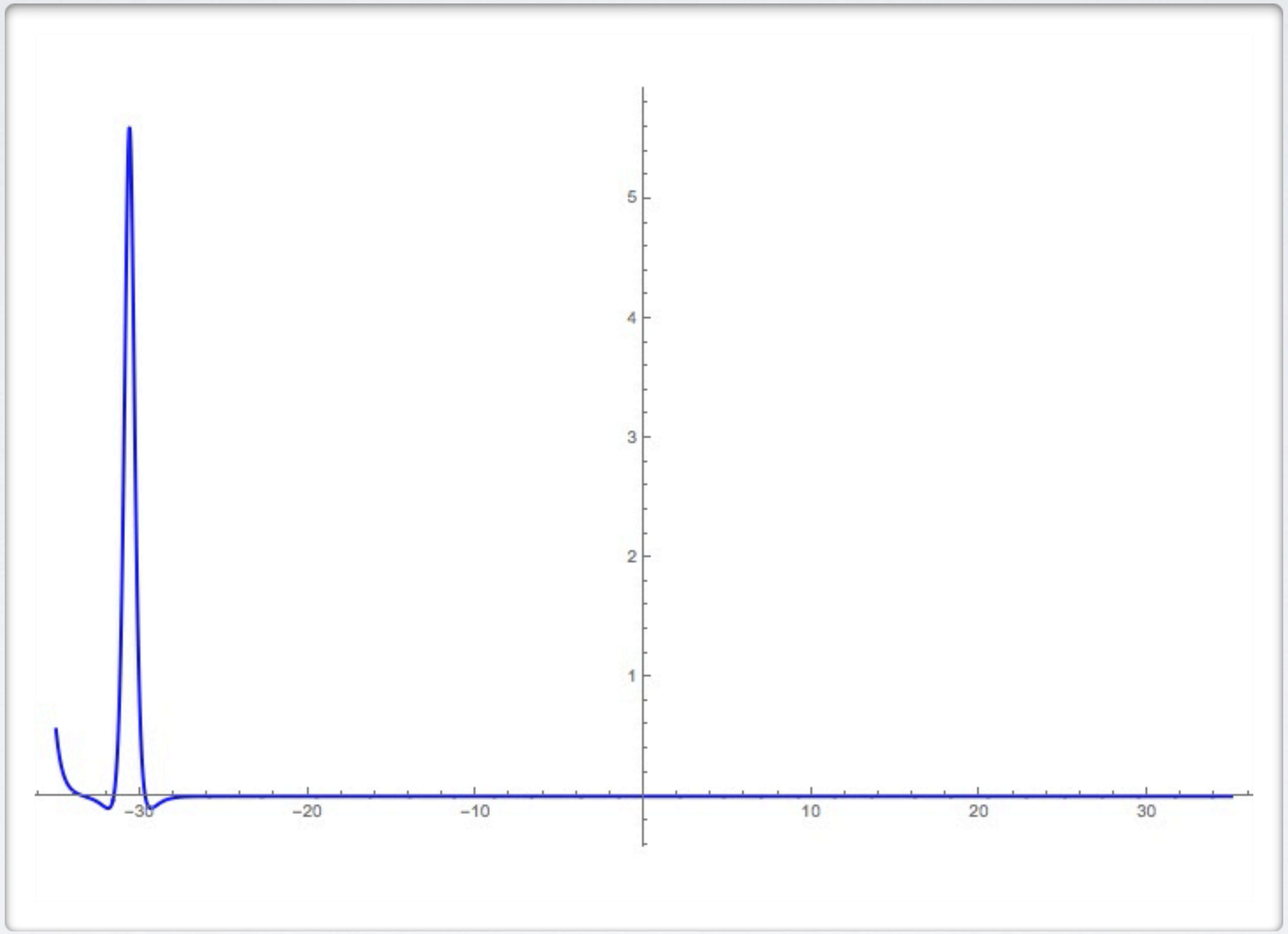


Reflectionless potentials for all energies!

Complex Solitons + QM + optics

M. Kac

"Can one hear the eigenvalues of a potential ?"



Why these potentials are interesting?

Complex Solitons + QM + optics

The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi$$

Formal coincidence

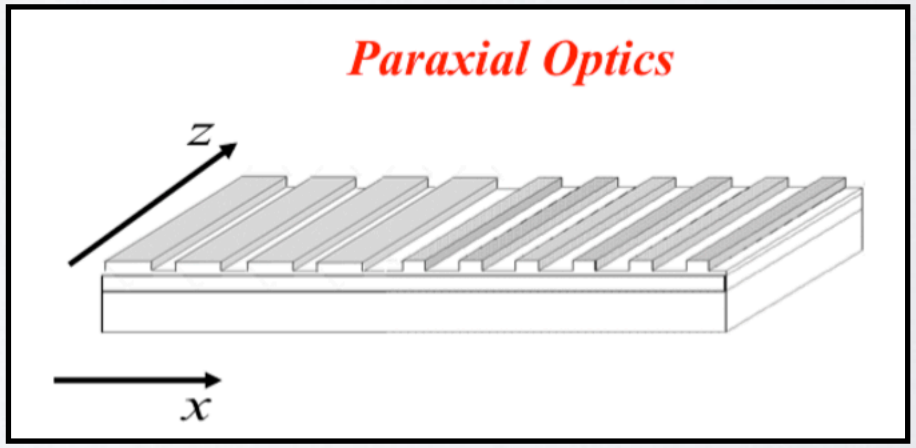
Complex Solitons + QM + optics

The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi$$

Formal coincidence

$$i\frac{\partial E}{\partial z} = \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0\,n(x)\,E$$



The paraxial equation of diffraction

Complex Solitons + QM + optics

The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi$$

A complex potential

Formal coincidence ↕

$$i\frac{\partial E}{\partial z} = \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0\,n(x)\,E$$

A refractive index with gain and loss

The paraxial equation of diffraction

Complex Solitons + QM + optics

The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi$$

The probability density

Formal coincidence ↕

$$i\frac{\partial E}{\partial z} = \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0\,n(x)\,E$$

