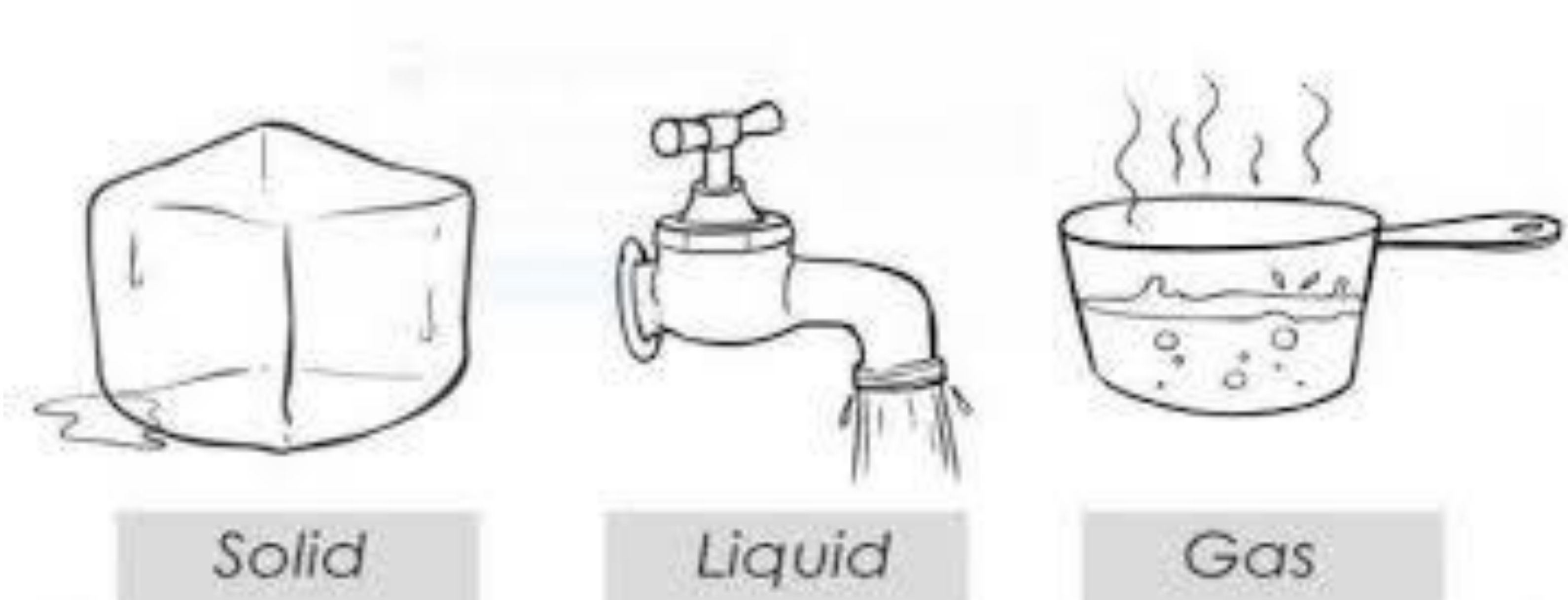
Complex phases in quantum mechanics



Carl M. Bender
Science Gallery, London
10 September 2024

Much of this talk is based on a recent EPL with
my longtime friend and collaborator **Daniel W. Hook**

"Complex phases in quantum mechanics"
CMB and D. W. Hook
Invited Perspective paper
Europhysics Letters **146**, 50001 (2024)

We live in a complex world

Complex mathematics was developed well before classical mechanics, gravity, fluid dynamics, E&M, relativity. But complex numbers do not appear in classical physics. Propagation, equilibrium, and diffusion are described by hyperbolic, elliptic, and parabolic equations, which are *real*

But… Quantum mechanics is *complex*:

$$i\psi_t(x,t) = -\psi_{xx}(x,t) + V(x)\psi(x,t)$$

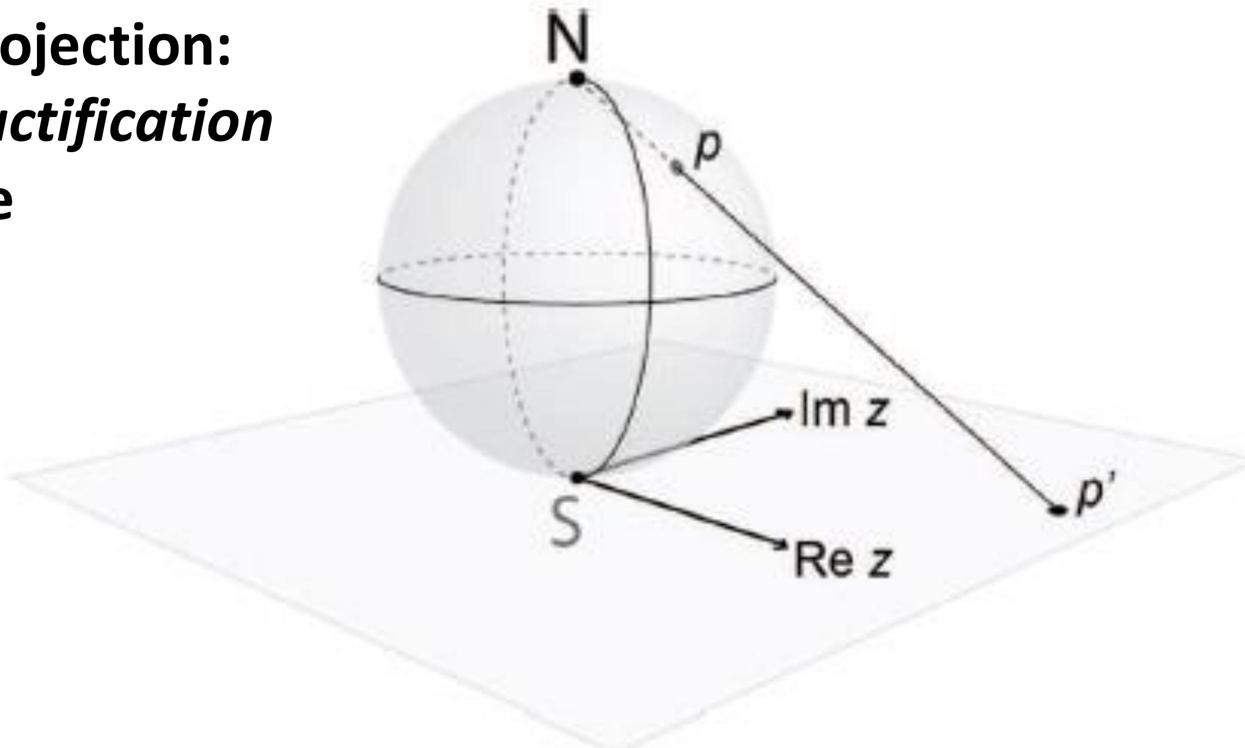
→ Quantum amplitude $\psi(x,t)$ is complex

→ Uncertainty principle follows from $[\hat{x},\hat{p}] = i$

→ Time reversal *T* involves complex conjugation (Wigner)

Complex plane and real line have different topology

Stereographic projection:
one-point compactification
of complex plane



$\infty \equiv \lim_{z \to 0} 1/z$

<u>1 point at ∞</u>

Two-point compactification **of real line:**

<u>2 points at ∞</u>

EXAMPLE: **Classical** **Hamiltonian** $H = p^2 - x^4$ **defines two** ***phases*****,**
phase 1 on the real line, phase 2 in the complex plane

Hamilton's equations: $x'(t) = 2p(t)$, $\quad p'(t) = 4[x(t)]^3$

Phase 1**:** **On real line a particle is initially at origin** $x(0) = 0$ **(top of the potential) with positive energy** $E > 0$

Time T **for this particle to slide down to** ∞ **is** ***finite*****:**

$$T = \int_0^\infty \mathrm{d}x(4E + 4x^4)^{-1/2} = 0.927037\ldots E^{-1/4}.$$

Q: Where is the particle when $t > T$ **?!**
A: Still at ∞**, where else?!**

$$-\infty \longleftarrow 0 \longrightarrow \infty.$$

This ***unstable*** **motion is** ***parity*** **symmetric (*****P*** **symmetric)**

Generalization

If x^4 **is replaced by** $|x|^a$ $(a>2)$**, time to reach** ∞ **is *finite***

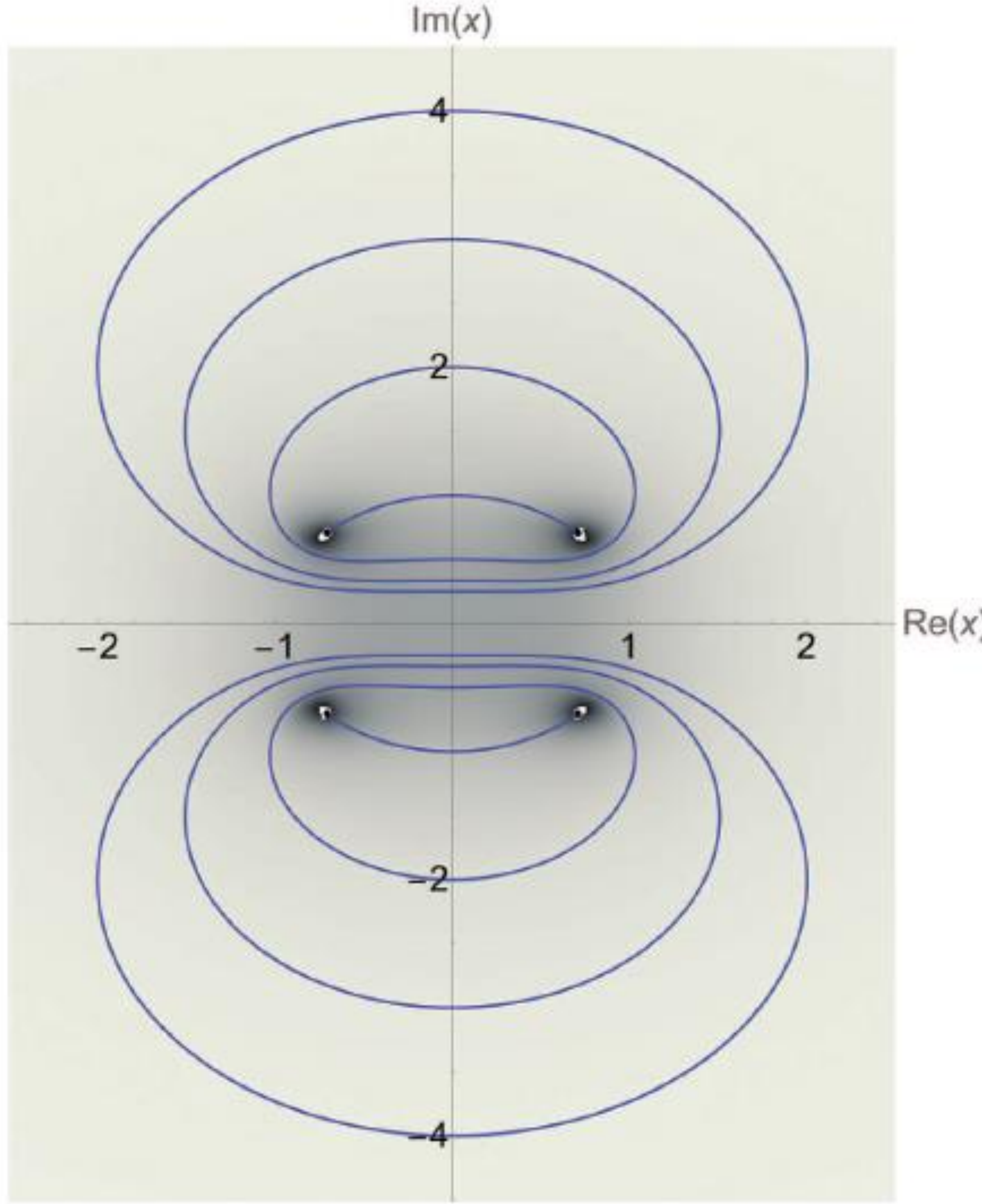
But if $0<a\leq 2$**, time to reach** ∞ **is *infinite* (Xeno's paradox!)**

Harmonic-oscillator value $a=2$ **is a transition point**

Phase 2: ***Periodic*** **classical motion enclosing turning points in complex plane**

