Large N Higgs and Finite Temperature

Paul Romatschke
CU Boulder

Non-Hermitian Quantum Mechanics

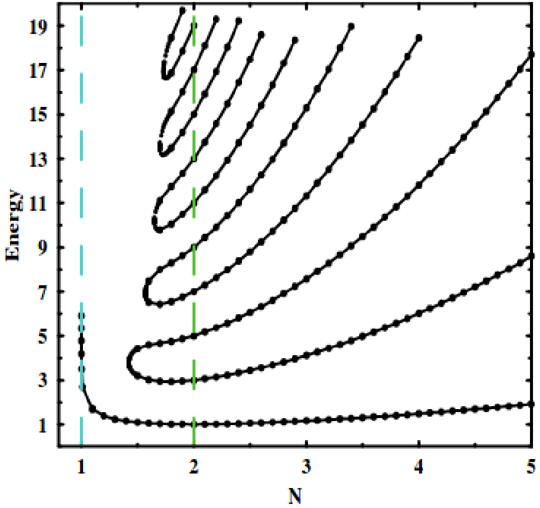
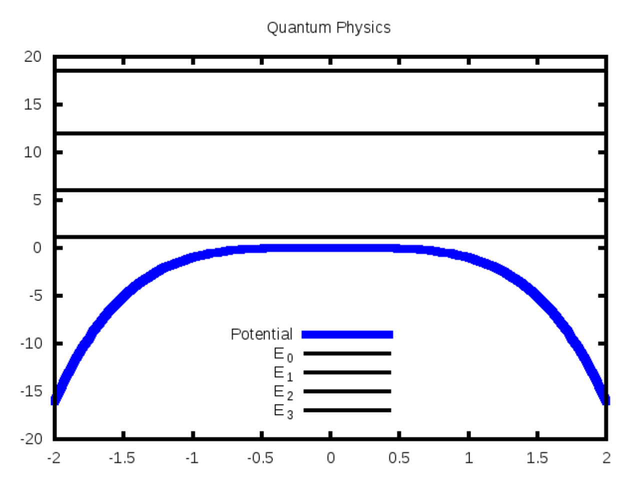


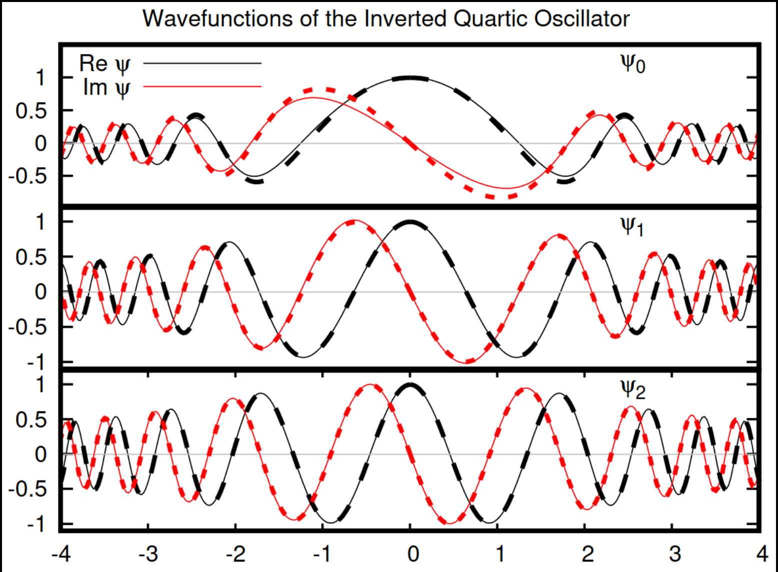
FIG. 1. Energy levels of the Hamiltonian $H = p^2 - (ix)^N$ as a function of the parameter N. There are three regions:

[Bender & Böttcher, 1997]

Classically unstable (but $\mathcal{PT}$ *symmetric*) potential $V(x) = -x^4$



$V = -x^4$: solutions of $\mathcal{H}\psi(x) = E\psi(x)$ with $x \in \mathbb{R}$:



[PR, 2408.12643]

$\mathcal{PT}$-symmetric Quantum Mechanics **is** Quantum Mechanics, not some "funny business"

