Resurgence and Nonperturbative Physics

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Applications of Quantum Field Theory to Hermitian and non-Hermitian Systems

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[DOE Division of High Energy Physics]

Physical Motivation: Decoding the QFT Path Integral

- path integral: the foundation of QFT
- theoretical challenges for conventional QFT methods:
  - high density ("sign problems")
  - high perturbative orders and highly nonlinear processes
  - non-equilibrium processes
  - strong fields & large gradients: short time/distance scales
  - coherence and decoherence
  - radiation reaction
  - quantum control & optimization
- "resurgence" unifies perturbative+nonperturbative QFT

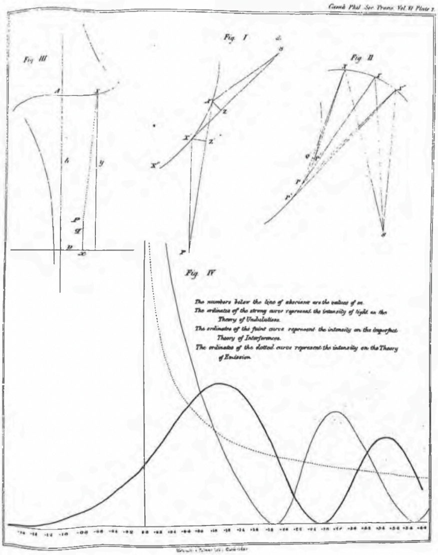
Airy: "Spurious Rainbows", and the Airy Function (1830s)

(Mika-Pekka Markkanen, via Wikimedia Commons)

Airy and Rainbows: The Original "Sign Problem" (Airy, 1836)

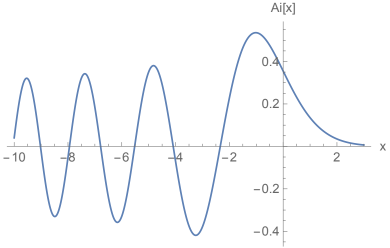
"On the intensity of light in the neighbourhood of a caustic"

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\phi\, e^{i\left(\frac{1}{3}\phi^3 + x\,\phi\right)}$$



"On the numerical calculation of a class of definite integrals and infinite series"

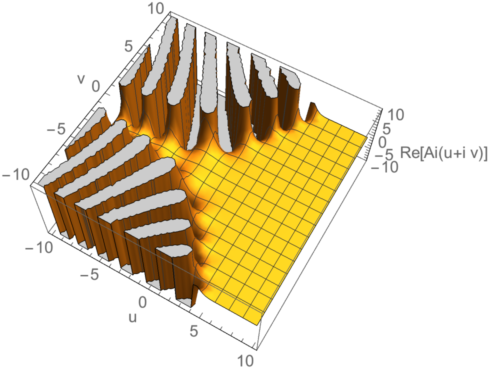
$$\mathrm{Ai}\left(x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\phi\, e^{i\left(\frac{1}{3}\phi^3 + x\,\phi\right)}$$



$$\mathrm{Ai}\left(x\right) \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\ \\ \dfrac{\sin\left(\frac{2}{3}\left(-x\right)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\left(-x\right)^{1/4}} & , \quad x \to -\infty \end{cases}$$

"Stokes, by mathematical supersubtlety, transformed Airy's integral into a form by which the light at any point of any of those thirty bands, and any desired greater number of them, could be calculated with but little labour" Lord Kelvin in Stokes's Obituary, 1903

"On the discontinuity of arbitrary constants which appear in divergent developments"



• *real physics* is often governed by *complex saddle points*

• Stokes phenomenon: as (the phase of) an external parameter varies, the saddle points move and the steepest descents contours are deformed. At certain phases, these contours jump and a saddle can appear or disappear

The Stokes Phenomenon in QFT

- basic feature of amplitude or S-matrix computations
- basic feature of QFT path integral

$$\int \mathcal{D}\phi \, \exp\left[\frac{i}{\hbar} S[\phi; m, g, \mu, B, E, \lambda, \tau, T, \ldots]\right]$$

- generator of perturbative (loop, gradient, ...) expansions

$$\sum_n a_n \, \hbar^n$$

- generator of nonperturbative (saddle) expansions

$$\sum_{\text{saddles}} e^{\frac{i}{\hbar} S_c} \, \det\left(\frac{\delta^2 S}{\delta \phi^2}\right) \sum (\text{fluctuations})$$