Closed orbits, period $2T$

Hamilton's eqns agree:
$+\infty$ and $-\infty$ are the same point in complex plane!
Particle goes from $+\infty$ to $-\infty$ in no time



On real axis in complex plane (phase 2):

Unidirectional motion, periodic with period $2T$:

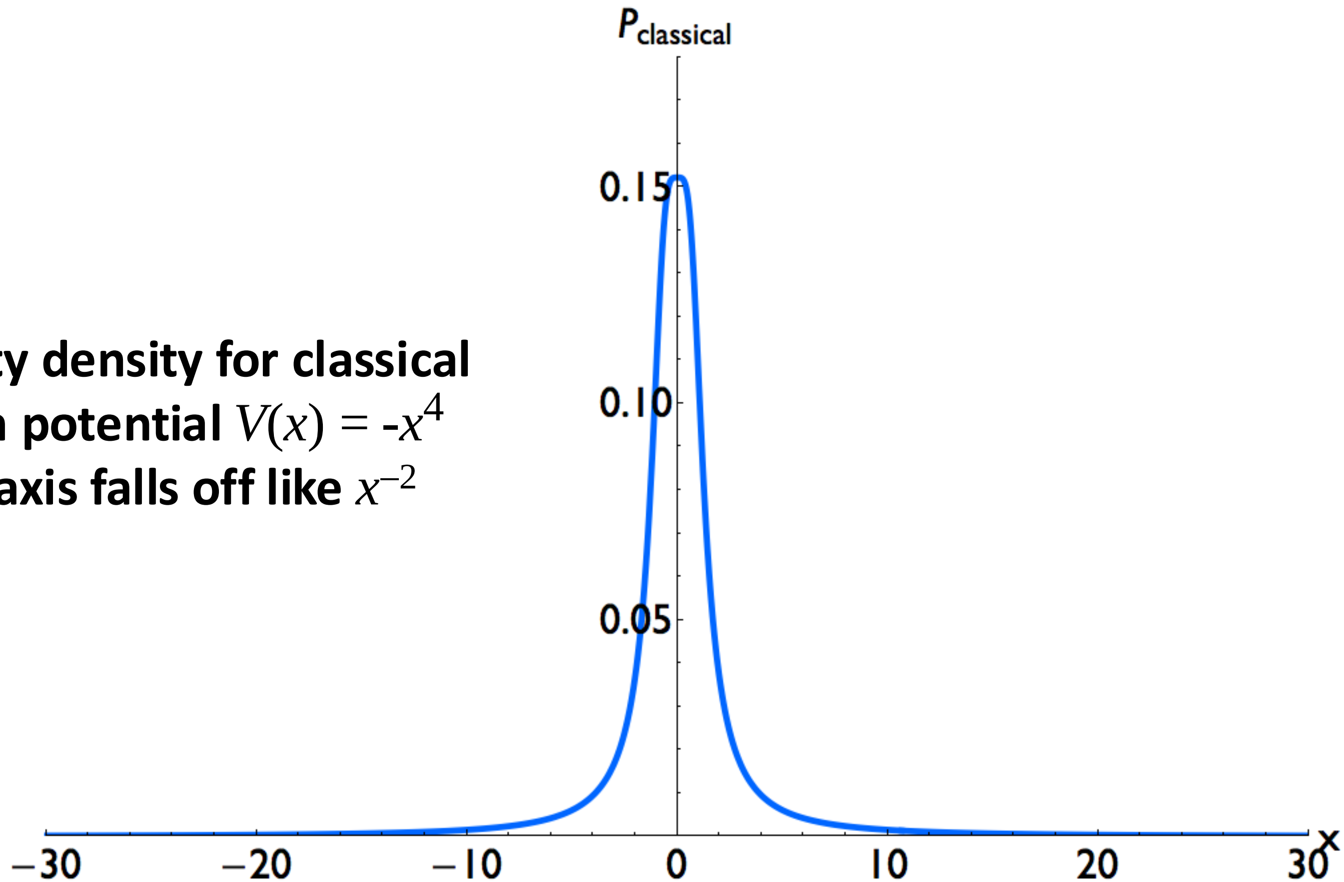
$$-\infty \longrightarrow 0 \longrightarrow \infty \quad \text{and} \quad -\infty \longleftarrow 0 \longleftarrow \infty.$$

This stable motion is *parity-time symmetric* (*PT* symmetric)

Two different phases distinguished by different global symmetries

Probability of finding the particle on the real axis in complex plane is inversely proportional to *speed* of particle:

Probability density for classical particle in potential $V(x) = -x^4$ on real-x axis falls off like x^{-2}



Classical particle on real axis in complex plane is in a
Dynamically-stable *PT*-symmetric bound state localized at the origin

A key point of this talk

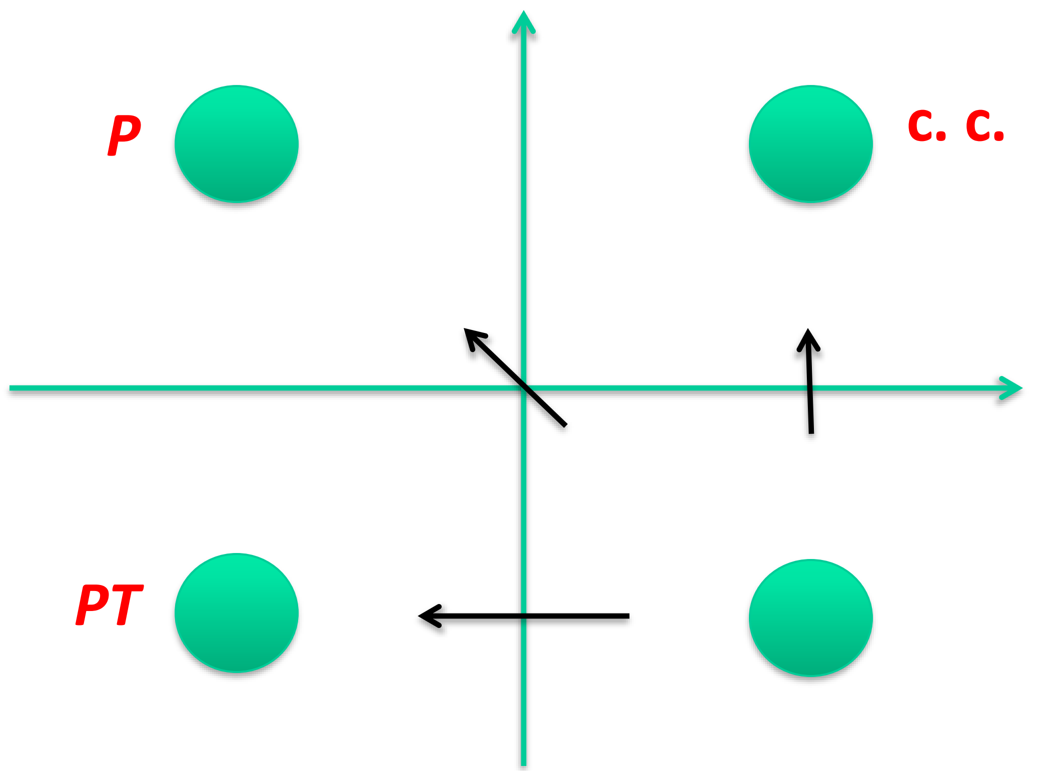
You can’t just look at a potential and say whether it is attractive or repulsive

It may be <u>BOTH</u>:
Attractive in one phase and repulsive in another phase

Phases distinguished by having different global symmetries…

Three global symmetry reflections in the complex-*x* plane:

(1) Complex conjugation c. c. is an *up-down* reflection: $x \to x^*$
(2) Parity *P* is a reflection *through the origin*: $x \to -x$
(3) Parity-time *PT* is a *left-right* reflection: $x \to -x^*$



One slide on *PT*-symmetric quantum theory

Idea of *PT* symmetry: Generalize quantum mechanics by replacing *mathematical* condition of Hermiticity

$$H^\dagger = H$$

with *physically-intuitive* condition of *space-time reflection symmetry*

$$[H, \mathcal{PT}] = 0$$

Bottom line:

If *H* is *PT* symmetric, its eigenvalues can be real and positive and time evolution can be unitary even if *H* is not Hermitian

PT symmetry combines quantum mechanics with the beautiful theory of complex variables

Q: How can there be *multiple* phases in quantum mechanics?!

A: Schrödinger equation is a *local* condition at *one point*; boundary conditions are required. In complex plane boundary conditions are imposed in pairs of regions (*Stokes sectors*). If sectors are globally *PT symmetric* (left-right symmetric), energy spectrum may be real

In the complex plane the *quantum-mechanical* theory defined by $H = p^2 \square x^4$ can also exist in *2 physically different phases* depending on how the Schrödinger eigenvalue problem is constructed:

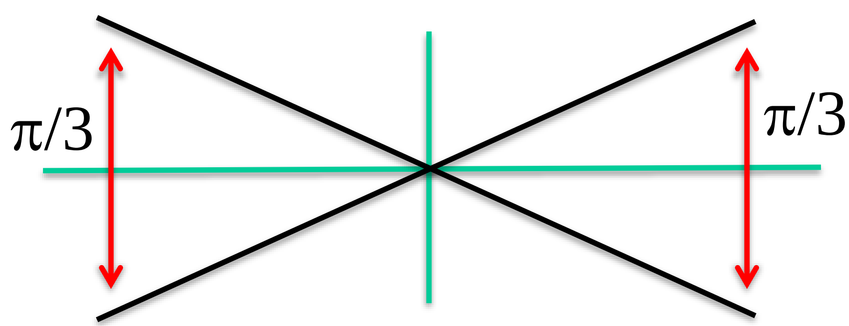
Phase 1: Construct the up-side down potential $V(x) = -x^4$ from $V(x) = e^{i\theta}x^4$ by rotating θ smoothly from 0 to π

At $\theta = 0$: **Eigenvalue problem is posed on real line; eigenfunctions vanish at $+\infty$ and $-\infty$**

WKB: Eigenfunctions vanish exponentially for real x like $\exp(-x^3/3)$ for large positive x and like $\exp(x^3/3)$ for large negative x

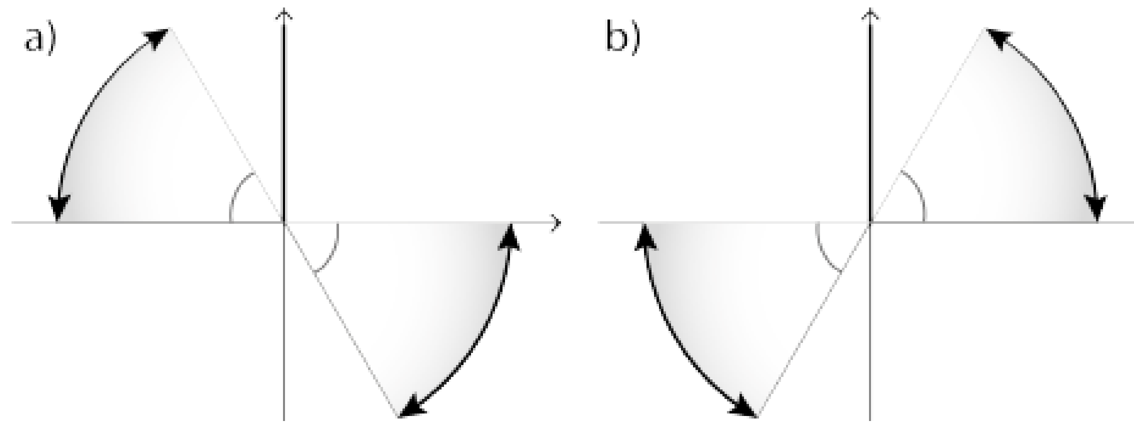
Eigenfunctions vanish exponentially in complex-x plane in *Stokes sectors* of angular opening $\pi/3$ centered about the real-*x* axis

This eigenvalue problem has global *P*, *PT*, and c. c. symmetry:



Now rotate θ smoothly from 0 to π (or $-\pi$)

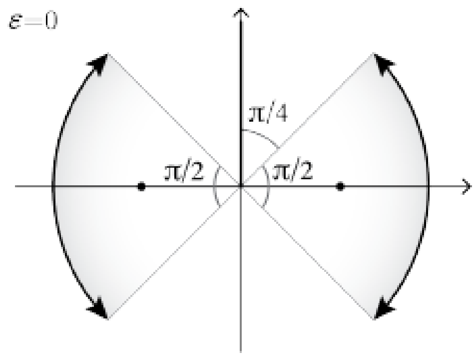
Stokes sectors for $\theta = \pi$ (or $-\pi$)