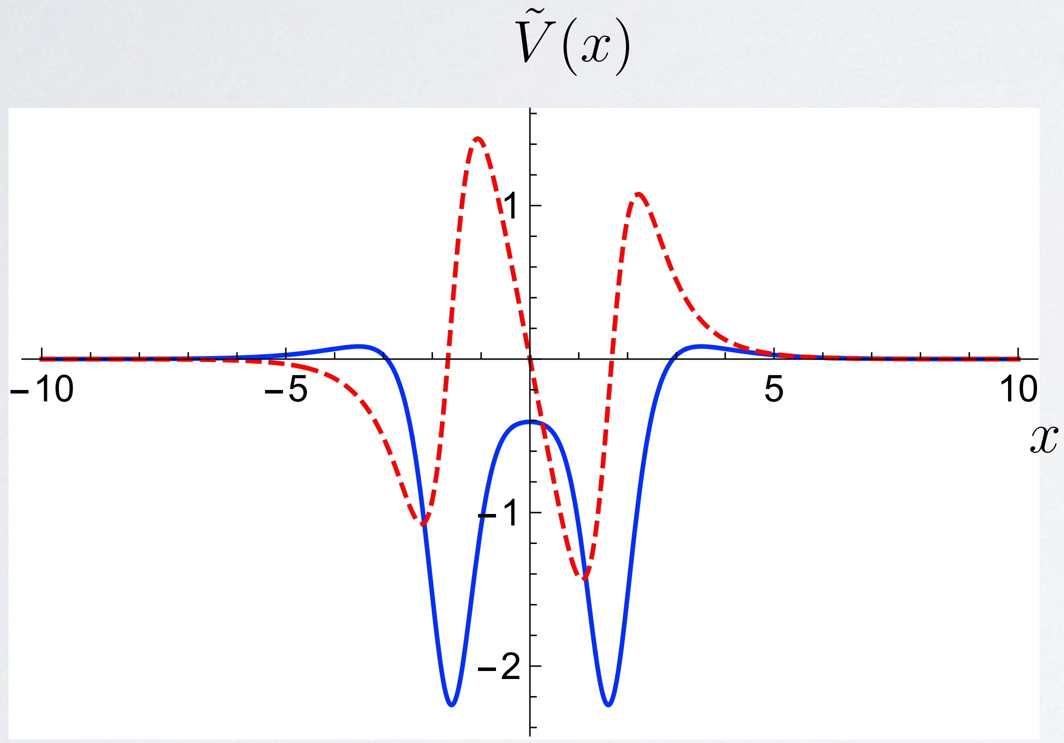
The power intensity

The paraxial equation of diffraction

Complex Solitons + QM + optics

The path of a beam of light through a material $|E(x,z)|^2$

Complex soliton



F. C., V. Jakubsky, M. Plyushchay, PRA (2015)

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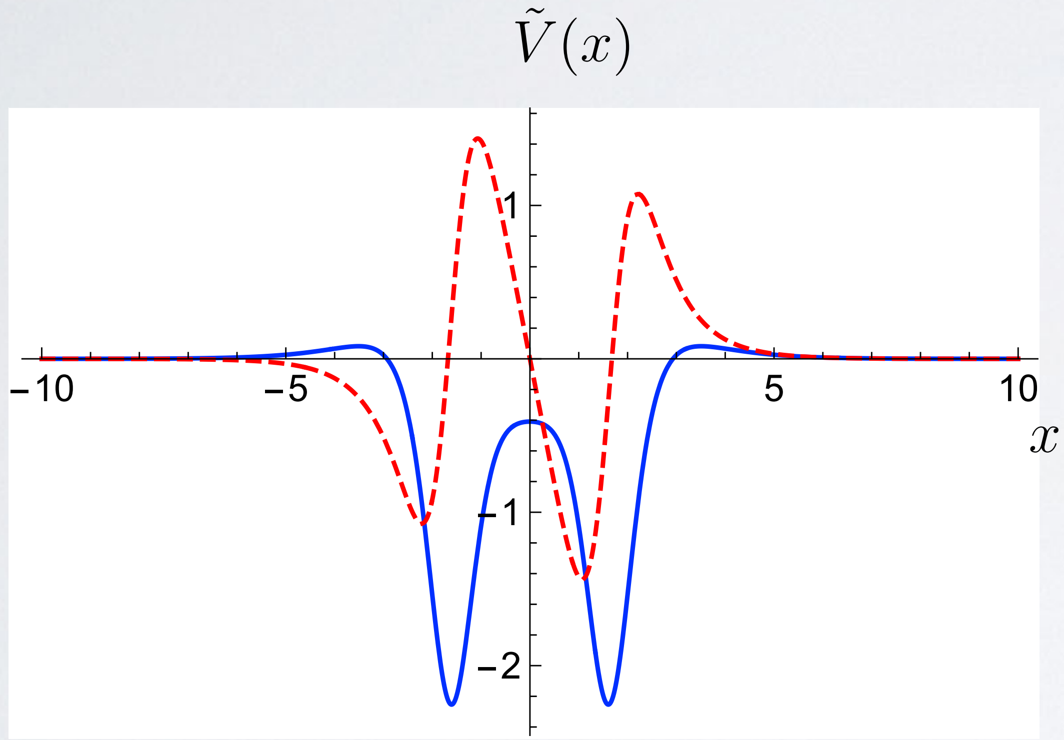