- these expansions look different, but they must agree !
- how they agree = resurgence

Resurgent Trans-Series

Resurgence: 'new' idea in mathematics (Écalle 1980s; Dingle 1960s; Stokes 1850s)

resurgence = unification of perturbative & non-perturbative physics

- perturbative series expansion ⟶ *trans-series* expansion

$$f(g) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} c_{k,l,p} \underbrace{(g)^p}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{S}{g}\right]\right)^k}_{\text{instantons}} \underbrace{(\ln[g])^l}_{\text{logarithms}}$$

- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, Chern-Simons, String Theory, ...
- "non-perturbative completion" (see Daniele Dorigoni's talk)
- define the path integral constructively as a trans-series

“Resurgence”

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle

Question: can we take advantage of this for QFT ?

Resurgence: generic large-order/low-order duality

- general feature of exponential integrals: e.g. Airy
- expansions about the two saddles are explicitly related

$$T_r^{\pm} = (\pm 1)^r \frac{\Gamma\left(r+\frac{1}{6}\right)\Gamma\left(r+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^r r!} = \left\{1, \pm\frac{5}{48}, \frac{385}{4608}, \pm\frac{85085}{663552}, \dots\right\}$$

- large order behavior of fluctuation coefficients:

$$T_r^{+} \sim \frac{(r-1)!}{(2\pi)\left(\frac{4}{3}\right)^r}\left(1 - \left(\frac{4}{3}\right)\frac{5}{48}\frac{1}{(r-1)} + \left(\frac{4}{3}\right)^2 \frac{385}{4608}\frac{1}{(r-1)(r-2)} - \dots\right)$$

- generic in nonlinear ODEs, PDEs, difference eqs, ...
- similar behavior in QM, matrix models, QFT ...

Borel summation: from series to transseries

- Borel transform of series, where $c_n \sim n! \quad , \quad n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series has finite radius of convergence (singularities)

- Borel summation of original asymptotic series:

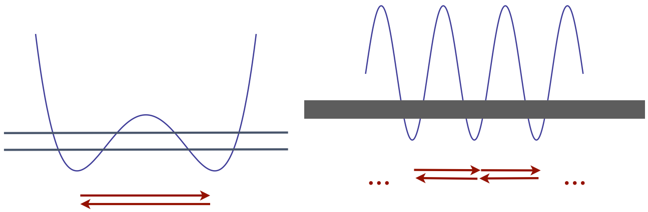
$$\mathcal{S} f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

- the singularities of $\mathcal{B}[f](t)$ provide a physical encoding of the global asymptotic behavior of $f(g)$

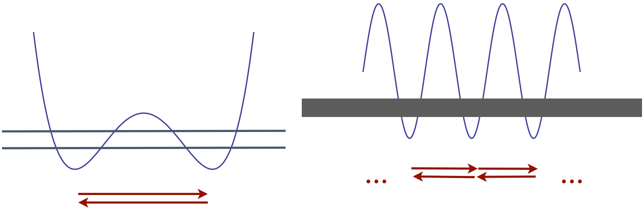
- Borel singularities = non-perturbative physical objects

- resurgence: perturbative sector encodes the non-perturbative sectors via the Borel transform

Resurgence in Infinite Dimensions: the QM Path Integral



Resurgence in Infinite Dimensions: the QM Path Integral



$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{S}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar}\, \mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

- one-instanton fluctuation factor:

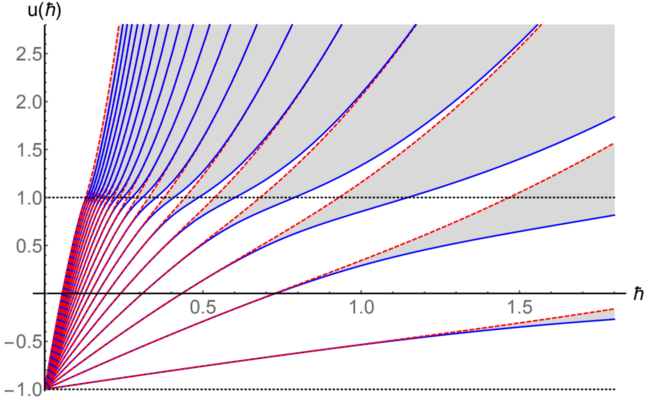
$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N + \frac{1}{2}\right)\, \hbar^2}{S} \right) \right]$$

- the entire trans-series can be decoded in terms of the perturbative series

Alvarez/Casares, GD/Ünsal, ...