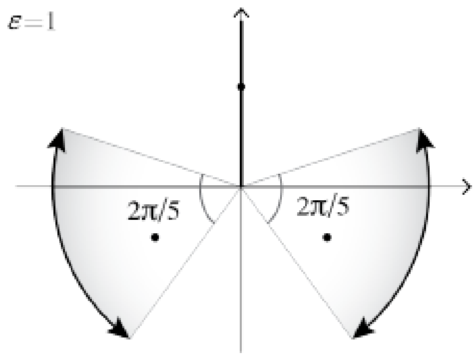


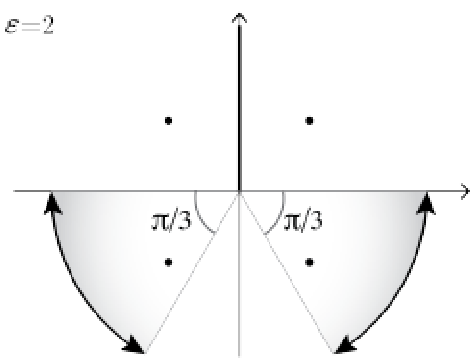
In these phases eigenvalue problems have global *P* symmetry but *not* *PT* or complex-conjugation symmetry

Eigenvalues are complex; states decay or grow with time.
(Think of these as *one* phase, not 2, because sign of i is not physically observable)

Phase 2: Construct the potential $V(x) = -x^4$ from $V(x) = x^2(ix)^\varepsilon$ by increasing ε smoothly from 0 to 2







Eigenvalue problem in this phase has global *PT* symmetry, not *P* or c. c. symmetry

Eigenvalues are all *real*; states are stable, do not decay or grow with time. Balanced loss and gain: Particles on real axis flow *into* potential on one side, *out* on other side

BTW: If we replace $V(x) = -x^4$ with $V(x) = x^2(-ix)^\varepsilon$ and increase ε smoothly from 0 to 2, Stokes sectors lie above real axis, flow is reversed, but eigenvalues are the same

In this phase we see unidirectional scattering

“Reflectionless Potentials and *PT* symmetry”
Z. Ahmed, CMB, and M. V. Berry
Journal of Physics A **38**, L627 (2005)

“*PT*-symmetry breaking and Laser-absorbing modes in optical scattering systems”
Y. D. Chong, L. Ge, and A. D. Stone
Physical Review Letters **106**, 093902 (2011)

“Scattering off *PT*-symmetric upside-down potentials”
CMB and M. Gianfreda
Physical Review A **98**, 052118 (2018)

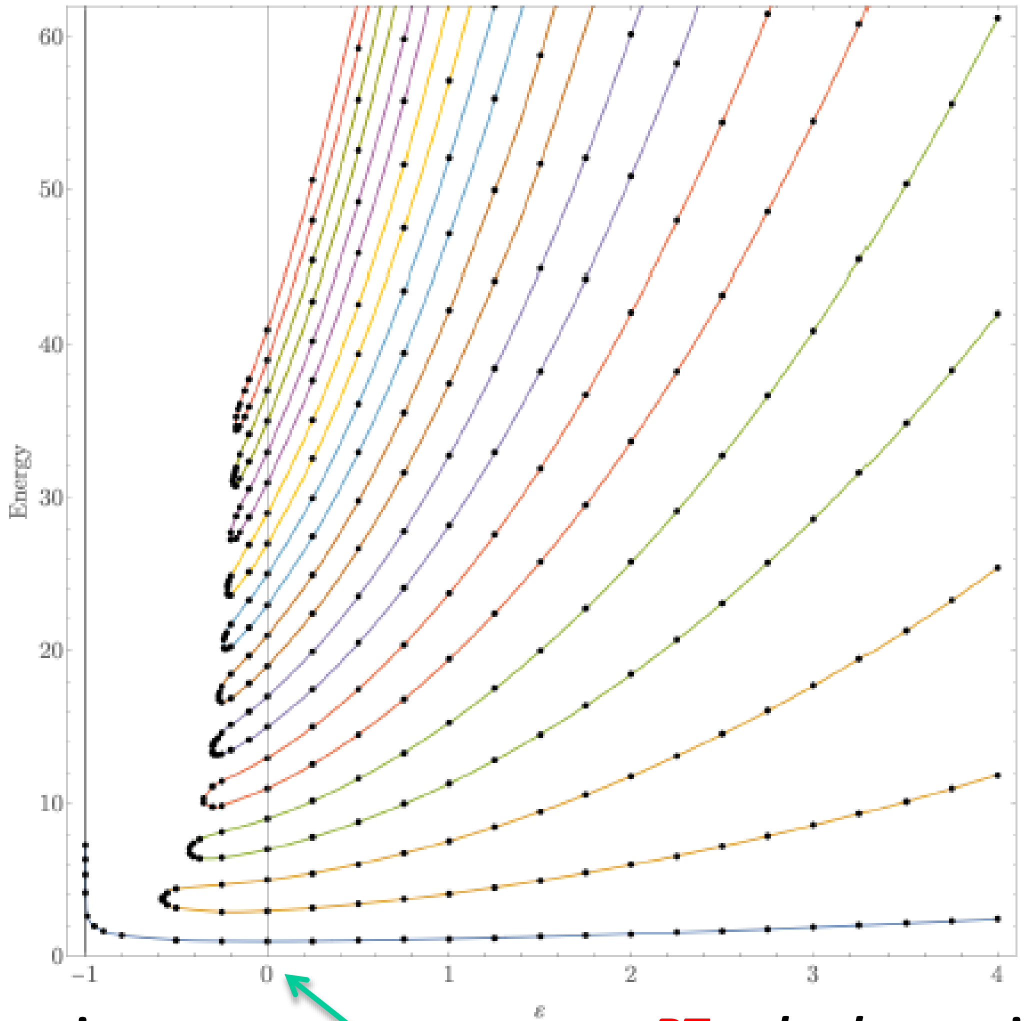
Experimental proposal for cold-atom scattering using cut-off $-|x|^p$ potential on real axis.
This potential is separately *P* and *T* symmetric:

“Experimentally realizable *PT* phase transitions in reflectionless quantum scattering”
M. B. Soley, CMB, and A. D. Stone
Physical Review Letters **130**, 250404 (2023)

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$

No reason to stop at $\varepsilon = 2$

(despite ZPG)



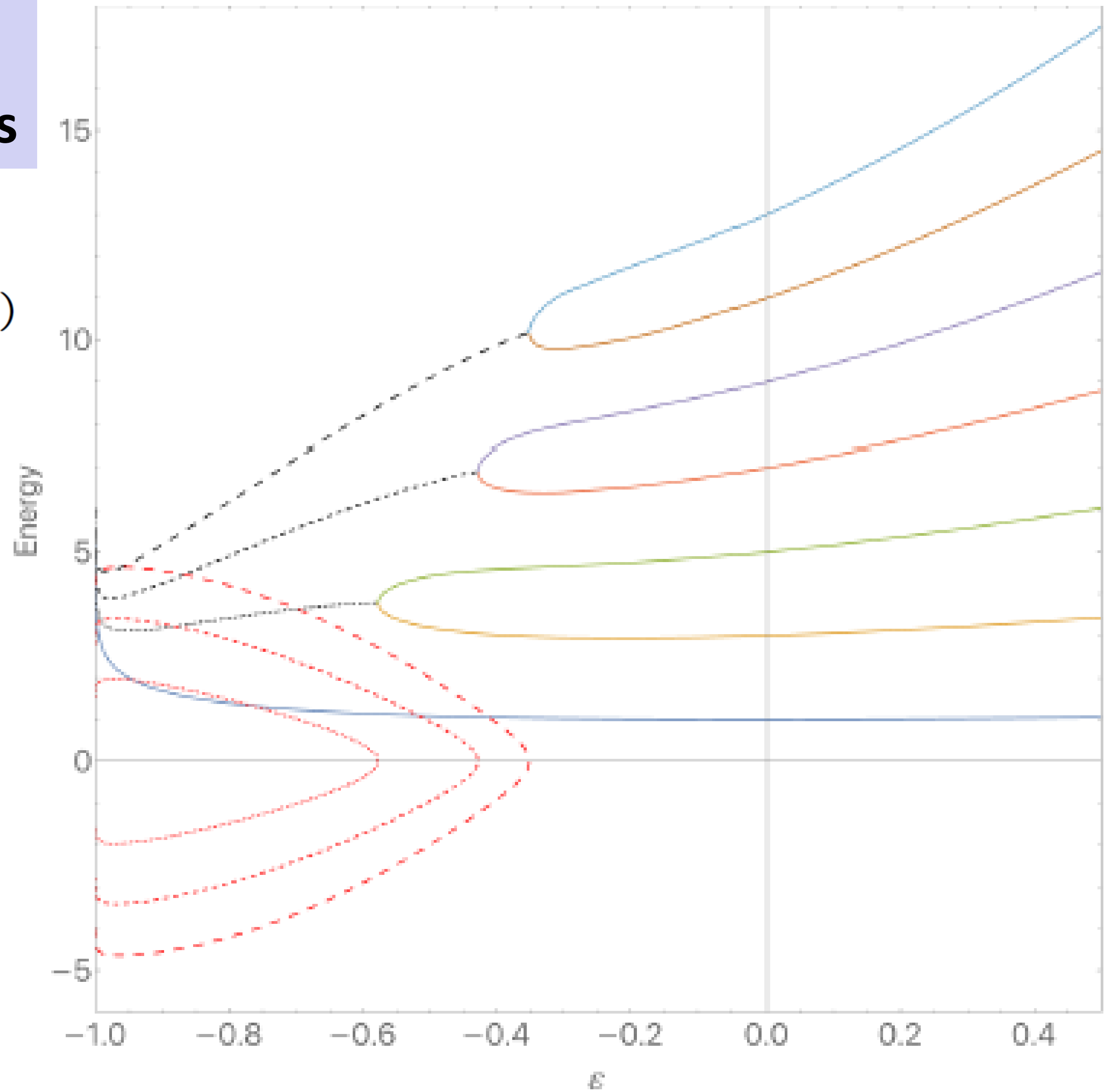
PT broken **region $\varepsilon<0$ (complex and real energies)**