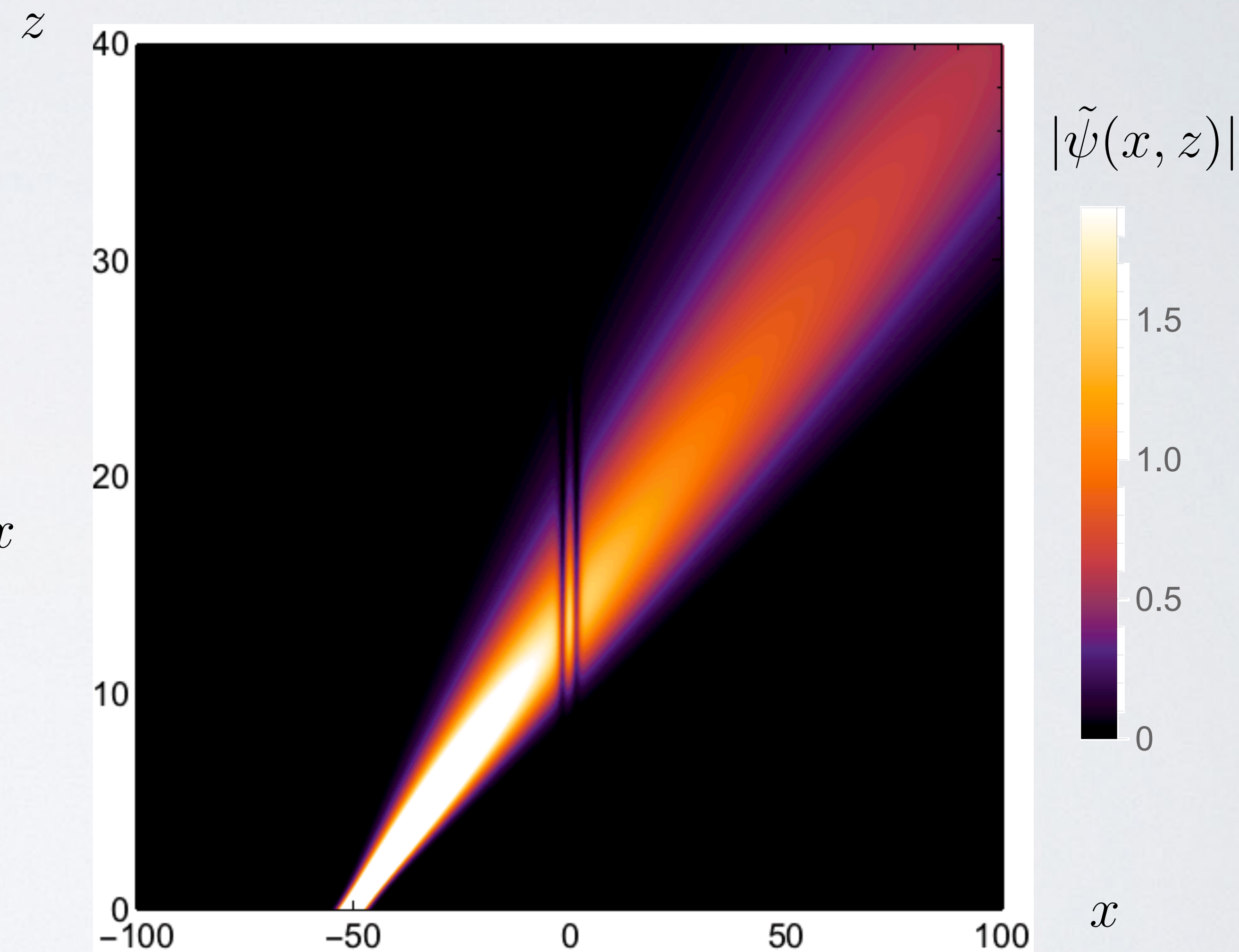

Reflectionless but detectable, non trivial phase shift

F. C., V. Jakubsky, M. Plyushchay, PRA (2015)

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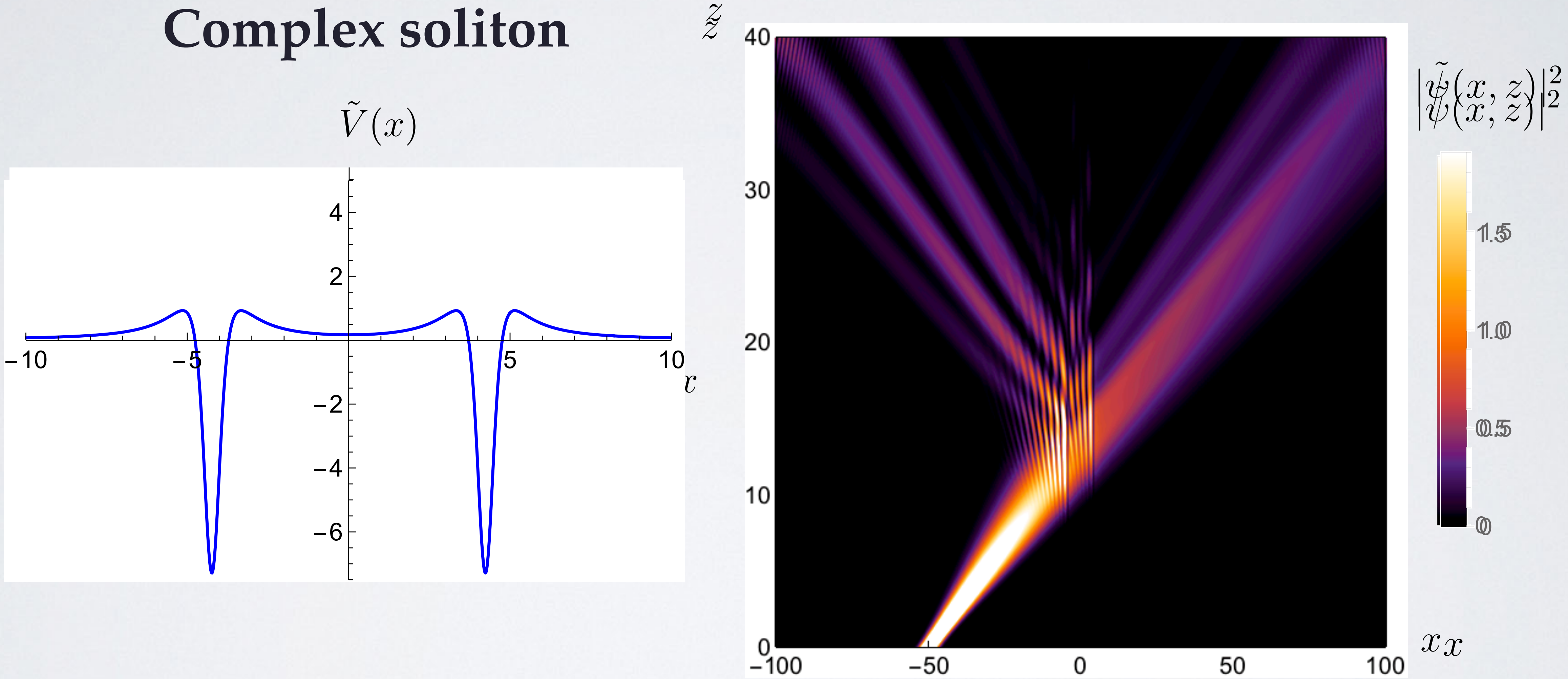

Invisible potentials in both directions

F. C., V. Jakubsky, M. Plyushchay, PRA (2015)

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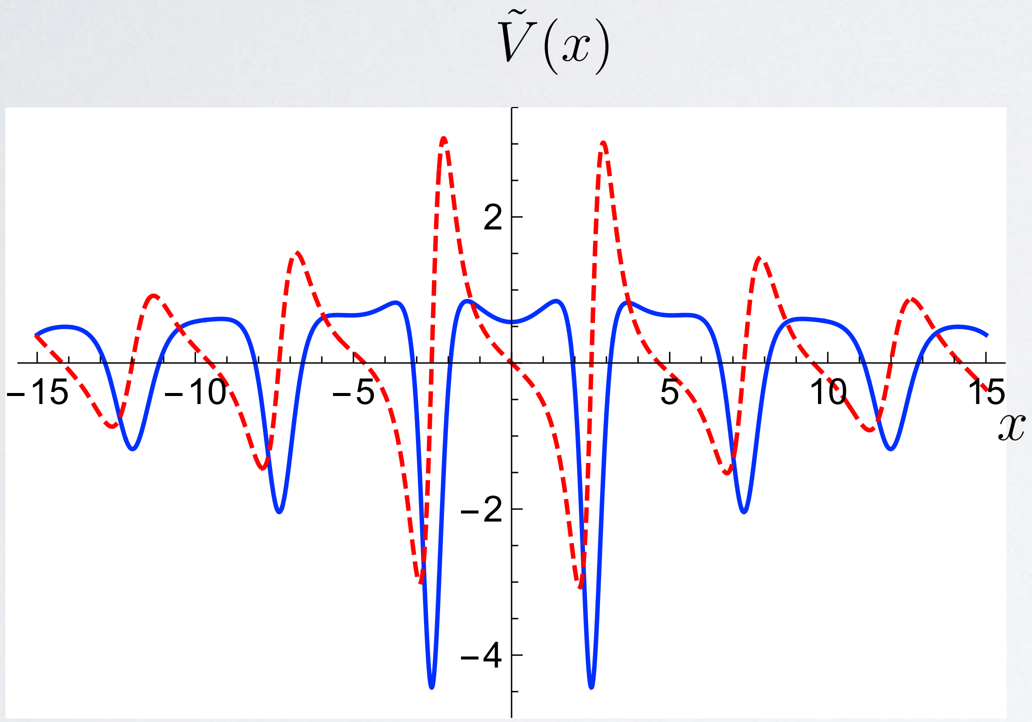

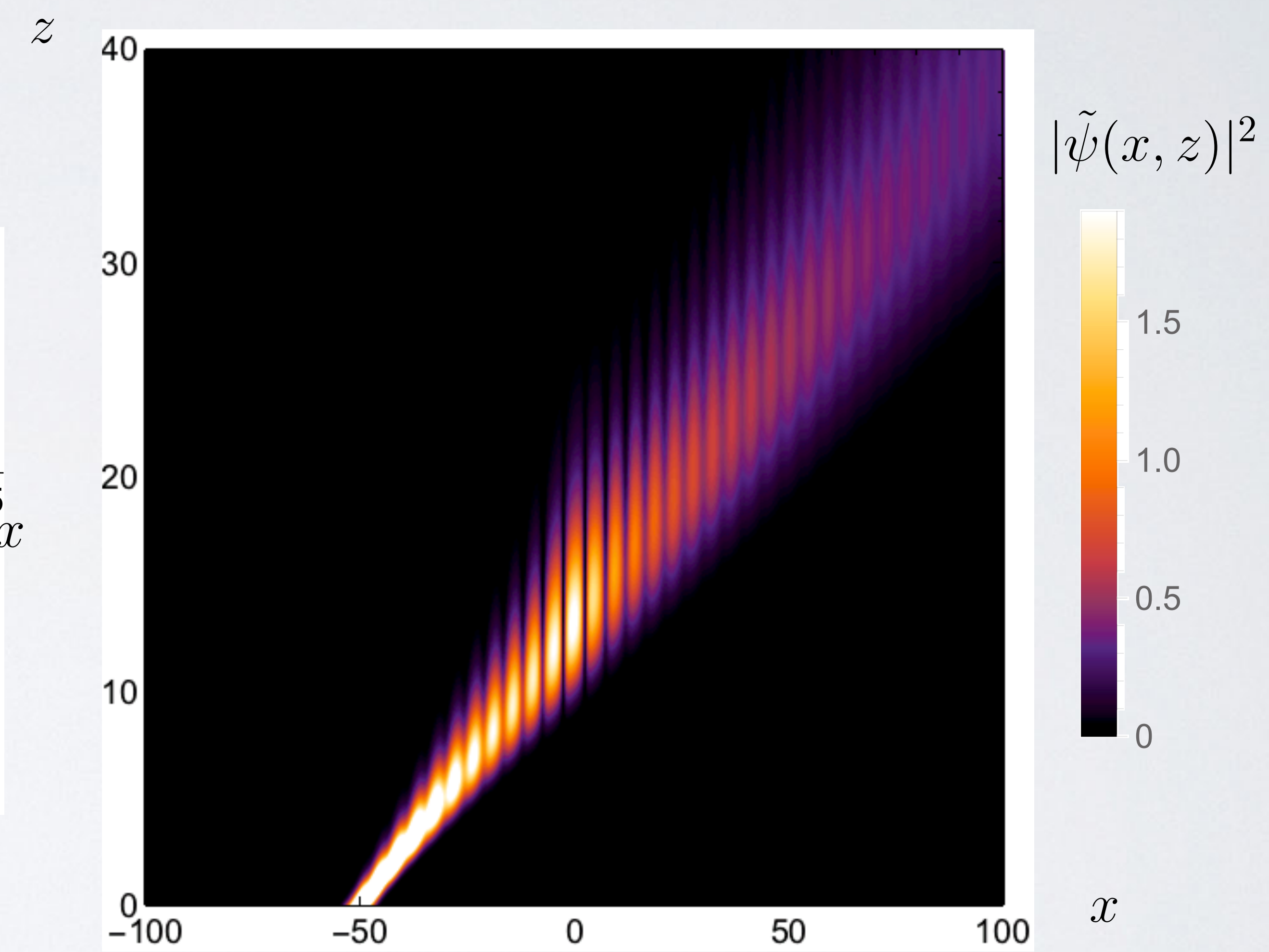

Invisible optical crystal with a bound state in the continuum (BIC)