Nonlinear Stokes Phenomenon in the Mathieu Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



- nonlinear Stokes transition: real/complex instantons

(Başar/GD/Ünsal 1603.04924, 1501.05671)

- cf. Nekrasov partition function for $\mathcal{N}=2$ SU(2) SYM

Gross-Witten-Wadia = 2d $U(N)$ Lattice Gauge Theory

$$Z(t, N) = \int_{U(N)} DU \, \exp\left[\frac{N}{t} \mathrm{tr}\left(U + U^\dagger\right)\right]$$

- 't Hooft coupling $t = g^2 N$
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal)

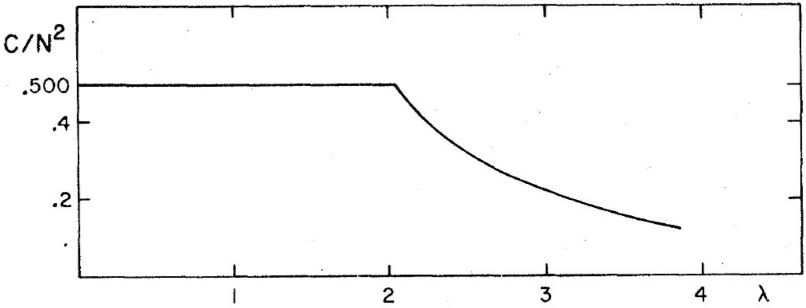


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

- "order parameter" $\Delta(t,N) \equiv \langle \det U \rangle$ satisfies a nonlinear ODE
- Rossi equation (Painlevé III):

$$t^2\Delta'' + t\Delta' + \frac{N^2\Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

- non-perturbative large N effects from the ODE

$$\Delta(t,N) = \sum_n \frac{c_n^{(0)}(t)}{N^{2n}} + e^{-N\,S(t)}\sum_n \frac{c_n^{(1)}(t)}{N^n} + e^{-2N\,S(t)}\sum_n \frac{c_n^{(2)}(t)}{N^n} + \dots$$

- all physical observables inherit this trans-series structure
- phase transition = nonlinear Stokes phenomenon
- universal reduction to Painlevé II across phase transition

Resurgence: Large N at Strong 't Hooft Coupling

- large N trans-series at strong coupling ($t > 1$)

$$\Delta(t, N) \approx \sigma_{\text{strong}}\, J_N\left(\frac{N}{t}\right) \sim \sigma_{\text{strong}}\, \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \ldots$$

- strong-coupling large N instanton action

$$S_{\text{strong}}(t) = \text{arccosh(t)} - \sqrt{1 - \frac{1}{t^2}}$$

- nonlinearity $\Rightarrow$ trans-series with all odd powers of

$$\sigma_{\text{strong}} \frac{e^{-N S_{\text{strong}}(t)}}{\sqrt{S'_{\text{strong}}(t)}}$$

Resurgence: Large N at Weak 't Hooft Coupling

- large N trans-series at weak-coupling ($t < 1$)

$$\Delta(t, N) \sim \sqrt{1-t} \sum_{n=0}^{\infty} \frac{d_n^{(0)}(t)}{N^{2n}} - \frac{\sigma_{\rm weak}}{2\sqrt{2\pi N}} \frac{t\, e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}} \sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} + \dots$$

- weak-coupling large N instanton action

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,{\rm arctanh}\left(\sqrt{1-t}\right)$$

- large-order growth of perturbative coefficients ($\forall\, t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}} \frac{\Gamma(2n - \frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}} \left[1 + \frac{(3t^2 - 12t - 8)}{96(1-t)^{3/2}} \frac{S_{\rm weak}(t)}{(2n - \frac{7}{2})} + \dots\right.$$

- (parametric) resurgence relations, for all t:

$$\sum_{n=0}^{\infty} \frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2 - 12t - 8)}{96(1-t)^{3/2}} \frac{1}{N} + \dots$$