PT **transition at $\varepsilon=0$**

PT unbroken **region $\varepsilon>0$ (all real energies)**

Detailed behavior of eigenvalues in BROKEN and UNBROKEN regions

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$



BROKEN REGION

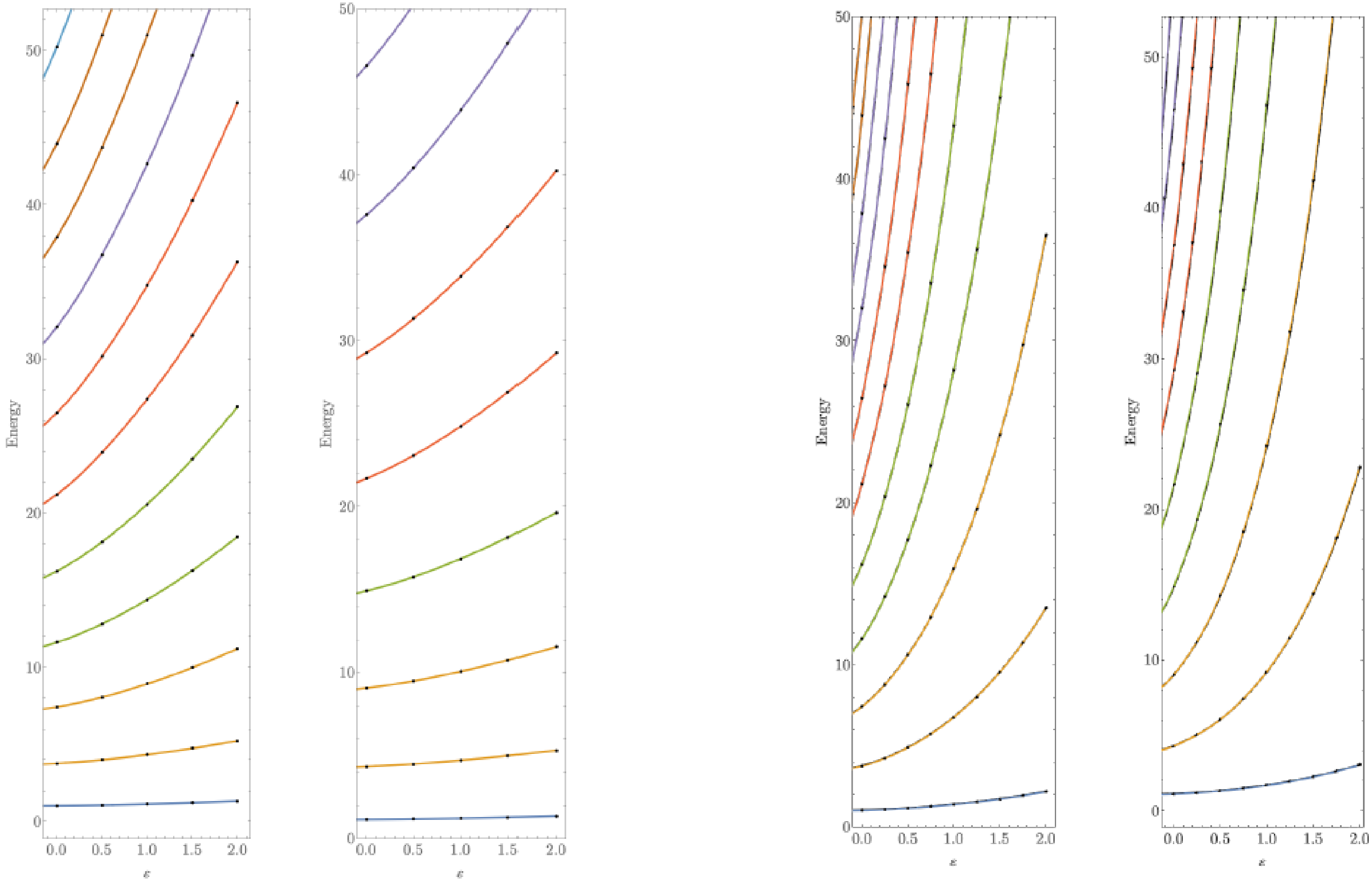
UNBROKEN REGION

Proof of spectral reality for ε > 0

P. Dorey, C. Dunning, and R. Tateo
J. Phys. A **34**, 5679 (2001)

P. Dorey, C. Dunning, and R. Tateo
J. Phys. A **40**, R205 (2007)

$$H = p^{2m} + x^{2n}(ix)^{\varepsilon} \quad (m,\, n = 1,\, 2,\, 3,\, \cdots)$$



m=1, n=2 m=1, n=3 m=2, n=1 m=3, n=1

Pioneering *experimental* work that inspired the *PT* community

First observation of *PT* transition using optical wave guides

"Observation of *PT*-symmetry breaking in complex optical potentials,"
A. Guo, G. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat,
V. Aimez, G. Siviloglou, and D. Christodoulides,*
Physical Review Letters **103**, 093902 (2009)
[Balanced loss and gain]

***2023 Schawlaw Laser Prize "For pioneering several areas in laser sciences, among them, the fields of *parity-time non-Hermitian optics*, accelerating Airy waves, and discrete solitons in periodic media"**

nature physics

LETTERS

PUBLISHED ONLINE: 24 JANUARY 2010 | DOI: 10.1038/NPHYS1515

Observation of parity–time symmetry in optics

Christian E. Rüter[1], Konstantinos G. Makris[2], Ramy El-Ganainy[2], Demetrios N. Christodoulides[2], Mordechai Segev[3] and Detlef Kip[1]*

One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables[1]. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity–time (*PT*) symmetry[2–7]. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories[8], non-Hermitian Anderson models[9] and open quantum systems[10,11], to mention a few. Although the impact of *PT* symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where *PT*-related notions can be implemented and experimentally investigated[12–15]. In this letter we report the first observation of the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential. We observe both spontaneous *PT* symmetry breaking and power oscillations violating left–right symmetry. Our results may pave the way towards a new class of *PT*-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

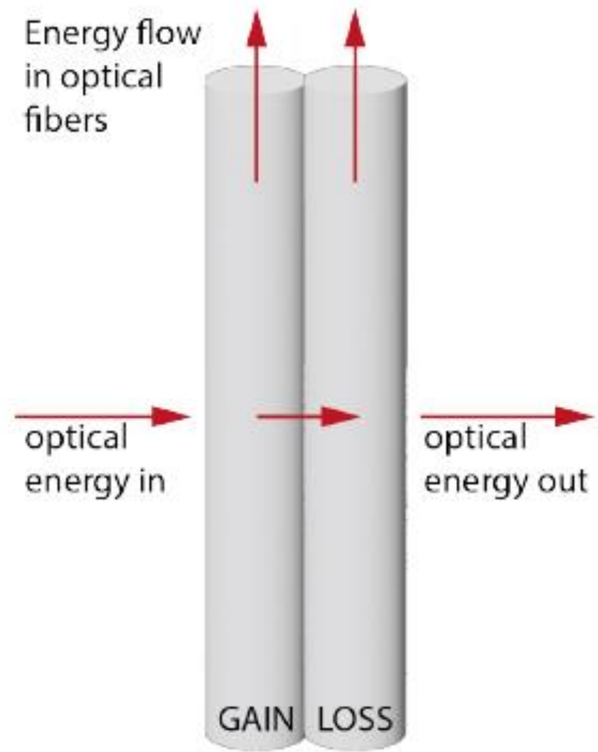
($\varepsilon > \varepsilon_{th}$), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous *PT* symmetry-breaking, that is, a 'phase transition' from the exact to broken-*PT* phase[7,20].

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in *PT*-symmetric complex potentials. In fact, such *PT* 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions[7,12–14]. Given that the complex refractive-index distribution $n(x) = n_R(x) + in_I(x)$ plays the role of an optical potential, we can then design a *PT*-symmetric system by satisfying the conditions $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$.

In other words, the refractive-index profile must be an even function of position x whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope E of the optical beam is governed by the paraxial equation of diffraction[13]:

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + in_I(x)]E = 0$$

Paraxial ray obeys effective Schrödinger equation



One fiber: $H_1 = [a + ib]$

$$i\frac{d}{dt}\psi = (a+ib)\psi. \qquad \psi(t) = Ce^{(-ai+b)t} \quad (C \text{ constant})$$

Two uncoupled fibers: $H_2 = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$

Two coupled fibers: $H_{\text{coupled}} = \begin{bmatrix} a+ib & g \\ g & a-ib \end{bmatrix}$

Hamiltonian is *PT* symmetric: $[H_{\text{coupled}}, \mathcal{PT}] = 0$

where $\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **and time reversal is *c. c.***

Secular equation real: $\det(H_{\text{coupled}} - IE) = E^2 - 2aE + a^2 + b^2 - g^2 = 0$

Energies real for sufficiently strong coupling: $E_{\pm} = a \pm \sqrt{g^2 - b^2}$

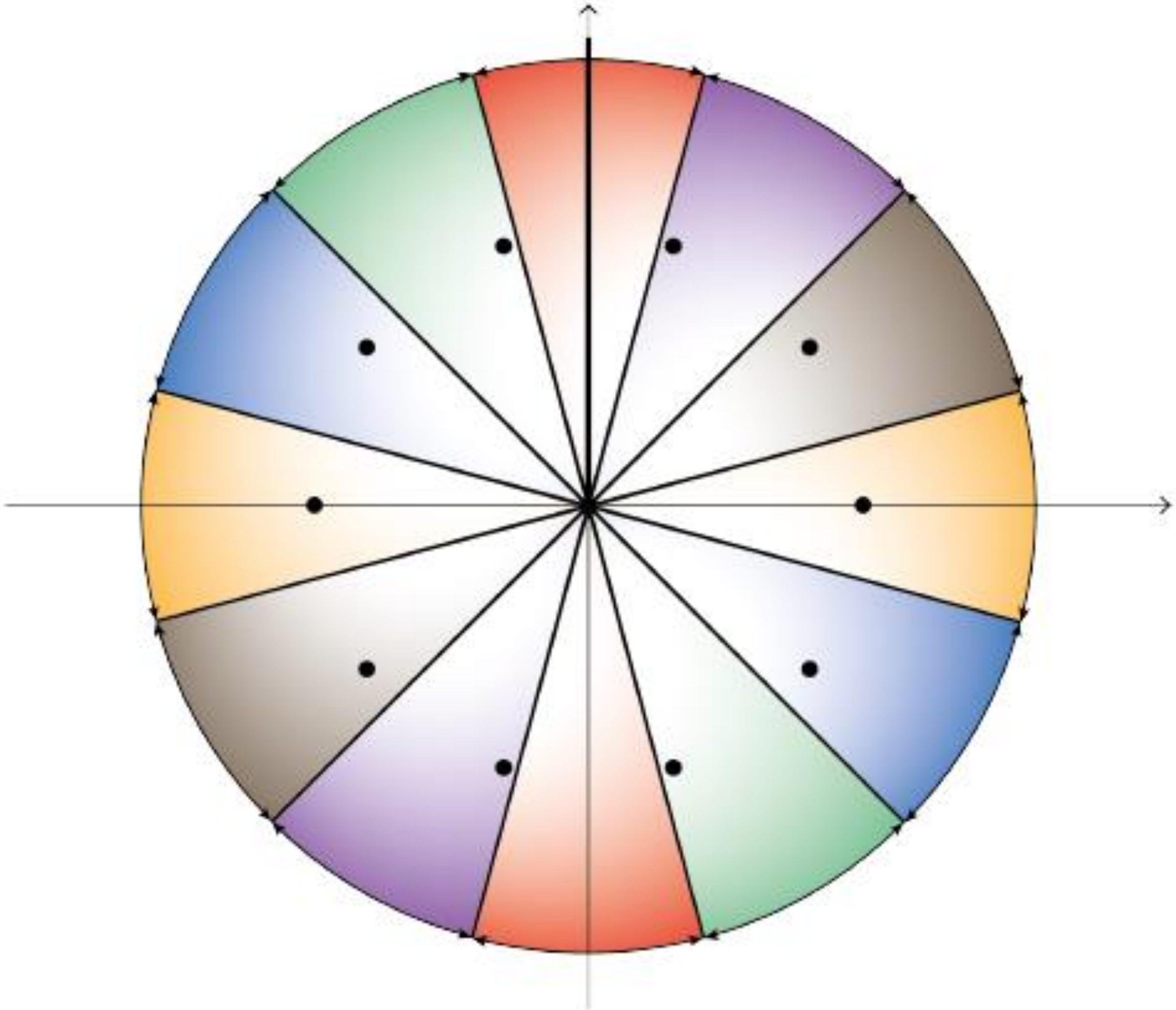
Regions of *unbroken* *PT* symmetry: $g > b$ **and** $g < -b$