$\mathcal{PT}$-symmetric Quantum Field Theory

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$\mathcal{PT}$-symmetric $-g\varphi^4$ theory

Wen-Yuan Ai,[1,*] Carl M. Bender,[2,†] and Sarben Sarkar[1,‡]

[1] *Theoretical Particle Physics and Cosmology Group, Department of Physics, King's College London, Strand, London WC2R 2LS, UK*

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The scalar field theory with potential $V(\varphi) = \frac{1}{2}m^2\varphi^2 - \frac{1}{4}g\varphi^4$ ($g > 0$) is ill defined as a Hermitian theory but in a non-Hermitian $\mathcal{PT}$-symmetric framework it is well defined, and it has a positive real energy spectrum for the case of spacetime dimension $D = 1$. While the methods used in the literature do not easily generalize to quantum field theory, in this paper the path-integral representation of

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Nevertheless very influential paper for field theorists

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- Consider N-component scalar $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$

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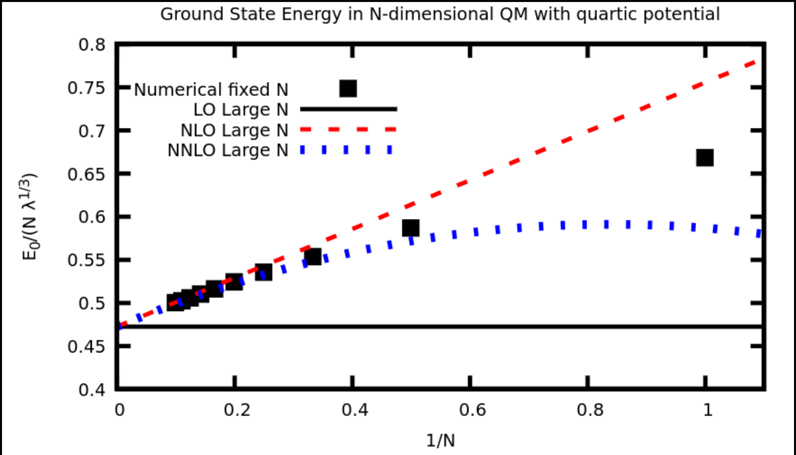
Expansion in $\frac{1}{N}$ allows systematic non-perturbative QFT solution!

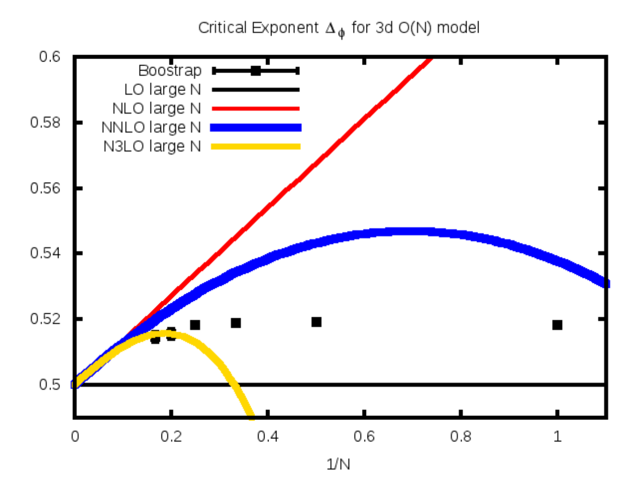
Quantitative tests of large N method (in less than 4d)

Quantum Mechanics with $V = (\vec{x})^2$ in N dimensions



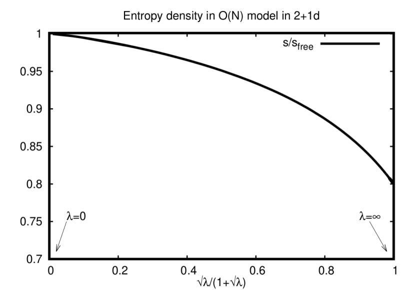
[PR, lecture notes on large N QFT: 2310.00048]

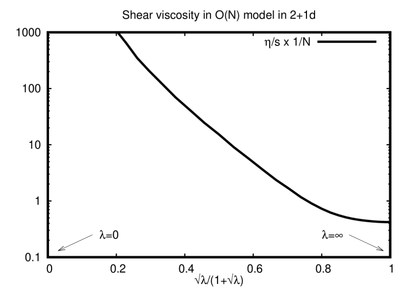
3d: superrenormalizable QFT



adapted from Kos et al., [1307.6856]