Resurgence in GWW: double-scaling limit = Painlevé II

- uniform limit of Bessel function:

$$\lim_{N\to\infty} J_N(N - N^{1/3}\kappa) = \left(\frac{2}{N}\right)^{1/3} \mathrm{Ai}\left(2^{1/3}\kappa\right)$$

- scaling of $J_N(N/t)$ as $t \to 1$: $\qquad N \to \infty$ with x fixed

$$t \sim 1 + \frac{x}{(2N^2)^{1/3}} \qquad ; \qquad \Delta(t, N) = \left(\frac{2t}{N}\right)^{1/3} y(x)$$

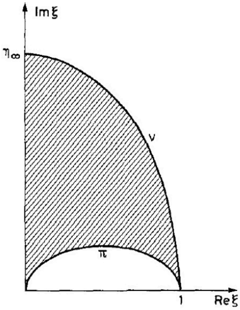
$$\Delta \quad \text{PIII equation} \quad \longrightarrow \quad \frac{d^2y}{dx^2} = x\, y(x) + 2\, y^3(x) \quad \text{(PII)}$$

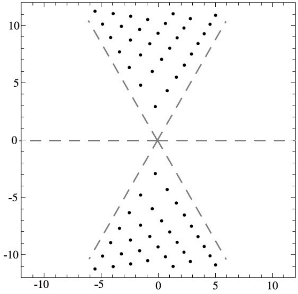
- Painlevé II = "nonlinear Airy equation"

- the immediate vicinity of the physical phase transition region is described by the Hastings-McLeod Painlevé II solution

Lee-Yang: complex zeros of $Z(t, N)$ pinch real axis at phase transition point in the thermodynamic ($N \to \infty$) limit

- double-scaling: bridge across transition (nonlinear Airy)





GWW zeros (Kolbig)   Painlevé II (Novokshenov; Huang)

• idea: “reconstruct” non-perturbative physics from a “reasonable” amount of perturbative input information

• the key to a more accurate analytic continuation from the original series is a more accurate analytic continuation of its Borel transform, especially near its singularities

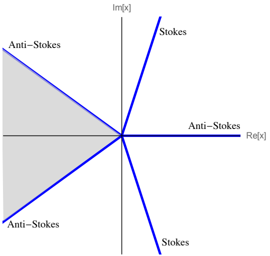
• technical problem: given a finite number (possibly small) of terms in a perturbative expansion, which is presumably asymptotic, what is the most effective way to analytically continue the truncated Borel transform?

[new optimal methods: O. Costin, GD 2003.07451, 2009.01962, 2108.01145]

Analytic Continuation of Painlevé I *tritronquée* (Costin, GD: 1904.11593)

- Painlevé I: $y''(x) = 6\,y^2(x) - x$
- series expansion as $x \to +\infty$

$$y(x) \sim -\sqrt{\frac{x}{6}}\left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{30}{(24\,x)^{5/4}}\right)^{2n}\right)$$



- 5-fold symmetry: $y(x) \approx \sqrt{x}\,\mathcal{P}\left(\frac{4}{5}x^{5/4}; \{2, g_3\}\right)$ (Boutroux)
- *tritronquée*: poles only in $\frac{2\pi}{5}$ wedge (Dubrovin et al)

$$y(x) \approx \frac{1}{(x - x_{\rm pole})^2} + \frac{x_{\rm pole}}{10}(x - x_{\rm pole})^2 + \frac{1}{6}(x - x_{\rm pole})^3$$
$$+ h_{\rm pole}(x - x_{\rm pole})^4 + \frac{x_{\rm pole}^2}{300}(x - x_{\rm pole})^6 + \ldots$$

- Q: does the expansion as $x \to +\infty$ "know" this ?

Transmutation: Asymptotic Series to Meromorphic Function

$$y(x) = \frac{1}{(x - x_{\text{pole}})^2} + \frac{x_{\text{pole}}}{10}(x - x_{\text{pole}})^2 + \frac{1}{6}(x - x_{\text{pole}})^3 + h_{\text{pole}}(x - x_{\text{pole}})^4 + \frac{x_{\text{pole}}^2}{300}(x - x_{\text{pole}})^6 + \dots$$