F. C., V. Jakubsky, M. Plyushchay, PRA (2015)

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Integrability & non-Hermitian physics

PRL **100,** 030402 (2008) PHYSICAL REVIEW LETTERS week ending 25 JANUARY 2008

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The nonlinear Schrödinger equation

$$iq_t + \alpha \left(q_{xx} - 2\kappa \left| q \right|^2 q \right) = 0$$

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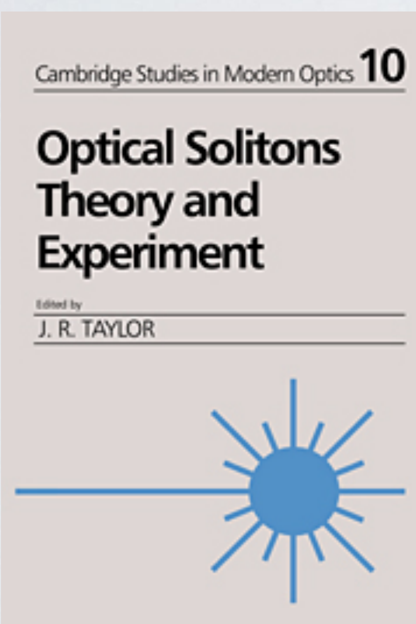
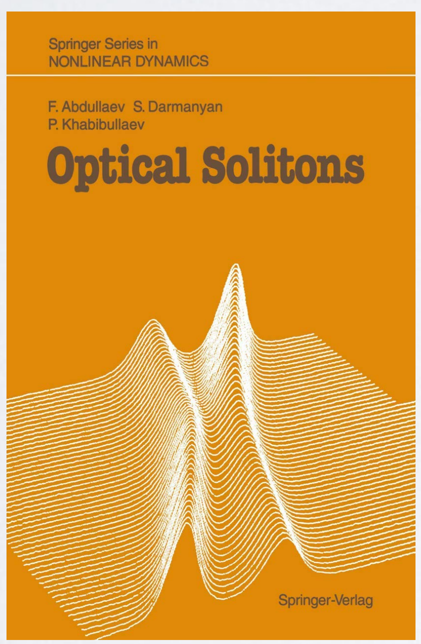
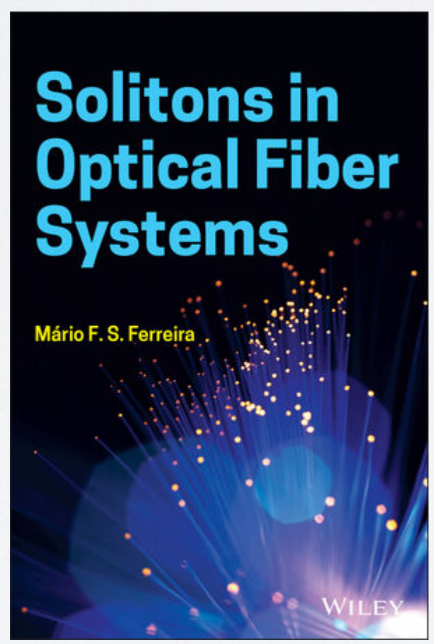
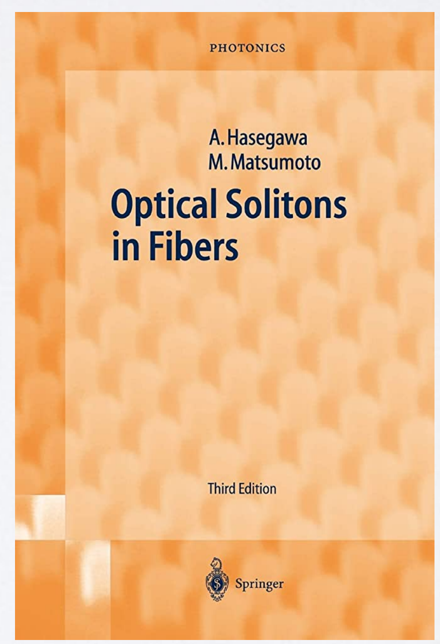
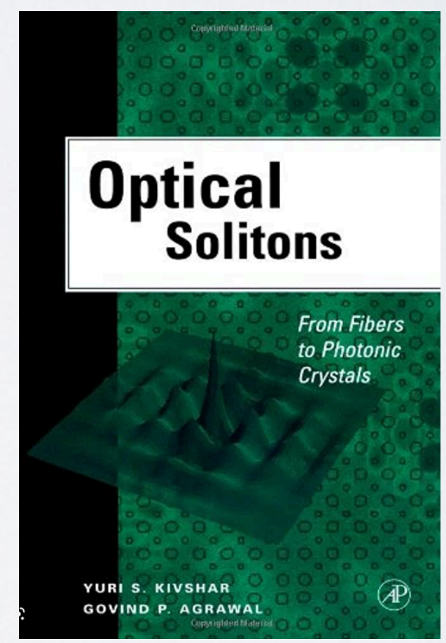
Propagation of light in fiber optics

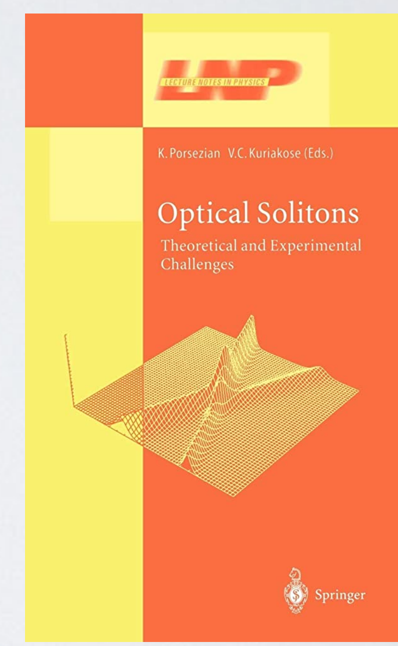












The nonlinear Schrödinger equation

$$iq_t + \alpha\left(q_{xx} - 2\kappa\,|q|^2\,q\right) = 0$$

and many, many more applications...

- Water waves
- Plasma physics
- Bose-Einstein condensates
- Superconductivity
- Gravity
- Classical and quantum field theory
- Non-Hermitian physics

The nonlinear Schrödinger equation

$$iq_t + \alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) = 0$$

What is the origin of this equation?