Large N 3d finite temperature





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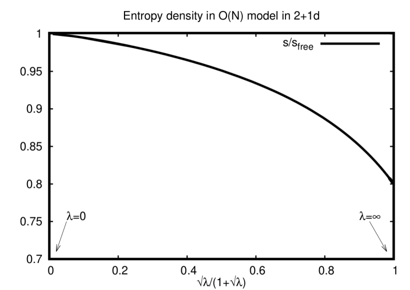

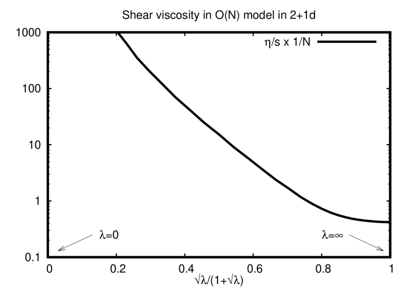

Infinite coupling, direct from QFT: $\frac{s_\infty}{s_{\rm free}} = \frac{4}{5}$, $\frac{\eta_\infty}{N s_{\rm infty}} \simeq 0.42$

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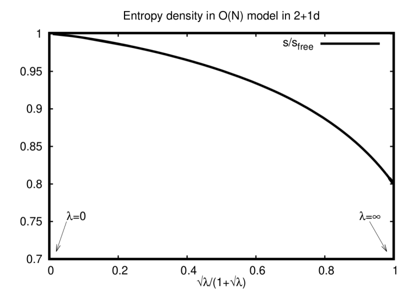

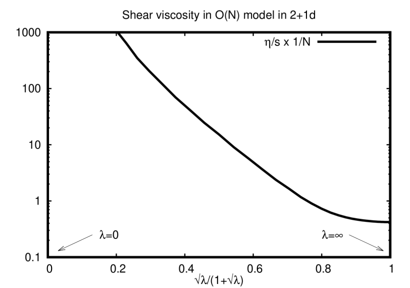

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Compare to $\frac{s_\infty}{s_{\rm free}} = \frac{3}{4}$, $\frac{\eta_\infty}{s_{\rm infty}} \simeq 0.08$ from AdS/CFT conjecture

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- ϕ is now entering quadratically, can be integrated out. Exact action

$$S_E = \frac{N}{2}\mathrm{Tr}\ln\left[-\partial^2 + i\zeta\right] + N\int d^d x \frac{\zeta^2}{4\lambda}\,.$$

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- Non-trivial to show: degenerate saddles for all constant auxiliary fields

$$\zeta(x) = \zeta_0$$

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- Note: often $\bar{\zeta} \in \mathbb{C}$, need Lefshetz-Picard for ζ_0 integration

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Some more pedagogical introduction to new Large N approach:

"Quantum Field Theory in Large-N Wonderland: Three Lectures"
Lecture Notes, 63rd Cracow School in Theoretical Physics
arXiv:2310.00048

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The auxiliary field is not (only) a mathematical trick, it has real physical meaning!

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Main difference to lower dimensions: UV divergencies, requiring renormalization
You **cannot** fix the coupling: it's not even an observable!

[Stevenson, 2409.01228]

Large N scalars in 4d in continuum

- $\vec{\phi}^4$ model is renormalized **non-perturbatively**

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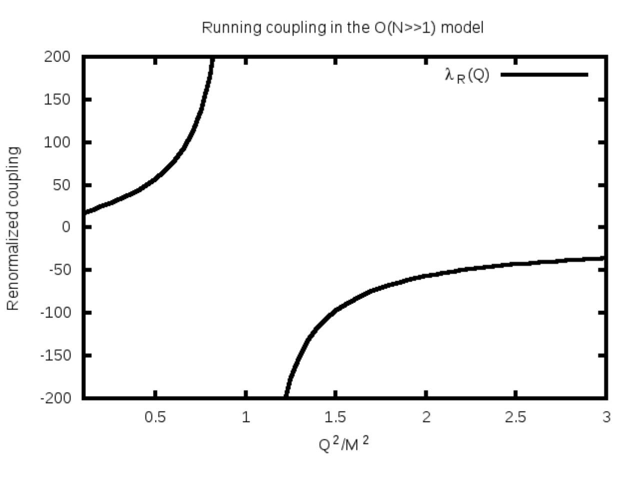
- Theory has "UV-fixed point", but bare coupling is **negative**