- Resurgent extrapolation of $y(x)$ near 1st pole:

$$\begin{aligned} y(x) \approx\ & \frac{0.9999999999999999999999999999999997886}{(x-x_1)^2} + 3.5 \times 10^{-35} - 2.4 \times 10^{-34}(x - x_1) \\ & -0.238416876956881663929914585244923803(x-x_1)^2 \\ & +0.166666666666666666666666666666657864(x-x_1)^3 \\ & -0.06213573922617764089649014164005140(x-x_1)^4 + 4 \times 10^{-31}(x-x_1)^5 \\ & +0.0189475357392909503157755851627665(x-x_1)^6 + \dots \end{aligned}$$

- exact non-perturbative connection formulas satisfied to high precision (O.Costin & GD: 1904.11593)

- extracted from a completely different expansion at $x \to +\infty$

Heisenberg-Euler Effective Action

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B}) \cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

($\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.

$|\mathfrak{E}_k| = \frac{m^2 c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} =$ „Kritische Feldstärke“.)

- the first (non-perturbative) QFT computation
- paradigm of “effective field theory” (non-linear)
- compute: $\ln\det(\not{D} + m)$, $\not{D} := \not{\partial} + ie\not{A}$
- generating function for multi-leg one-loop amplitudes

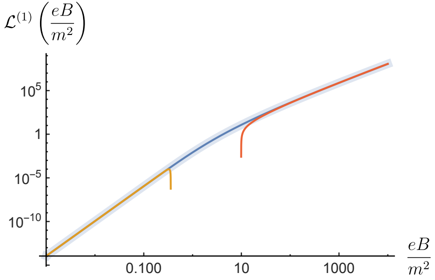
Resurgent Extrapolation of Heisenberg-Euler   GD/Harris 2101.10409

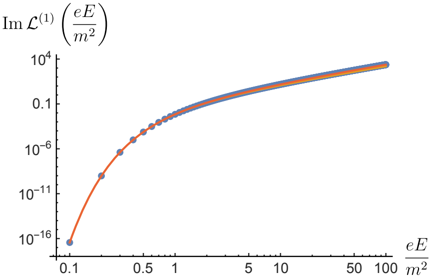
$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) = -\frac{B^2}{2}\int_0^\infty \frac{dt}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right) e^{-m^2 t/(eB)}$$

$$\sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^\infty (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}}\zeta(2n+4)\left(\frac{eB}{m^2}\right)^{2n} \quad , \quad eB \ll m^2$$

$$\sim \frac{1}{3}\cdot\frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right) + \ldots \quad , \quad eB \gg m^2$$

- small $B \to$ large B; small $B \to$ large E (from 10 terms!)





- exponentially suppressed terms are also accessible
- also at 2 loop (no Borel representation)