Lax Pair formalism $\Leftrightarrow$ Zero curvature formalism Zakharov and Shabat (1972)

$$\Psi_t = V\Psi,\ \Psi_x = U\Psi \quad \Leftrightarrow \quad \partial_t U - \partial_x V + [U, V] = 0$$

$$\Psi_{tx} = \Psi_{xt}$$

Gauge field equations

The nonlinear Schrödinger equation

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$$\Psi_t = V\Psi, \quad \Psi_x = U\Psi \quad \Leftrightarrow \quad \partial_t U - \partial_x V + [U, V] = 0$$

$$iq_t + \alpha q_{xx} - 2\alpha q^2 r = 0,$$

$$ir_t - \alpha r_{xx} + 2\alpha q r^2 = 0$$

U ... $-i\alpha qr - 2i\alpha\lambda^2$

$i\alpha q$ ...

$$r(x,t) = \kappa q^*(x,t)$$

$$V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

$$C = -i\alpha r_x + 2\alpha\lambda r$$

$$H = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix}$$

Finding the objects U and V is equivalent to find an integrable system!!

$$H\psi = E\psi$$

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The Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy

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$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}$$

$$V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

- Sine-Gordon
- Korteweg de Vries (KdV)
- mKdV
- Non-linear Schrödinger
- Hirota

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The nonlocal nonlinear Schrödinger equation

PRL **110,** 064105 (2013) PHYSICAL REVIEW LETTERS week ending 8 FEBRUARY 2013

Integrable Nonlocal Nonlinear Schrödinger Equation

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[1]*Department of Applied Mathematics, University of Colorado, Campus Box 526, Boulder, Colorado 80309-0526*

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(Received 22 August 2012; published 7 February 2013)

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$$iq_t + \alpha q_{xx} - 2\alpha q^2 r = 0,$$

$$ir_t - \alpha r_{xx} + 2\alpha q r^2 = 0$$

The nonlocal nonlinear Schrödinger equation

A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$

A time-reversed pair, $r(x,t) = \kappa q^*(x,-t)$

A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$

A $\mathcal{PT}$**-symmetric pair,** $r(x,t) = \pm q^*(-x,-t)$

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The nonlocal Hirota equation

J. Cen, F.C, A. Fring, JMP (2019)

$$q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta \left[q_{xxx} - 6qrq_x \right] = 0,$$

New classes of integrable systems & solitons solutions

The nonlocal nonlinear Schrödinger equation

A parity transformed conjugate pair, $r(x,t) = \kappa q^*(-x,t)$

A time-reversed pair, $r(x,t) = \kappa q^*(x,-t)$

A real parity transformed conjugate pair, $r(x,t) = \pm q(-x,t)$

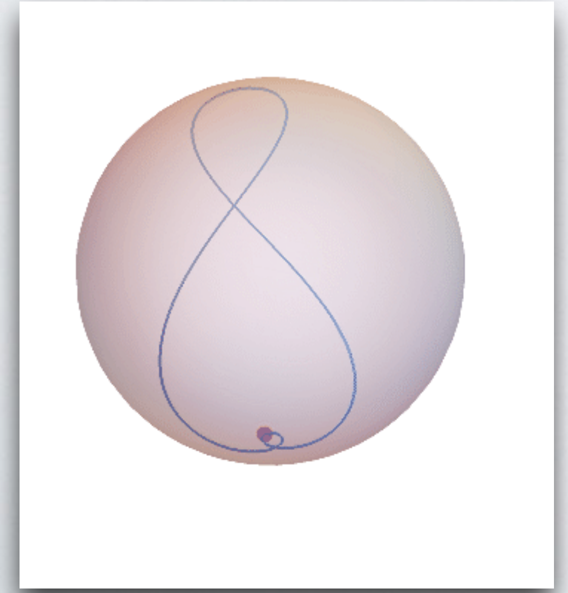
A $\mathcal{PT}$**-symmetric pair,** $r(x,t) = \pm q^*(-x,-t)$

The nonlocal Hirota equation

J. Cen, F.C, A. Fring, JMP (2019)

$$q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta\left[q_{xxx} - 6qrq_x\right] = 0,$$

New classes of integrable systems & solitons solutions



$$\mathbf{s}_t = -\alpha\mathbf{s}\times\mathbf{s}_{xx} - \frac{3}{2}\beta\left(\mathbf{s}_x\cdot\mathbf{s}_x\right)\mathbf{s}_x + \beta\mathbf{s}\times\left(\mathbf{s}\times\mathbf{s}_{xxx}\right)$$

Gauge transformations

Extended Landau-Lifschitz equation

J. Cen, F.C, A. Fring, JPhA (2020)

New solutions and models....

....but something else ???

The Bullough-Dodd model (Tzitzéica)

$$\mathcal{L}_{\mathrm{BD}} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - e^{\varphi} - \frac{1}{2}e^{-2\varphi} + \frac{3}{2}$$

$$\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0$$

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$$\varphi_I^\pm(x,t) = \ln\left[\frac{\cosh\left(\beta + \sqrt{k^2-3}t + kx\right) \pm 2}{\cosh\left(\beta + \sqrt{k^2-3}t + kx\right) \mp 1}\right]$$

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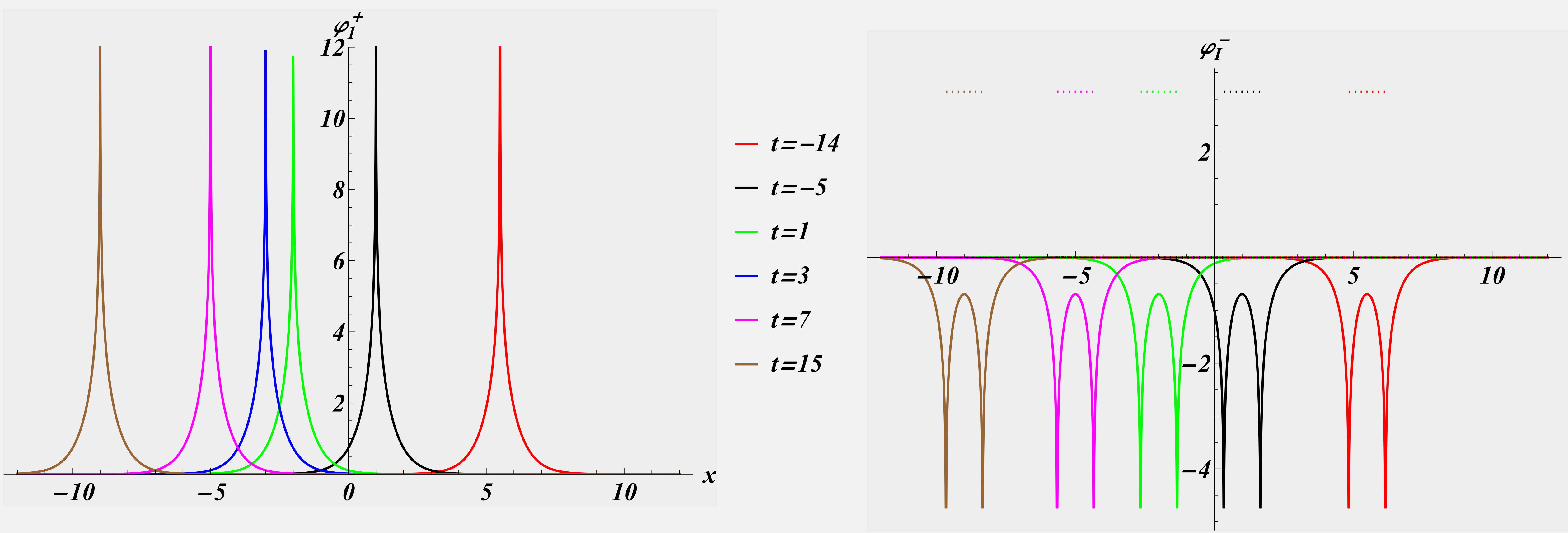
When β is real the solutions are singular.....