Region of *broken* *PT* symmetry: $-b < g < b$

In unbroken *PT* symmetry regions system is in *dynamic equilibrium*

THREE-PHASE HAMILTONIAN SYSTEM

$$H = p^2 + x^{10}$$

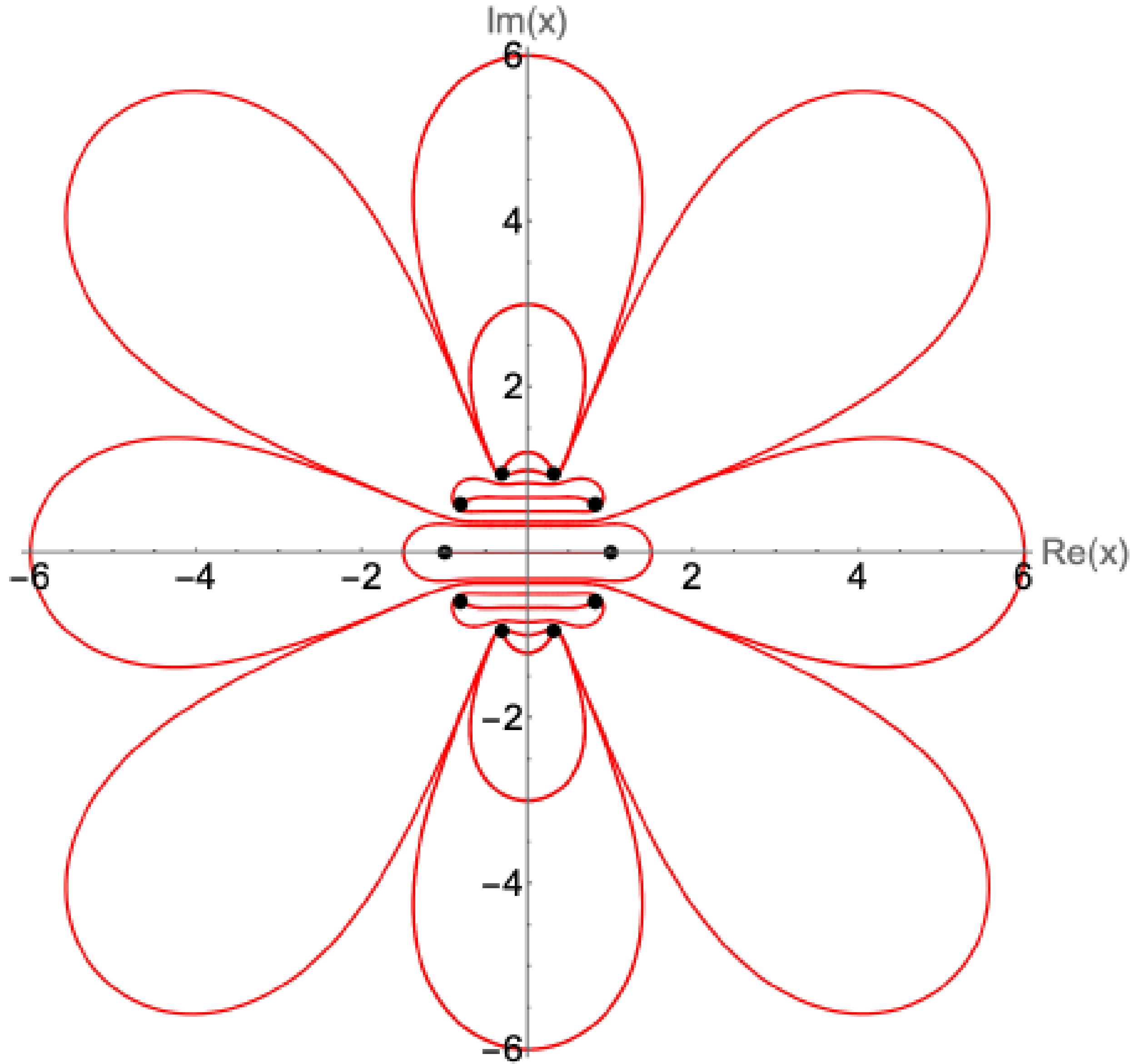


Quantum theory has 3 pairs of *PT*-symmetric *Stokes Sectors*

(Dots are classical turning points)

AT THE CLASSICAL LEVEL

Families of classical paths densely fill 3 regions of complex plane bounded by *separatrices*



AT THE QUANTUM LEVEL

$$-y''(x) + x^{10}y(x) = Ey(x)$$

WKB: Eigenvalues in ratio
1 : 1.424 : 7.780

$$E_n \sim \left[\frac{(6n+3)\sqrt{\pi}\,\Gamma(3/5)}{\Gamma(1/10)}\right]^{5/3} \quad (n\to\infty)$$

$$E_0 = 0.737..., \quad E_1 = 4.596..., \quad E_2 = 10.768..., \quad E_3 = 18.866..., \quad E_4 = 28.680...$$

$$E_n \sim \left[\frac{(6n+3)\sqrt{\pi}\,\Gamma(3/5)}{\cos(\pi/5)\Gamma(1/10)}\right]^{5/3} \quad (n\to\infty)$$

$$E_0 = 1.049..., \quad E_1 = 6.543..., \quad E_2 = 15.330..., \quad E_3 = 26.858..., \quad E_4 = 40.831...$$

$$E_n \sim \left[\frac{(6n+3)\sqrt{\pi}\,\Gamma(3/5)}{\cos(2\pi/5)\Gamma(1/10)}\right]^{5/3} \quad (n\to\infty)$$

$$E_0 = 5.214..., \quad E_1 = 32.539..., \quad E_2 = 76.235..., \quad E_3 = 133.568..., \quad E_4 = 203.053...$$

Three *noninteracting* phases described by one Hamiltonian

Reminds us of *electron*, *muon*, and *tau*

These particles decay into one another if energy is *<u>complex</u>*

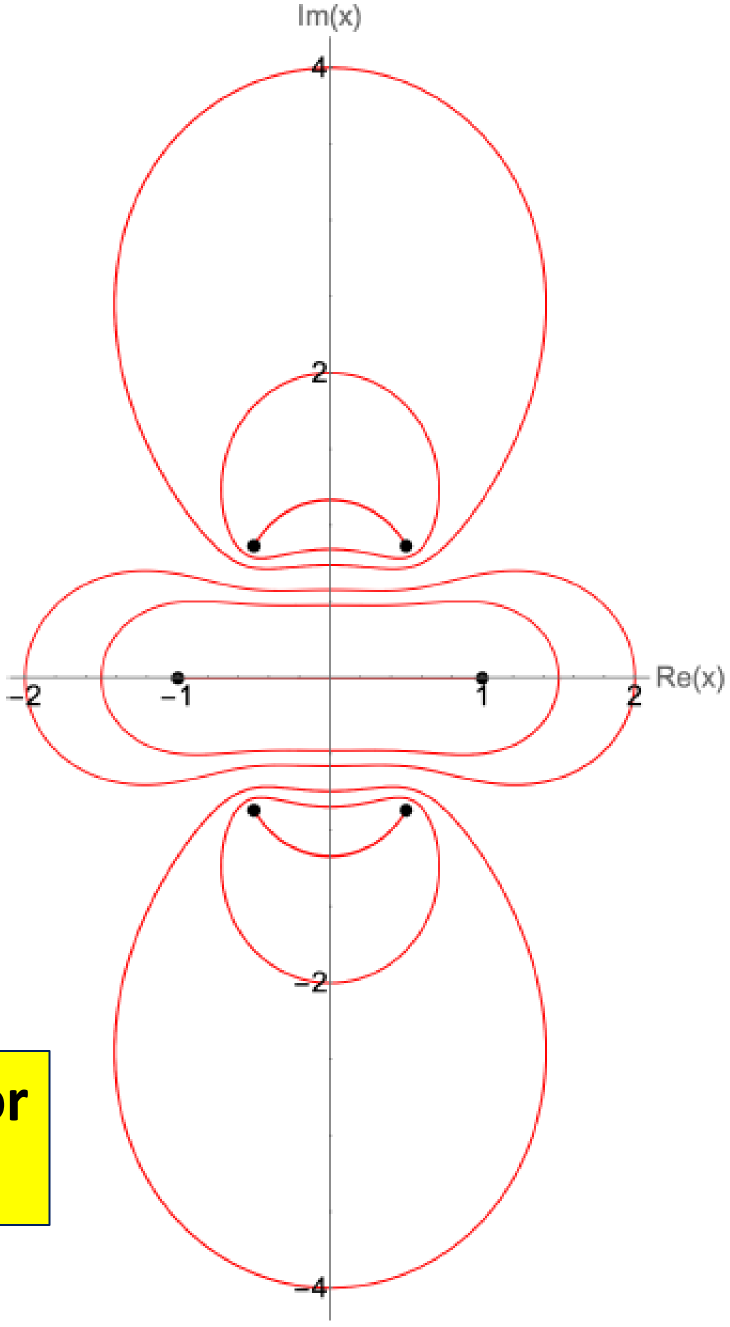
***If energy is complex, quantum phases mix*;**
classical paths cross from region to region

"Families of particles with different masses in *PT*-symmetric quantum field theory"
CMB and S. P. Klevansky
Physical Review Letters **105**, 031601 (2010)

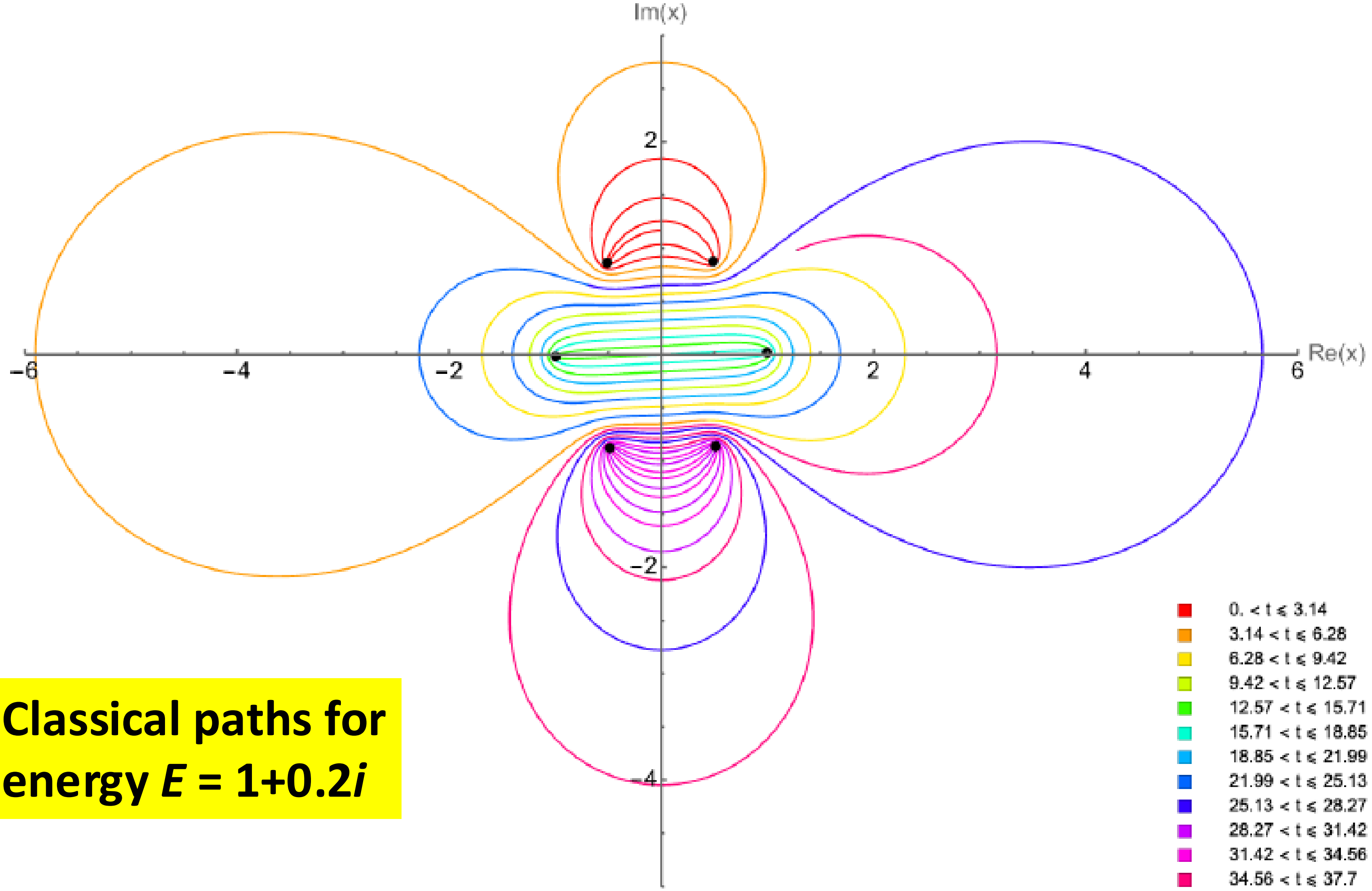
Hamiltonian with two phases:

$H = p^2 + x^6$

Two real positive spectra at the quantum level

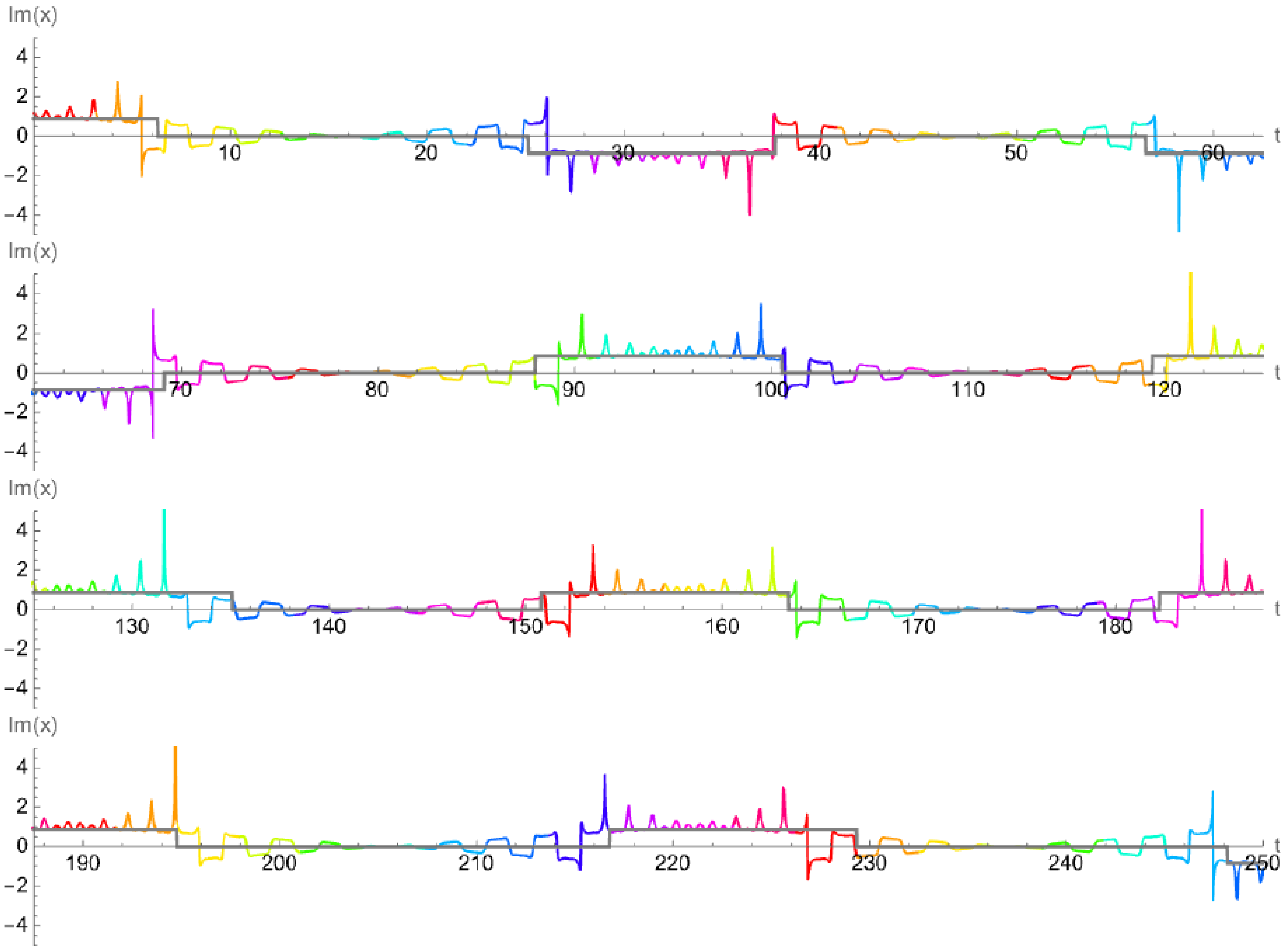


Classical paths for real energy $E = 1$



Classical paths for energy $E = 1+0.2i$

Initial position of particle $x = 1.167i$



Im(x)
4
2
0
−2
−4
t
10
20
30
40
50
60
Im(x)
4
2
0
−2
−4
t
70
80
90
100
110
120
Im(x)
4
2
0
−2
−4
t
130
140
150
160
170
180
Im(x)
4
2
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−2
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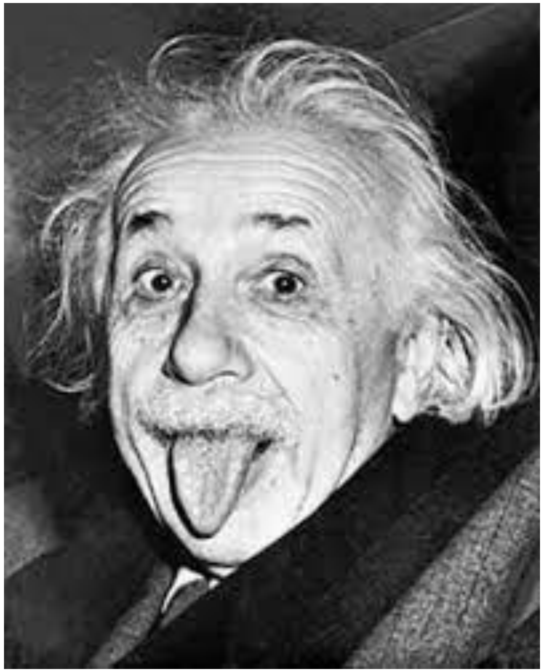
t
190
200
210
220
230
240
250

Extension to QFT



Homogeneous real Lorentz group:

6-parameter group of ***real*** 4 x 4 matrices transforming the space-time point (x, y, z, t) into (x', y', z', t') but leaving $x^2 + y^2 + z^2 - t^2$ invariant

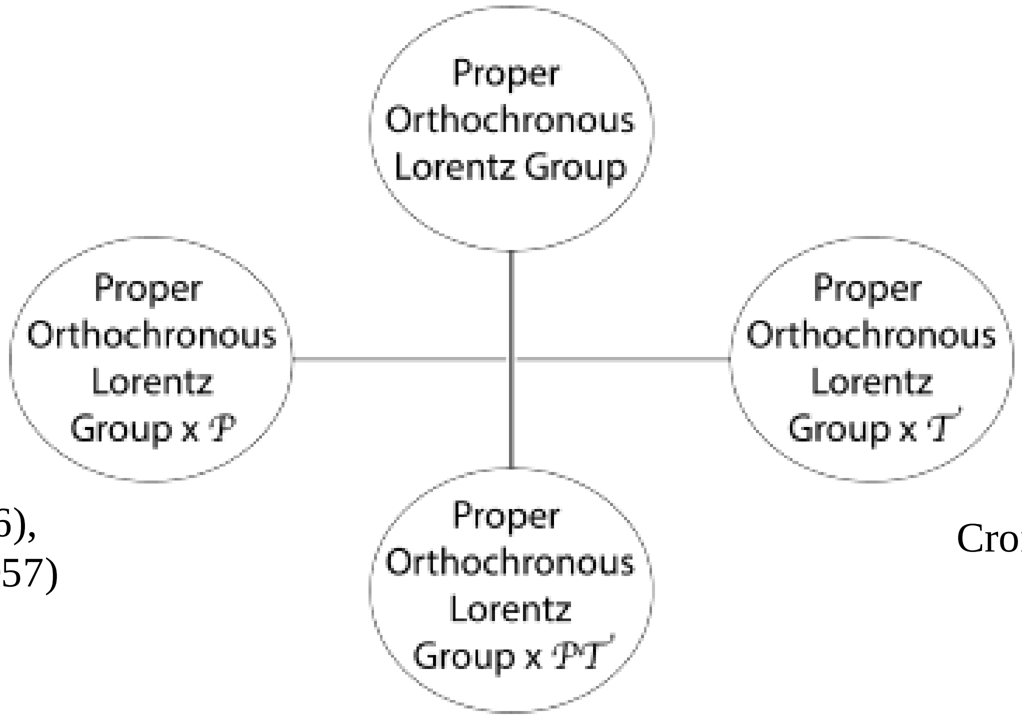
P and ***T*** are elements of the Lorentz group:

P: $(x, y, z, t) \rightarrow (-x, -y, -z, t)$

T: $(x, y, z, t) \rightarrow (x, y, z, -t)$

Note: Fundamental (2-component spinor) representation is ***complex*** (Wigner)

Real Lorentz group has 4 disconnected parts



Lee & Yang (1956),
C. S. Wu et al (1957)

Cronin & Fitch et al. (1980)

Complex Lorentz group is up-down and left-right connected so *PT* symmetry appears naturally in the complex plane

PCT, Spin and Statistics, and all that
R. S. Streater and A. S. Wightman

Repeat: Message of this talk

One Hamiltonian may define several different physical phases characterized by different global symmetries; different phases are *not* analytic continuations of one another

Field theory example (remember the x^4 anharmonic oscillator):

Hermitian $g\phi^4$ quantum field theory (ϕ scalar, g>0) has one phase; stable, not asymptotically free, trivial in 4 dimensions

Non-Hermitian $-g\phi^4$ quantum field theory (g>0) has two phases:

(1) ϕ scalar, *P*-symmetric, unstable
(2) ϕ *pseudoscalar*, *PT*-symmetric, stable, asymptotically free, nontrivial in 4 dimensions

Seminal paper by Dyson

"Divergence of perturbation theory in quantum electrodynamics"
Physical Review **85,** 631 (1952)

Dyson's idea: Replace electric charge *e* by *ie*
(that is, replace *fine-structure constant* α by -α)

Dyson says: Coulomb force changes sign, so quantum vacuum becomes unstable. Abrupt (*nonanalytic*) change at α=0 implies perturbation series in powers of α *diverge*

We return to Dyson's paper shortly, but first a short digression:

Perturbation series DO diverge, but for a simpler reason …

GRAPH COUNTING

Perturbation series diverge because there are *factorially* many Feynman diagrams. In bosonic theories diagrams add in phase

(This is just *combinatorics*, not physics)

Dyson's paper inspired an industry of graph counting:

Hurst (1952)
Petermann (1953)
Thirring (1953)
Glimm and Jaffe (1968)
CMB and Wu (1969)

Perturbation series for the N^{th} eigenvalue of the anharmonic oscillator in the coupling constant g:

$$E_N(g) \sim \sum_{n=0}^{\infty} a_n g^n$$

For the ground-state energy $E_0(g)$:

$$a_n \sim \frac{m}{2}\frac{\sqrt{6}}{\pi^{3/2}}(-1)^{n+1}\Gamma(n+\tfrac{1}{2})\left(\frac{24g}{m^3}\right)^n \quad (n \to \infty)$$

CMB & T. T. Wu
"Large-order behavior of perturbation theory"
Physical Review Letters **27**, 461 (1971)

Complex analysis explains ***why*** perturbation series must diverge

Square-root singularities (now called exceptional points) near $g = 0$ in complex-coupling-constant plane …

CMB and T. T. Wu
"Analytic Structure of Energy Levels in a Field-Theory Model"
Physical Review Letters **21**, 406 (1968)

Complex analysis explains quantization!

Example: **Two-state system having energies a and b…**

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Couple **states with coupling constant g:**

$$H = \begin{pmatrix} a & g \\ g & b \end{pmatrix}$$

Energies for this two-state system

$$\det \begin{pmatrix} a - E & g \\ g & b - E \end{pmatrix} = 0$$

$$E^2 - (a+b)E + ab - g^2 = 0$$

$$E(g) = \frac{a+b}{2} \pm \frac{1}{2}\sqrt{(a-b)^2 + 4g^2}$$

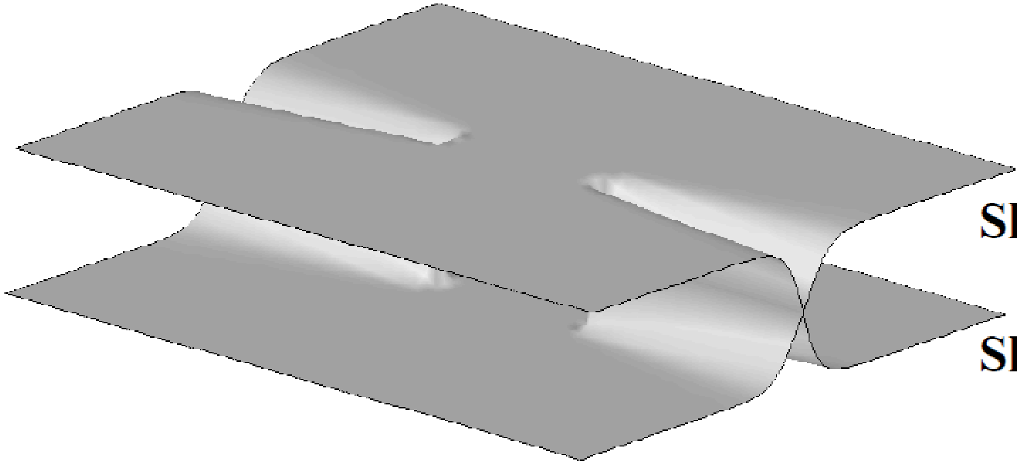
Square-root singularities in complex-*g* plane at

$$g = \pm \frac{|a-b|}{2} i$$

(*exceptional points*, originally called *Bender-Wu singularities*)