Inhomogeneous Fields: further divergence

- EFT expansion grows rapidly: one of <u>many</u> pages at 6th order

$$
\begin{aligned}
&-\frac{89}{693}\,F_{\kappa\lambda}F_{\mu\nu\rho}F_{\sigma\rho\lambda}F_{\mu\kappa\sigma\nu}-\frac{89}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu\rho}F_{\mu\rho\sigma}F_{\sigma\nu}-\frac{89}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\rho\nu\sigma}F_{\rho\sigma\mu}\\
&-\frac{94}{1155}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\kappa\sigma}F_{\rho\nu}F_{\sigma\mu}+\frac{97}{3080}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\kappa}F_{\rho\sigma}F_{\nu\mu}F_{\sigma\rho}+\frac{100}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\lambda\sigma}F_{\rho\nu}F_{\sigma\mu\kappa}\\
&+\frac{101}{1386}\,F_{\kappa\lambda\mu}F_{\nu\mu\lambda}F_{\kappa\rho\sigma}F_{\nu\sigma\rho}-\frac{101}{6930}\,F_{\kappa\lambda\mu}F_{\mu\nu\rho}F_{\rho\lambda\sigma}F_{\sigma\nu\kappa}-\frac{101}{6930}\,F_{\kappa\lambda\mu}F_{\nu\kappa\rho}F_{\sigma\nu\mu}F_{\lambda\sigma\rho}\\
&+\frac{101}{9240}\,F_{\kappa\lambda\mu}F_{\nu\rho\sigma}F_{\kappa\mu\lambda}F_{\nu\sigma\rho}-\frac{103}{630}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\nu\kappa}F_{\mu\rho\sigma}F_{\sigma\rho}-\frac{103}{693}\,F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\mu\sigma\rho\kappa}F_{\sigma\nu}\\
&+\frac{103}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu\rho}F_{\rho\sigma}F_{\kappa\sigma\nu}-\frac{103}{1386}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\nu\rho}F_{\rho\sigma}F_{\sigma\mu}-\frac{103}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\sigma}F_{\mu\nu\kappa}F_{\sigma\rho}\\
&-\frac{103}{1386}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\lambda\sigma}F_{\nu\mu}F_{\rho\sigma\kappa}-\frac{109}{1260}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho\sigma}F_{\sigma\rho}F_{\kappa\nu\mu}+\frac{109}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\rho\nu}F_{\sigma\mu\kappa}\\
&-\frac{115}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\kappa\sigma}F_{\rho\mu}F_{\sigma\nu}-\frac{122}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\rho\nu\sigma}F_{\rho\kappa}F_{\sigma\mu}-\frac{122}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu}F_{\rho\nu\sigma}F_{\rho\sigma\kappa}\\
&-\frac{124}{3465}\,F_{\kappa\lambda}F_{\mu\nu\rho}F_{\rho\mu\lambda\sigma}F_{\sigma\nu\kappa}-\frac{128}{693}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\mu\kappa}F_{\rho\sigma}F_{\sigma\nu}+\frac{128}{1155}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\kappa\rho}F_{\mu\nu\sigma}F_{\sigma\rho}\\
&-\frac{128}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\rho\sigma}F_{\kappa\sigma\nu}F_{\rho\mu}+\frac{130}{693}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\kappa\rho}F_{\nu\sigma}F_{\mu\sigma\rho}-\frac{134}{3465}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\kappa\rho\sigma}F_{\nu\sigma\rho}\\
&+\frac{151}{13860}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\rho\sigma}F_{\nu\mu}F_{\sigma\rho}+\frac{152}{3465}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\sigma\nu\mu}F_{\sigma\rho\lambda\kappa}+\frac{157}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\nu\rho}F_{\mu\sigma}F_{\sigma\rho\kappa}\\
&-\frac{163}{1155}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\mu\sigma}F_{\sigma\rho\nu\kappa}-\frac{163}{1155}\,F_{\kappa\lambda}F_{\mu\nu\lambda\rho}F_{\nu\mu\sigma}F_{\sigma\rho\kappa}-\frac{163}{3465}\,F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\rho\mu\kappa\sigma}F_{\sigma\nu}\\
&+\frac{163}{5544}\,F_{\kappa\lambda}F_{\lambda\kappa}F_{\mu\nu}F_{\rho\sigma}F_{\sigma\rho}F_{\nu\mu}+\frac{164}{3465}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\kappa\sigma}F_{\rho\mu}F_{\sigma\nu}-\frac{166}{3465}\,F_{\kappa\lambda}F_{\lambda\mu\nu\rho}F_{\mu\rho\sigma}F_{\sigma\nu\kappa}\\