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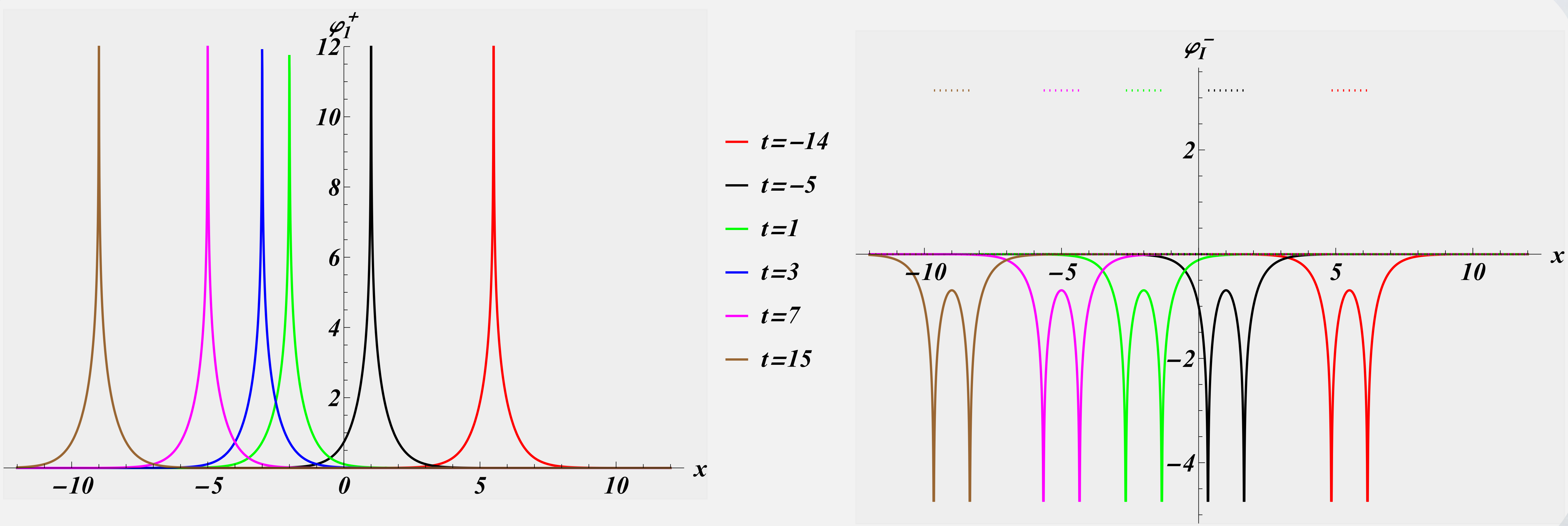

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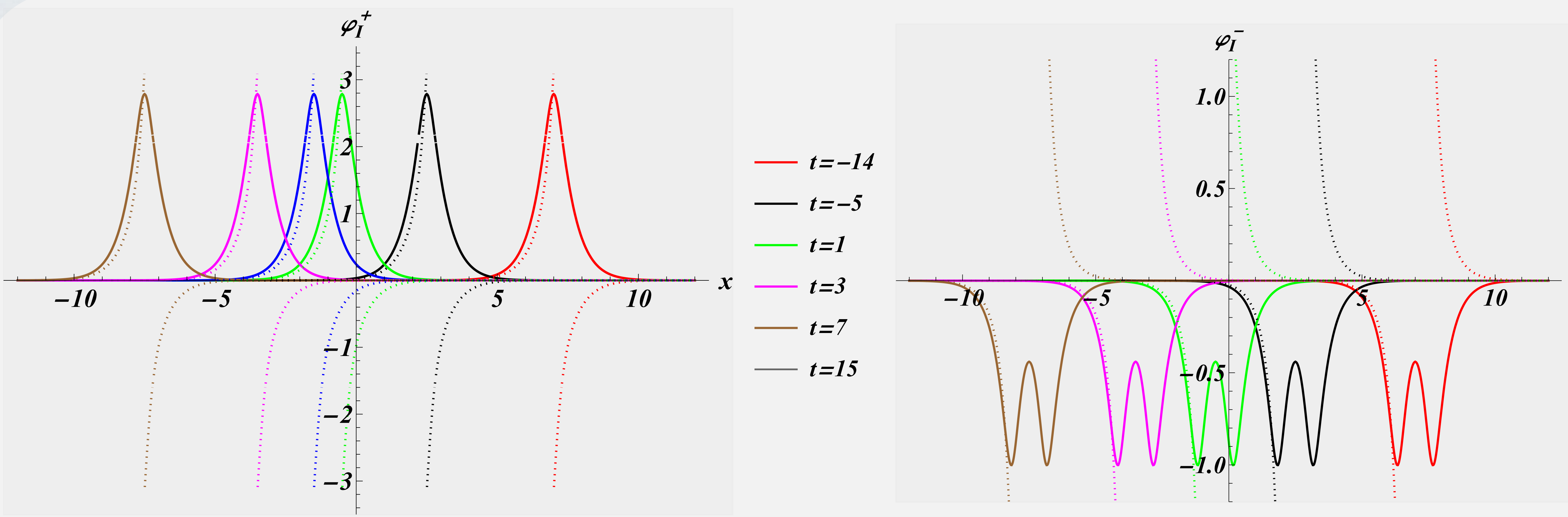
When β is imaginary the solutions become regular.....



The Bullough-Dodd model (Tzitzéica)

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When β is imaginary the solutions become regular.....

$$\mathcal{PT} : x \to -x, \quad t \to -t, \quad i \to -i, \quad \varphi \to \varphi.$$

$$\mathcal{PT} : \varphi_I^{\pm} \to \varphi_I^{\pm}$$

the energies are real $E[\varphi_I^{\pm}] = -6|k|$

PT-SYMMETRY **regularizes solitons and explains the reality of conserved charges...**

......something else ???

Field theories and linear stability

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi),$$

Lagrangian density in 1+1 D

$$\ddot{\varphi} - \varphi'' + \frac{\partial V(\varphi)}{\partial \varphi} = 0$$

Euler-Lagrange equations

Field theories and linear stability

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Lagrangian density in 1+1 D

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Euler-Lagrange equations

Small perturbation $\varphi \to \varphi_s + \varepsilon\chi$ $\varepsilon \ll 1$

$$\ddot{\varphi}_s - \varphi_s'' + \left.\frac{\partial V(\varphi)}{\partial\varphi}\right|_{\varphi_s} + \varepsilon\left(\ddot{\chi} - \chi'' + \chi\left.\frac{\partial^2 V(\varphi)}{\partial\varphi^2}\right|_{\varphi_s}\right) + \mathcal{O}(\varepsilon^2) = 0$$

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$$\chi(x,t) = e^{i\lambda t}\Phi(x)$$

Ansatz

Field theories and linear stability

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$$\chi(x,t) = e^{i\lambda t}\Phi(x)$$

Ansatz

$$-\Phi_{xx} + V_1\Phi = \lambda^2\Phi$$

$$V_1(x) := \left.\frac{\partial^2 V(\varphi)}{\partial\varphi^2}\right|_{\varphi_s}$$

Sturm-Liouville eigenvalue problem / Schrödinger equation with potential

Field theories and linear stability

Let's see some very well-known examples

$$-\Phi_{xx} + V_1 \Phi = \lambda^2 \Phi$$

$$V_1(x) := \left. \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \right|_{\varphi_s}$$

Field theories and linear stability

Let's see some very well-known examples

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Quantum meaning of classical field theory*

R. Jackiw

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Recent researches have shown that it is possible to obtain information about the physical content of nontrivial quantum field theories by semiclassical methods. This article reviews some of these investigations. We discuss how solutions to field equations, treated as classical, c-number nonlinear differential equations, expose unexpected states in the quantal Hilbert space with novel quantum numbers which arise from topological properties of the classical field configuration or from the mixing of internal and space-time symmetries. Also imaginary-time, c-number solutions are reviewed. It is shown that they provide nonperturbative information about the vacuum sector of the quantum theory.

Rev. Mod. Phys. 49, 681 (1977)

Field theories and linear stability

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Let's see some very well-known examples

Sine-Gordon $V(\varphi) = -\cos\varphi$

static kink $\varphi_s = 4\arctan e^x$

Field theories and linear stability

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$$V_1(x) = -\frac{2}{\cosh^2 x} + 1$$

one-soliton reflectionless potential

Field theories and linear stability

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two-soliton reflectionless potential

$$V_1(x) = -\frac{6}{\cosh^2 x} + 4$$

Field theories and linear stability

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two-soliton reflectionless potential

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The Bullough-Dodd solutions and linear stability

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Bullough-Dodd static solutions

$$\phi_I^{\pm}(x) = \ln\left[\frac{\cosh\left(\beta+\sqrt{3}x\right)\pm 2}{\cosh\left(\beta+\sqrt{3}x\right)\mp 1}\right]$$

$$V_1^{+}(x) = 1 - \frac{3}{1-\cosh\left(\beta+\sqrt{3}x\right)} + \frac{8\sinh^4\left[\frac{1}{2}\left(\beta+\sqrt{3}x\right)\right]}{\left[2+\cosh\left(\beta+\sqrt{3}x\right)\right]^2},$$

The Bullough-Dodd solutions and linear stability

two-soliton reflectionless potential

$$V_1^+(x) = 1 - \frac{3}{1-\cosh\left(\beta+\sqrt{3}x\right)} + \frac{8\sinh^4\left[\frac{1}{2}\left(\beta+\sqrt{3}x\right)\right]}{\left[2+\cosh\left(\beta+\sqrt{3}x\right)\right]^2},$$

singularity $x_0 = -\beta\sqrt{3}$ when $\beta \in \mathbb{R}$

no singularities **PT-SYMMETRY** $\beta \in i\mathbb{R}$.