E(*g*) is a *smooth* (analytic) function defined on a two-sheeted Riemann surface

Complex-*g* surface



On this Riemann surface energy levels are continuous, *not* discrete

Quantization is topological – quantized energy levels correspond to discrete sheets of a Riemann surface

Exceptional points (where energy levels cross) cause divergence of perturbation series but *complex-variable* techniques (Padé, Borel) *sum* divergent perturbation series and get arbitrarily small error

End of digression!! Return to Dyson's paper about replacing *e* with *ie*

Claim **Like $-x^4$ potential, which has 2 phases with different global symmetries, Dyson's QED Hamiltonian with $e \to ie$ describes TWO phases having different global symmetries:**

(1) Unstable-vacuum *P*-symmetric phase in which A^μ is a *vector*

(2) Stable-vacuum *PT-symmetric* phase in which A^μ is an *axial* vector (like a theory of magnetic charge). Interaction $ieA^\mu J_\mu$ is *PT*-symmetric. Spectrum is real to leading order in α

"*PT*-Symmetric Quantum Electrodynamics"
CMB, I. Cavero-Pelaez, K. A. Milton, and K. V. Shajesh
Physics Letters B **613**, 97 (2005)

Two *indications* that Dyson's QED with $-\alpha$ is stable

(1) Renormalization group analysis [critical point of the beta function (Adler); finite-QED Johnson-Baker-Willey program]:

$$F_1(\alpha) = \frac{4}{3}\left(\frac{\alpha}{4\pi}\right) + 4\left(\frac{\alpha}{4\pi}\right)^2 - 2\left(\frac{\alpha}{4\pi}\right)^3 - 46\left(\frac{\alpha}{4\pi}\right)^4 + \cdots$$

Two terms: Jost & Luttinger (only negative α)
Third term: Rosner α = 28.789
Fourth term: Gorishny α = 4.804

But Bender & Milton noted that the negative α theory appears to have a critical point: α = -4.187, -3.657, -3.590, ... (!)

(2) Casimir force analysis (Glashow's idea, Boyer's calculation):
For conventional QED there is no solution for positive α

But for Dyson $-\alpha$ phase Casimir force balance gives α = -0.09235 (!)

"A Nonunitary Version of Massless Quantum Electrodynamics Possessing a Critical Point"
CMB and K. A. Milton, *Journal of Physics A* ***32***, L87-L92 (1999)

Conjecture regarding phases of gravity

Like QED, perhaps there are two phases of classical and quantum gravity, Hermitian and *PT*-symmetric, which are not analytic continuations of one another, one with $g^{\mu\nu}$ a tensor and the other with $g^{\mu\nu}$ an axial tensor

One phase real and attractive, other phase complex and repulsive

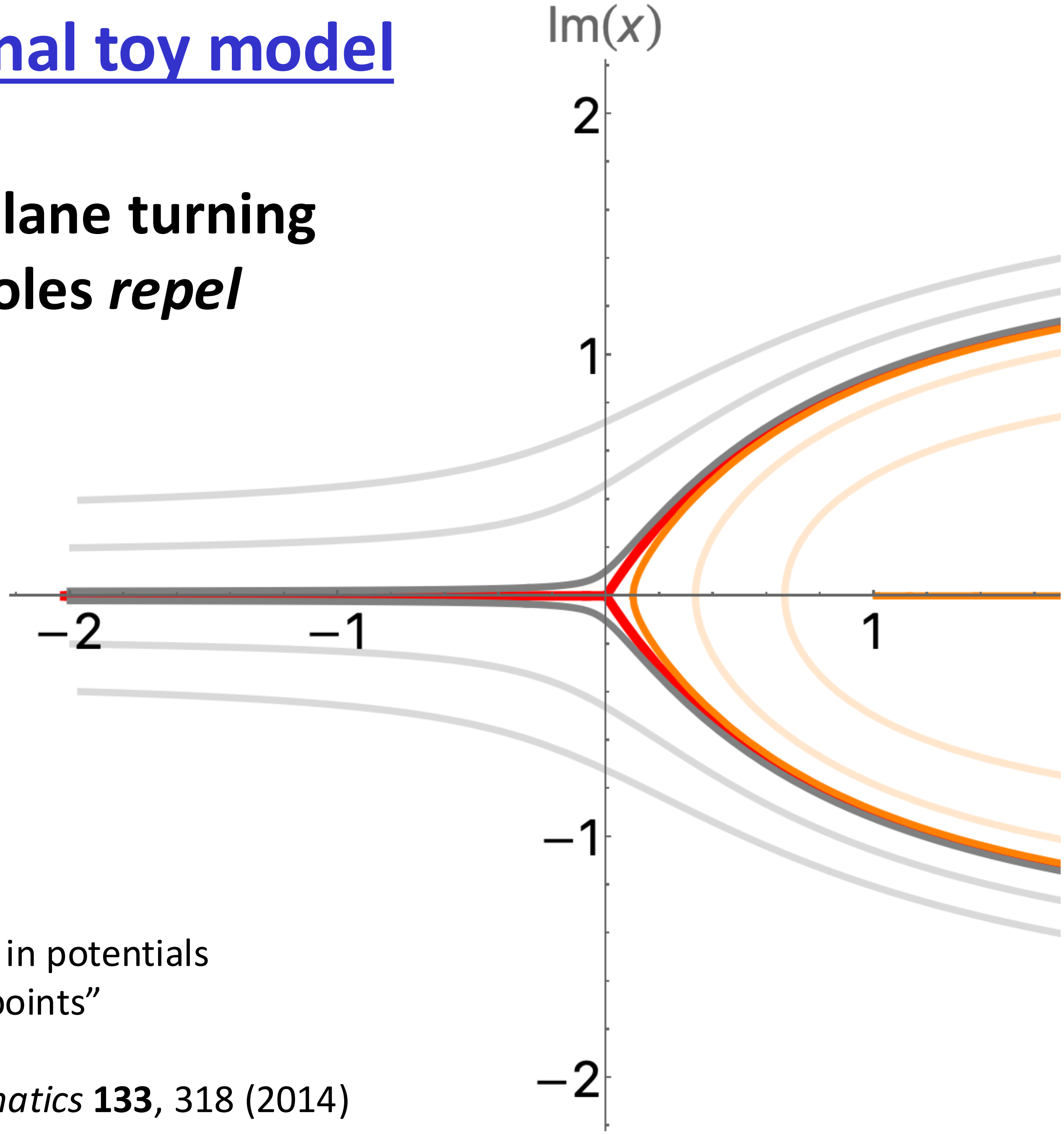
Might explain accelerating expansion of universe (dark energy)

"Making sense of non-Hermitian Hamiltonians"
CMB
Reports on Progress in Physics **70**, 947 (2007)

PT Symmetry in Quantum and Classical Physics
CMB *et al*
World Scientific, Singapore, 2019

One-dimensional toy model

In the complex plane turning points *attract*, poles *repel*



$$\boldsymbol{H = p^2 + 1/x}$$

$$\boldsymbol{E = 1}$$

"Complex classical motion in potentials with poles and turning points"
CMB and D. W. Hook
Studies in Applied Mathematics **133**, 318 (2014)

Mavromatos and Sarkar argue that a Chern-Simons gravity theory may have two phases. They study string-inspired effective axion anomalously coupled to Abelian gauge fields and gravity

M & S claim renormalization group analysis suggests that there is a Landau pole separating two phases:

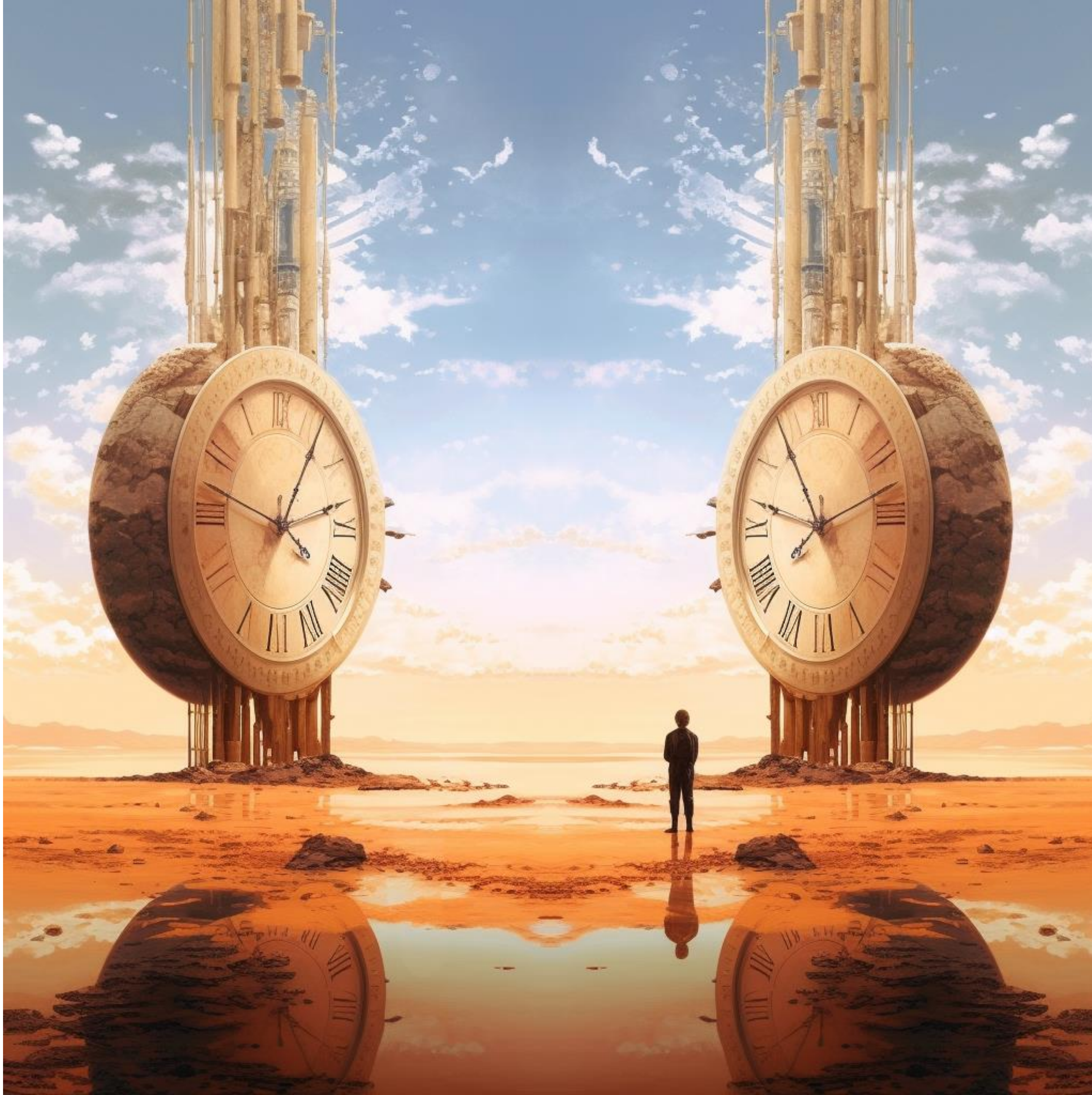
Ultraviolet (short-distance) Hermitian phase with *attractive gravity*