&-\frac{166}{3465}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\rho\nu\sigma}F_{\sigma\rho\mu\kappa}-\frac{166}{3465}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\mu\sigma}F_{\rho\kappa\sigma\nu}-\frac{166}{3465}\,F_{\kappa\lambda}F_{\mu\nu\lambda\rho}F_{\nu\mu\sigma}F_{\kappa\sigma\rho}\\
&+\frac{169}{6930}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho\sigma}F_{\nu\kappa}F_{\mu\sigma\rho}-\frac{178}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho\sigma}F_{\mu\kappa}F_{\nu\sigma\rho}+\frac{179}{1386}\,F_{\kappa\lambda}F_{\mu\nu\lambda\kappa}F_{\nu\rho\sigma}F_{\mu\sigma\rho}\\
&+\frac{181}{2772}\,F_{\kappa\lambda}F_{\mu\lambda\kappa}F_{\nu\rho\sigma}F_{\nu\mu\sigma\rho}+\frac{181}{2772}\,F_{\kappa\lambda}F_{\mu\nu\rho\sigma}F_{\nu\sigma\rho}F_{\mu\lambda\kappa}+\frac{212}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\kappa\rho}F_{\sigma\nu\mu}F_{\sigma\rho}\\
&+\frac{218}{3465}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\mu\rho}F_{\rho\sigma}F_{\sigma\nu}+\frac{218}{3465}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu}F_{\kappa\rho}F_{\nu\sigma}F_{\sigma\rho}-\frac{218}{3465}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho\sigma}F_{\nu\mu\kappa\sigma\rho}\\
&-\frac{218}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu\nu}F_{\nu\rho\sigma}F_{\sigma\kappa}F_{\rho\mu}+\frac{221}{1980}\,F_{\kappa\lambda}F_{\mu\nu\rho}F_{\sigma\rho\nu}F_{\sigma\mu\lambda\kappa}+\frac{227}{6930}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\nu\rho\sigma}F_{\nu\kappa}F_{\sigma\rho}\\
&-\frac{233}{83160}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\sigma}F_{\lambda\kappa}F_{\nu\mu}F_{\sigma\rho}-\frac{235}{2772}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\kappa}F_{\nu\rho\sigma}F_{\sigma\rho}F_{\nu\mu}-\frac{236}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\lambda\sigma}F_{\nu\kappa}F_{\rho\sigma\mu}\\
&+\frac{256}{3465}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu}F_{\nu\rho}F_{\mu\sigma}F_{\sigma\rho}-\frac{262}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\mu\rho}F_{\nu\kappa\sigma}F_{\sigma\rho}+\frac{263}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\rho\kappa}F_{\sigma\nu\mu}\\
&+\frac{269}{3465}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\rho\kappa\sigma}F_{\rho\sigma\nu}+\frac{269}{3465}\,F_{\kappa\lambda}F_{\mu\nu\rho}F_{\mu\sigma\rho\lambda}F_{\sigma\nu\kappa}-\frac{274}{3465}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\sigma\mu\kappa}F_{\sigma\rho\nu}\\
&-\frac{284}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\mu\kappa}F_{\nu\rho\sigma}F_{\nu\sigma\rho}-\frac{284}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\rho\sigma}F_{\mu\kappa}F_{\nu\sigma\rho}-\frac{289}{2310}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\rho\sigma}F_{\nu\kappa}F_{\sigma\rho}\\
&-\frac{326}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\lambda\nu}F_{\mu\rho}F_{\nu\kappa\sigma}F_{\sigma\rho}+\frac{331}{6930}\,F_{\kappa\lambda}F_{\mu\nu}F_{\rho\sigma\lambda\kappa}F_{\sigma\rho\nu\mu}+\frac{349}{1155}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\rho\mu\sigma}F_{\kappa\sigma\nu}\\
&-\frac{349}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\mu\kappa}F_{\nu\rho\sigma}F_{\sigma\rho}-\frac{356}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho}F_{\sigma\nu\kappa}F_{\sigma\rho\mu}+\frac{358}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\nu\rho}F_{\rho\mu\sigma}F_{\sigma\nu\kappa}\\
&-\frac{368}{3465}\,F_{\kappa\lambda}F_{\mu\nu\rho}F_{\lambda\sigma\rho\nu}F_{\kappa\sigma\mu}-\frac{376}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\lambda\mu}F_{\kappa\nu\rho}F_{\rho\sigma}F_{\mu\sigma\nu}-\frac{397}{3465}\,\mathrm{i}\,F_{\kappa\lambda}F_{\mu\nu}F_{\lambda\rho\sigma}F_{\sigma\nu}F_{\kappa\rho\mu}
\end{aligned}
$$