Exact Running coupling in O(N) Model



[PR, 2305.05678]

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  [PR 2401.06847]

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  [PR 2401.06847]
- Scattering amplitudes are well-behaved

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Only downside: this is not a gauge theory, so most hep theorists don't care!

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Large N Higgs Model – Abelian Case

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Problem 1: U(1) is continuous symmetry, cannot be spontaneously broken (Elitzur's theorem): **there cannot be a scalar VEV!**

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Problem 3: Treatment is classical, m_H receives large radiative corrections, yet $m_H \ll m_{\mathrm{Planck}}$ (**hierarchy problem**)

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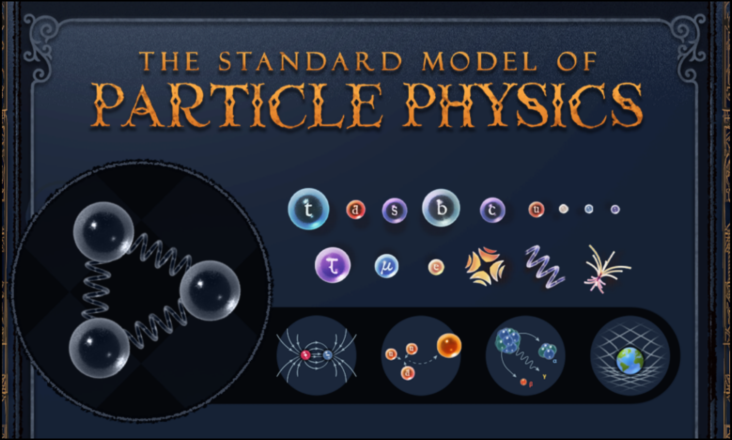
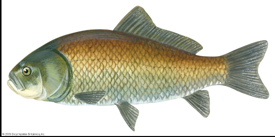
Problem 4: Fixing parameters to LHC data, Higgs **this vacuum is unstable** at $> 98\%$ cl [1307.3536]


THE STANDARD MODEL OF
PARTICLE PHYSICS



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  - Is $\mathcal{PT}$-symmetric courtesy of effective potential $V = -\phi^4$

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- Let's consider *pure scalar* model (no gauge fields) at finite temperature

Large N 4d scalar QFT at finite temperature

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$$\mathcal{V}(m) = \frac{m^4}{64\pi^2} \ln \frac{\Lambda_{\overline{\mathrm{MS}}}^2 e^{\frac{3}{2}}}{m^2} + \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta m)}{n^2}$$

[PR, 2211.15683]

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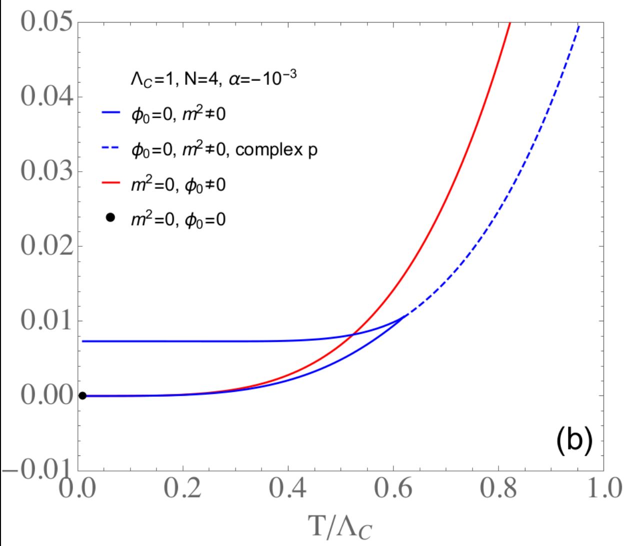
Rich phase structure in 4d Large N Scalar QFT



[Su, Weller & PR, in prep]

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