There exist linear stable perturbations !

We found new complex solitons with broken **PT** which are unstable!

F.C, A. Fring, T. Taira, NPB (2022)

PT-SYMMETRY

It is not a mere artifact which measure reality energy conditions but also the physical sense of theories...

Non-Hermitian field integrable field theories

New classes of integrable systems & solitons solutions

Non-physical solitons are regularized (removing singularities!)

All conserved charges are real, even though solitons are complex

An additional symmetry ensures the reality of the integrals of motion

The linear stable pertubations display also **PT-symmetry**

Are these features available only in this kind of integrable theories?

No, they can be applied everywhere.....

J. Cen, F.C, A. Fring, T. Taira, PLA(2022)

F.C, A. Fring, T. Taira, NPB (2021)

F.C, A. Fring, T. Taira, JHEP (2022)

F.C, A. Fring, T. Taira, NPB (2022)

Applications of Field Theory to Hermitian and Non-Hermitian Systems

Non-Hermitian ideas provide new phenomena in (integrable) field theories

Outline

3. **Applications to black hole physics**

The Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy

Lax Pair formalism $\Leftrightarrow$ Zero curvature formalism Zakharov and Shabat (1972)

$$\Psi_t = V\Psi, \ \Psi_x = U\Psi \quad \Leftrightarrow \quad \partial_t U - \partial_x V + [U, V] = 0$$

$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}$$

$$V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

- Sine-Gordon
- Korteweg de Vries (KdV)
- mKdV
- Non-linear Schrödinger
- Hirota

Integrable systems and gravity

Zero curvature formalism
Zakharov and Shabat (1972)

$$\partial_t U - \partial_x V + [U, V] = 0$$

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🥹 **This structure is intimately related with AdS_3 gravity**

M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

Integrable systems and gravity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = 0$$

Integrable systems and gravity

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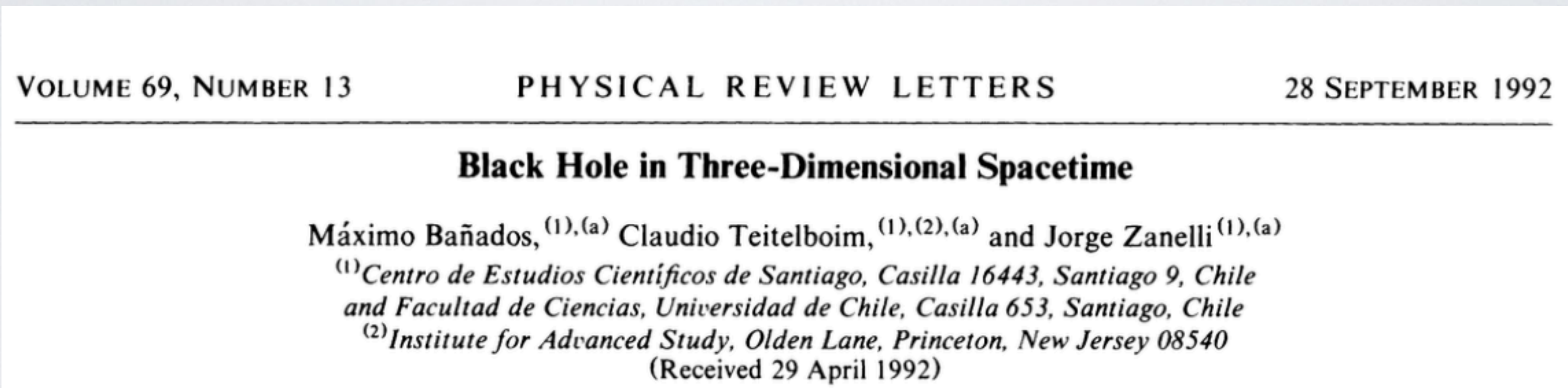
BTZ black hole


VOLUME 69, NUMBER 13 PHYSICAL REVIEW LETTERS 28 SEPTEMBER 1992

Black Hole in Three-Dimensional Spacetime

Máximo Bañados,[(1),(a)] Claudio Teitelboim,[(1),(2),(a)] and Jorge Zanelli[(1),(a)]
[(1)]*Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile*
and Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile
[(2)]*Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540*
(Received 29 April 1992)

- Laboratory to understand the main features of black holes
- One of the best evidences for the AdS/CFT correspondence
- Very useful in several contexts beyond holography: string theory, QFT, ...

Integrable systems and gravity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = 0$$

Can be formulated as two independent Chern-Simons copies

Gauge connections

$$\mathcal{A}^{\pm} = \omega \pm e/\ell \quad \longrightarrow \quad \mathcal{F}^{\pm} = d\mathcal{A}^{\pm} + \mathcal{A}^{\pm} \wedge \mathcal{A}^{\pm}$$

spin connection & dreibein

$SL(2,\mathbb{R})$

A. Achucarro and P. K. Townsend, PLB (1986)

E. Witten, NPB (1988)

Integrable systems and gravitation

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spin connection & dreibein

Zero curvature condition

This is done choosing specific boundary conditions for the gravitational field.

A. Achucarro and P. K. Townsend, PLB (1986)

E. Witten, NPB (1988)

Integrable systems and gravity

$$\mathcal{A}^{\pm} = \omega \pm e/\ell \quad \longrightarrow \quad \mathcal{F}^{\pm} = d\mathcal{A}^{\pm} + \mathcal{A}^{\pm} \wedge \mathcal{A}^{\pm} = 0$$

Zero curvature condition

Coussaert, Henneaux, P. van Driel (1995)

Boundary conditions $\mathcal{A} = b^{-1}(d+a)b \qquad b(\rho) = \exp\left[\log\left(\frac{\rho}{\ell}\right) L_0\right]$

$$a = a_{\varphi} d\varphi + a_t dt$$

$$[L_n, L_m] = (n-m) L_{n+m}$$

$SL(2,\mathbb{R})$ Generators

Integrable systems and gravity

$$a = a_\varphi d\varphi + a_t dt$$

$$a_\varphi \sim 2i\lambda L_0 - q(\varphi,t)L_1 + r(\varphi,t)L_{-1} = U = \begin{pmatrix} -i\lambda & q(\varphi,t) \\ r(\varphi,t) & i\lambda \end{pmatrix}$$

$$a_t \sim -2A(\varphi,t)L_0 - B(\varphi,t)L_1 + C(\varphi,t)L_{-1} = V = \begin{pmatrix} A(\varphi,t) & B(\varphi,t) \\ C(\varphi,t) & -A(\varphi,t) \end{pmatrix}$$