Fliegner et al, hep-th/9707189

Inhomogeneous Background Fields

GD & Z.Harris: 2212.04599

- precise comparison: test method on soluble cases

$$B(x) = B\,\text{sech}^2\left(x/\lambda\right) \qquad E(t) = E\,\text{sech}^2\left(t/\tau\right)$$

- analytic continuations: $B^2 \mapsto -E^2$, $\lambda^2 \mapsto -\tau^2$

- Keldysh inhomogeneity parameter

$$\gamma = \frac{\ell_B^2}{\lambda_C \lambda} = \frac{m}{eB\lambda} \mapsto \frac{m}{eE\tau}$$

- exact Dirac spectrum, so can be solved in various ways

- weak B field expansion

$$\frac{S(B,\lambda)}{L^2\lambda T} = \frac{m^4}{\pi^2} \sum_{n\geq 0} a_n(\gamma) \left(\frac{B}{m^2}\right)^{2n+4}$$

- $a_n(\gamma)$: polynomial in inhomogeneity parameter γ

- three independent Borel singularities can be seen in the large order growth of the perturbative coefficients $a_n(\gamma)$

Resurgence for Inhomogeneous Background Fields

- large order growth of $a_n(\gamma)$: $|t_1| = 1/(\sqrt{1+\gamma^2}+1)$

$$a_n(\gamma) \sim (-1)^n \Gamma(2n+\tfrac{3}{2}) \frac{3\sqrt{2\pi}}{|t_1|^{2n+3/2}} (1+\gamma^2)^{5/4}$$

$$\times \left[1 - \frac{5}{4}\frac{(1-\frac{3}{4}\gamma^2)}{\sqrt{1+\gamma^2}}\frac{|t_1|}{(n+\frac{1}{4})} + \frac{105}{32}\frac{(1+\frac{1}{4}\gamma^2)^2}{(1+\gamma^2)}\frac{|t_1|^2}{(n+\frac{1}{4})(n-\frac{1}{4})} + \dots\right]$$

- non-perturbative imaginary part

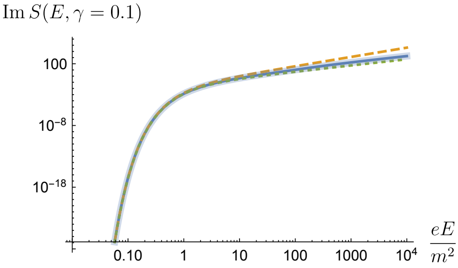
$$\frac{\mathrm{Im}S(E,\tau)}{L^3\tau} \sim \frac{m^4}{8\pi^3}\left(\frac{E}{m^2}\right)^{5/2}(1+\gamma^2)^{5/4}\exp\left(-\frac{\pi m^2}{E}\frac{2}{\sqrt{1+\gamma^2}+1}\right)$$

$$\times \left[1 - \frac{5}{4}\frac{(1-\frac{3}{4}\gamma^2)}{\sqrt{1+\gamma^2}}\left(\frac{E}{\pi m^2}\right) + \frac{105}{32}\frac{(1+\frac{1}{2}\gamma^2+\frac{1}{16}\gamma^4)}{(1+\gamma^2)}\left(\frac{E}{\pi m^2}\right)^2 + \dots\right]$$

- & all Borel singularities & all multi-instantons

Resurgent Extrapolation for Inhomogeneous Background Fields

- analytic continuation: $B \to i\,E$ and $\lambda \to i\,\tau$
- weak B field to strong E field (+ strong inhomogeneity)
- input: just 15 perturbative input terms



(blue=exact; blue shaded=extrapolation; orange=WKB; green =LCFA)

- accurate agreement over many orders of magnitude
- far superior to WKB or LCFA

Resurgence in Chern-Simons Theory (Costin, GD,Gruen, Gukov 2310.12317)

- Chern-Simons = topological quantum field theory
- sensitive probe of the topology of its 3-manifold

resurgence: decode topological information from perturbative data

Topology	Resurgence
flat connection	path integral saddle
Chern-Simons invariant	Borel singularity
Adjoint Reidemeister torsion	residue

Exact CS Invariant	Normalized CS Invariant	Padé-Borel	Padé-Conformal -Borel	Singularity Elimination
-0.002943401	1	1	1	1
-0.485874320	165.072391	not resolved	not resolved	161.05
0.053933576	-18.323554	not resolved	absent	absent
$0.123303626 \pm 0.035424464i$	$-41.891542 \mp 12.03527i$	$-42 \mp 12i$	$-41.8814 \mp 12.0371i$	$-41.891542 \mp 12.03527i$
0.235159766	-79.893881	not resolved	not resolved	-79.89
-0.171882873	58.3960000	not resolved	58.3754	58.3960000

- resurgent continuation across the natural boundary (!)

Conclusions

• nonperturbative QFT requires new theoretical ideas and methods

• Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series, which 'encode' analytic continuation information

• resurgent extrapolation: strong-field and non-perturbative and non-adiabatic information can be decoded efficiently from perturbative data

• QM, matrix models, Chern-Simons, ... ✓

• 2d sigma models ✓

• integrable/localizable SUSY QFT ✓

• 4d QFT ? [13 loop for $O(N)$ ϕ^4 (Borinsky, Panzer, Balduf,...)]

• there is extra un-tapped 'magic' in perturbation theory