Infrared (long-distance) *PT*-symmetric phase with *repulsive gravity*

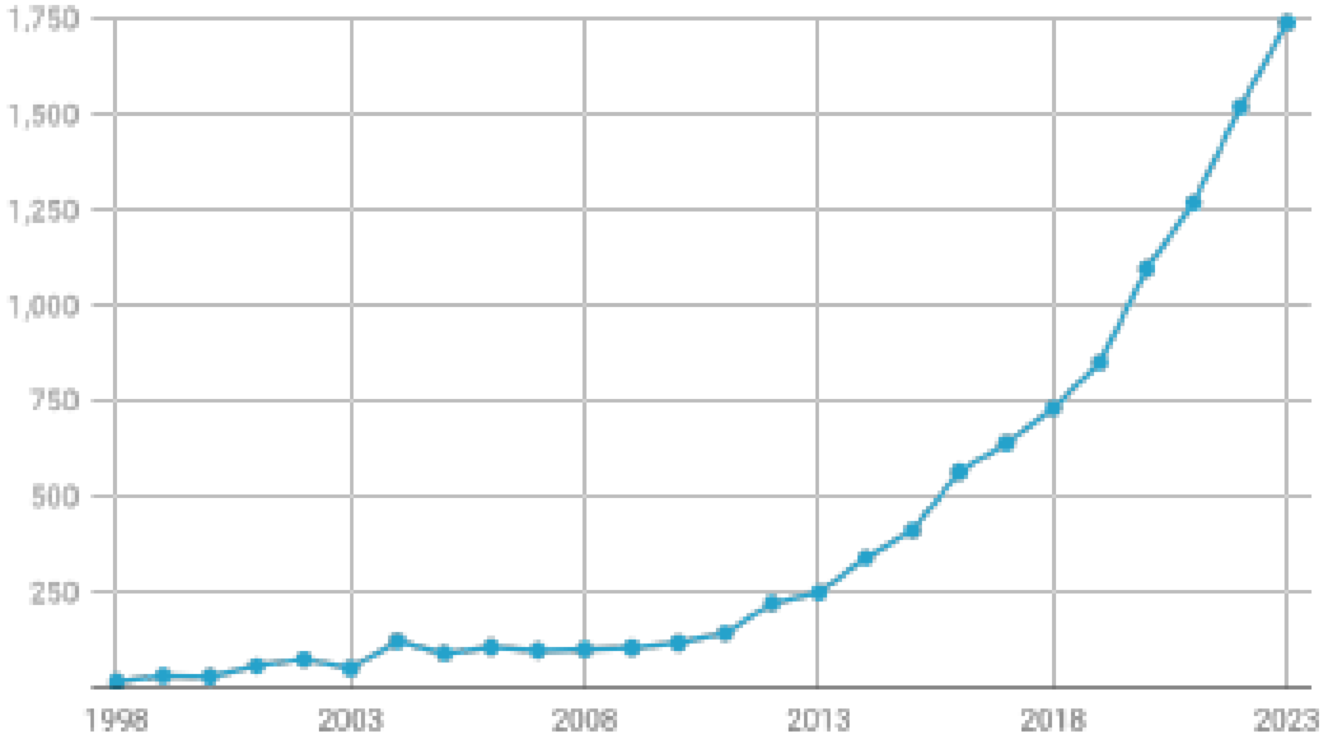
N. Mavromatos and S. Sarkar
Physical Review D **110**, 045004 (2024)

Looking into the future…

Many research opportunities!



Annual publications



Over 10,000 publications referencing ***PT*** symmetry since 1998 (**not** including papers on the arXiv)

for listening to my talk

Shakespearian Sonnet on *PT* symmetry

In realms unseen, where quantum whispers speak,
A novel symmetry doth enter stage,
With parity and time, a dance unique,
To bind the quantum actors, new age.

The *PT*-symmetry, a twofold guise,
Where *P* reflects in space, yet *T*, in time,
Together sing a tale of compromise,
In quantum world, a balance, most sublime.

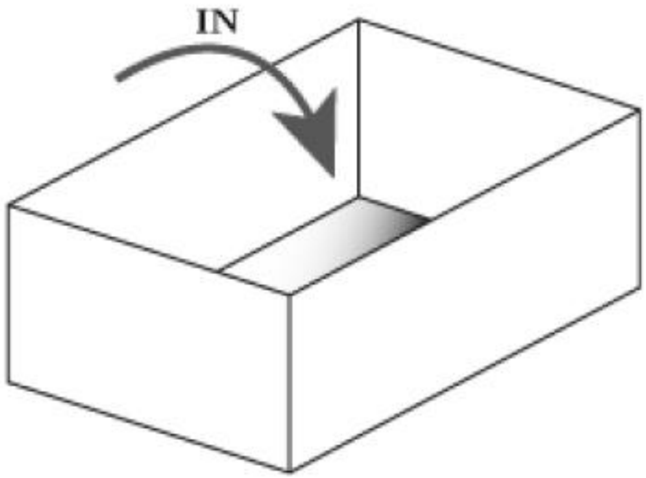
Unbroken, this domain of spectral dance,
The loss and gain of energy entwined,
Their quantum states, a harmonizing trance,
In eerie beauty, to each other bind.

And thus, in deep embrace of mystic schemes,
***PT*-symmetric quantum magic gleams.**

---Composed by ChatG*PT*

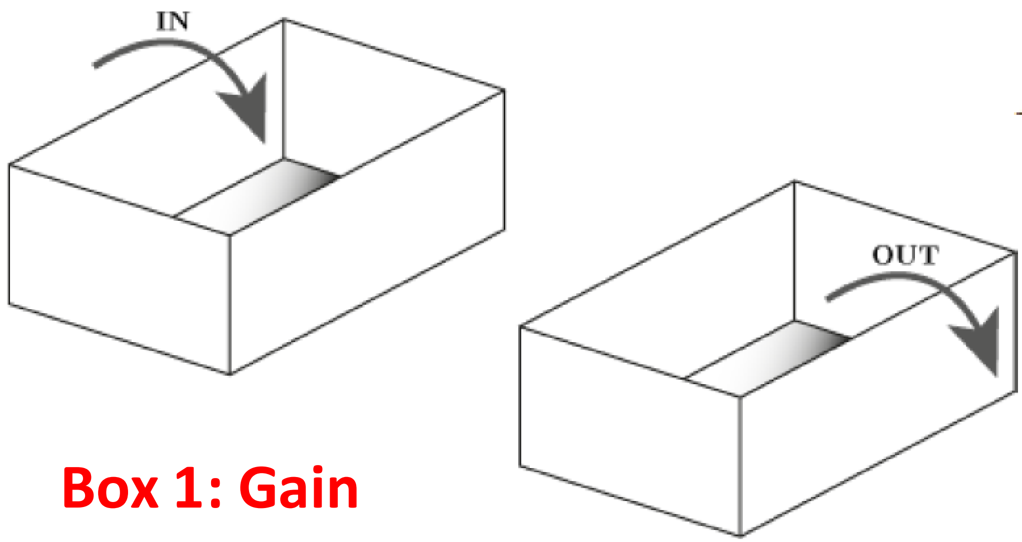
Intuitive explanation of the *PT* transition

Imagine a closed box with *gain*. The 1 x 1 Hamiltonian for this system is non-Hermitian: $H = [a+ib]$



Box 1: Gain

Two noninteracting closed boxes, one with *gain*, other with *loss*:



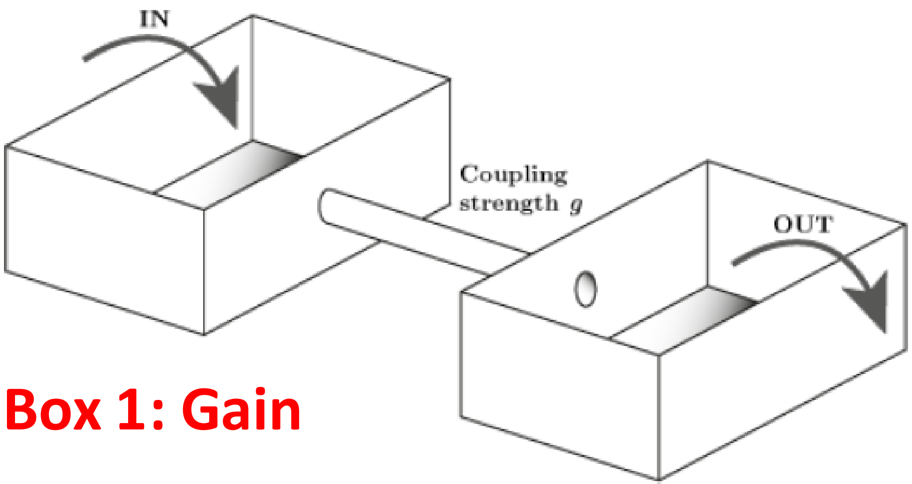
$$H_{\text{combined}} = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$$

Box 1: Gain

Box 2: Loss

System *not* in equilibrium

Couple the boxes:



Box 1: Gain

Box 2: Loss

$$H_{\text{coupled}} = \begin{bmatrix} a+ib & g \\ g & a-ib \end{bmatrix}$$

Hamiltonian is not Hermitian but it is *PT* symmetric:

Time reversal: $\mathcal{T}$ = complex conjugation

Parity: $\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Eigenvalues satisfy a *real* secular equation:

$$\det(H_{\text{coupled}} - IE) = E^2 - 2aE + a^2 + b^2 - g^2 = 0$$

$$E_{\pm} = a \pm (g^2 - b^2)^{1/2}$$

Transition at $|g| = |b|$
Energy is REAL if $|g| > |b|$

System in equilibrium for sufficiently large coupling

Unbroken PT symmetry means balanced loss and gain…

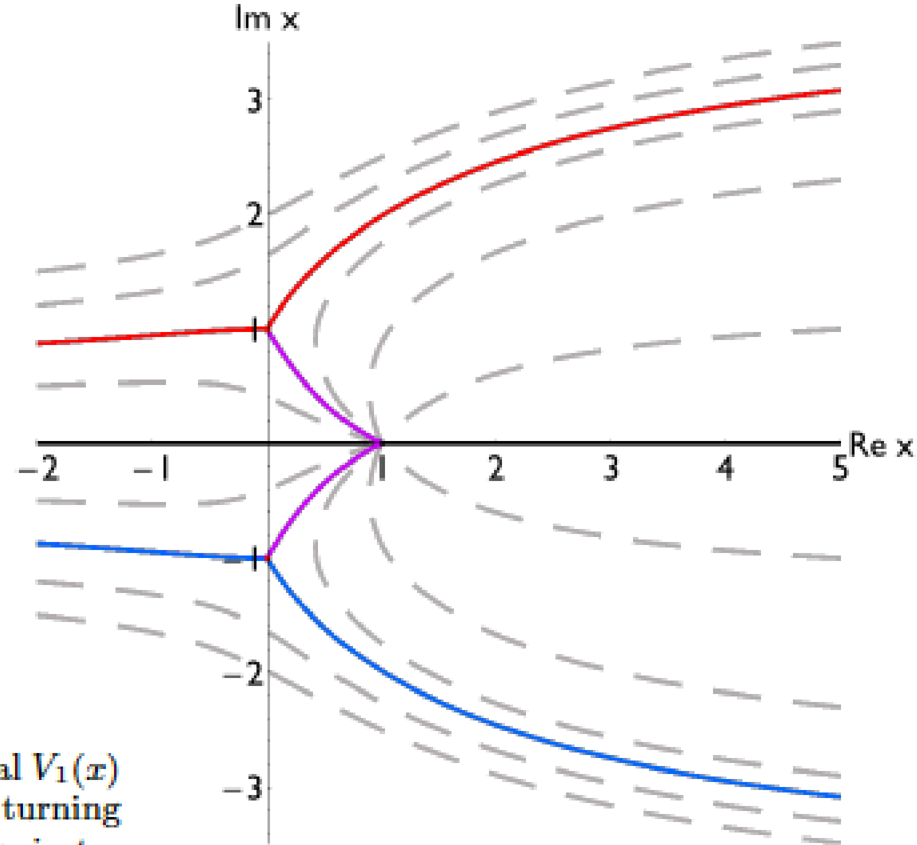


FIG. 8: Complex classical trajectories for the potential $V_1(x)$ in (11). The energy is $E = \frac{1}{2}$, so there is just one turning point at $x = 1$ and two poles at $x = \pm i$. Twelve trajectories (dashed lines) and separatrices (solid lines) are shown. A separatrix emerges from $x = i$ at 60°, −60°, and 180°. Trajectories start at $\mathrm{Re}\, x = -2$, and $\mathrm{Im}\, x = 0$, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$. Trajectories also start at $\mathrm{Re}\, x = 5$ and $\mathrm{Im}\, x = \pm 3.3$, ± 2.9, ± 2.3, ± 1, and 0. Three branches of the separatrix curve are shown in both the upper-half and the lower-half plane. The separatrices intersect at 120° angles at the poles. The separatrices end at the Zeno point (the turning point) $x = 1$, and they leave the plot at $x = -2 \pm 0.87i$ and at $x = 5 \pm 3.08i$.

$$V_1(x) = \frac{x}{x^2 + 1},$$