M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

Integrable systems and gravity

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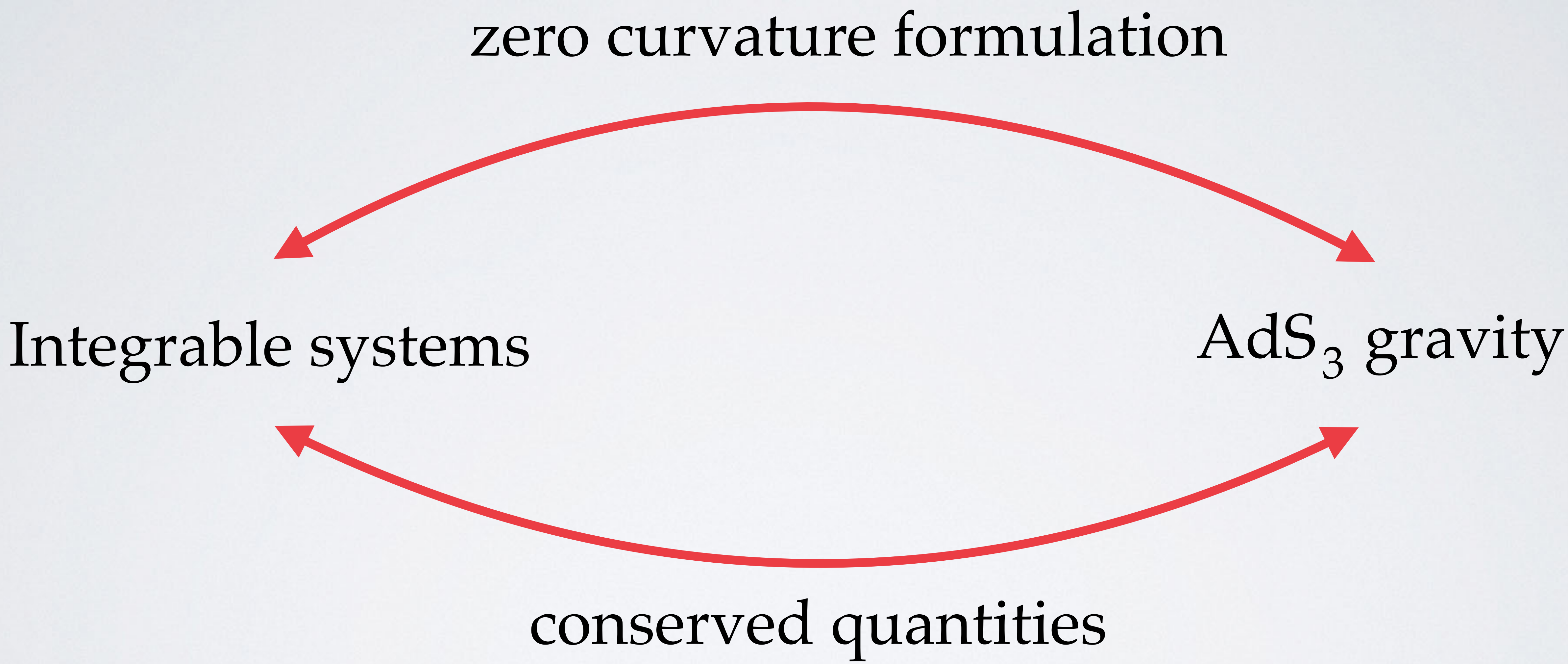
$$\mathcal{F}^\pm = d\mathcal{A}^\pm + \mathcal{A}^\pm \wedge \mathcal{A}^\pm \quad \longleftrightarrow \quad \partial_t U - \partial_\varphi V + [U,V] = 0$$

General Relativity — AKNS hierarchy

M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

Integrable systems and gravity

We have developed a dictionary



M. Cárdenas, F. C. K. Lara, M. Pino, PRL (2021)

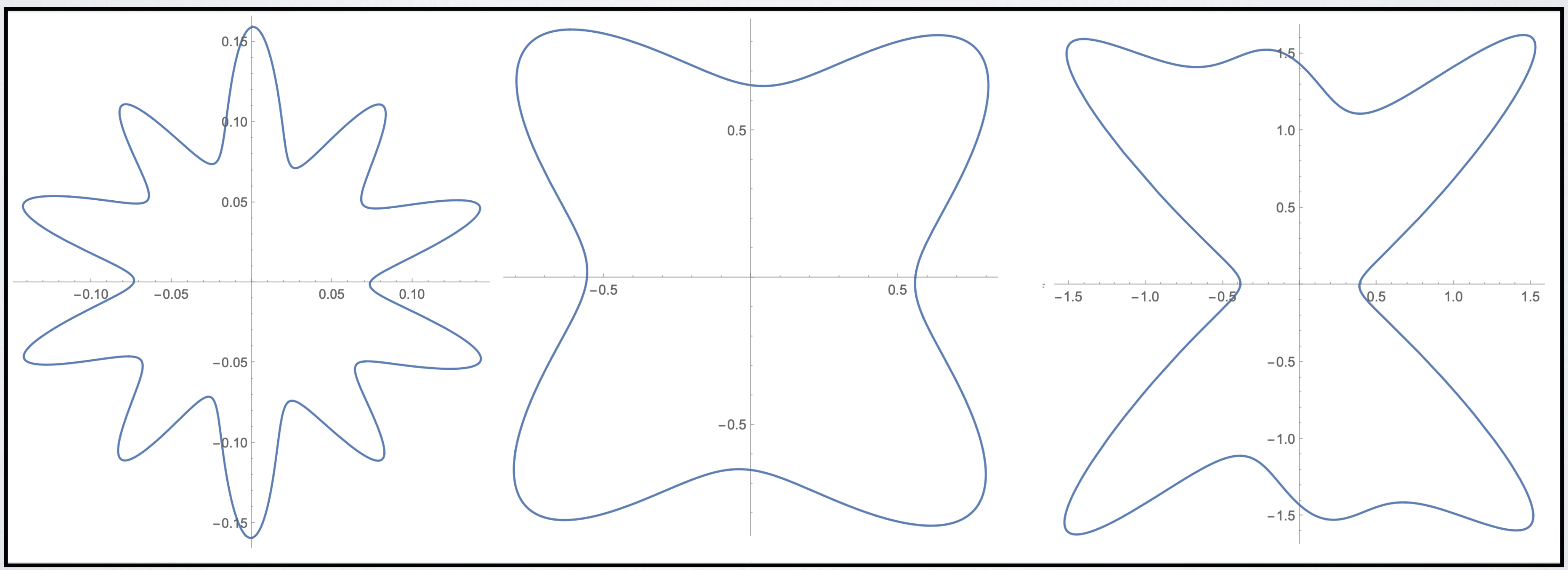
Integrable systems and gravity

M. Cárdenas, F. C., M. Pino, *work in progress*

Soliton solutions in gravity

$$ds^2 = \left(\ell\frac{d\rho}{\rho} - \ell(\lambda^+ + \lambda^-)d\phi + (A^- - A^+)dt\right)^2 + \left[\frac{\rho}{\ell}(p^+\ell d\phi - B^+dt) + \frac{\ell}{\rho}(r^-\ell d\phi + C^-dt)\right]\left[\frac{\rho}{\ell}(p^-\ell d\phi + B^-dt) + \frac{\ell}{\rho}(r^+\ell d\phi - C^+dt)\right]$$

Black hole horizon: Black flowers



Integrable systems and gravity

M. Cárdenas, F. C., M. Pino, *work in progress*

Soliton solutions in gravity

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$$\left[\frac{\rho}{\ell}(p^+\ell d\phi - B^+dt) + \frac{\ell}{\rho}(r^-\ell d\phi + C^-dt)\right]\left[\frac{\rho}{\ell}(p^-\ell d\phi + B^-dt) + \frac{\ell}{\rho}(r^+\ell d\phi - C^+dt)\right]$$

Integrable systems and gravity

- Non-local PT - inspired solutions
- Flat-limit of the AKNS boundary conditions
- Conformal symmetry AdS/CFT
- Interpretation of the quantum linear problem and black hole entropy
- Gravity analogue for integrable systems related by gauge transformations
- Higher dimensions and further hierarchies
- Different generating solution schemes
- How are these results connected with self-dual Yang-Mills description of integrable systems & Ward conjecture?

Outline

- Many open problems and new physics in solitons theory
- Meaning of complex and non-local solitons
- New ways to investigate gravity and more to be explored
- Electromagnetic solutions, self dual Yang-Mills and beyond
- Applications for AdS/CFT integrability?

4 **Discussion**

CONICYT
Ministerio de Educación
Gobierno de Chile

Fondecyt
Fondo Nacional de Desarrollo Científico y Tecnológico